Reaction diffusion equations in heterogeneous media

Propagation in unbounded domains

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Homogeneous equations

Equation

$$u_t = \Delta u + f(u) \text{ in } \mathbb{R}^N,$$

with f of Fisher-KPP type.

Travelling waves, spreading properties... KPP, Aronson-Weinberger, Fife - McLeod

Planar travelling fronts:

$$u(t,x) = \phi(x \cdot \vec{e} - ct) \text{ where } \phi : \mathbb{R} \to \mathbb{R},\ |\vec{e}| = 1$$

$$\begin{cases} -\phi'' - c\phi' & = & f(\phi) \\ \phi(-\infty) = 1, \quad \phi(+\infty) & = & 0 \end{cases}$$

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Spreading properties refer to

Cauchy problem

$$\begin{cases} u_t = \Delta u + f(u) \text{ in } \mathbb{R}^N \\ u(0,x) = u_0(x) \end{cases}$$

with $u_0 \geq 0$, $u_0 \not\equiv 0$ having compact support.

- Propagation: $u(t,x) \to 1$ as $t \to \infty$ (say $f(1) = 0$),
- Asymptotic speed of spreading: c^* such that for all $c < c*$

$$\sup_{|x| \leq ct} |u(t,x) - 1| \to 0 \quad \text{as} \quad t \to \infty$$

and for all $c > c*$

$$\sup_{|x| \geq ct} u(t,x) \to 0 \quad \text{as} \quad t \to \infty$$

In the Fisher - KPP case, the asympt. speed of spreading is $c^* = 2\sqrt{f'(0)}$ (Aronson-Weinberger).

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Non homogeneous reaction-diffusion equations

Homogeneous equation in “non homogeneous geometry"

$$\begin{cases} u_t = \Delta u + f(u) & \text{in } \Omega, \\ \nu \cdot \nabla u = 0 & \text{on } \partial\Omega \end{cases}$$

General (non homogeneous) equation with coefficients depending on t, x:

$$\begin{cases} u_t = \nabla(A(t,x)\nabla u) + q(t,x) \cdot \nabla u + f(t,x,u) & \text{in } \Omega, \\ A(t,x)\nabla u \cdot \nu = 0 & \text{on } \partial\Omega \end{cases}$$

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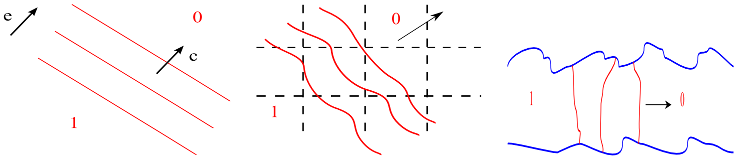
Generalised transition waves
(HB – François Hamel)



(a) Planar travelling front
(b) Pulsating front (periodic framework)
(c) Almost periodic environment (H. Matano)

Observation

There is a family of hypersurfaces $(\Gamma_t)_{t\in\mathbb{R}}$ such that :

$$\forall\,\lambda \in (0,1),\ \exists\, C_\lambda,\ \forall\, t \in \mathbb{R},\ \{x,\ u(t,x)=\lambda\} \subset \{x,\ d(x,\Gamma_t) \le C_\lambda\}$$

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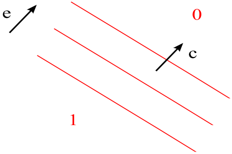
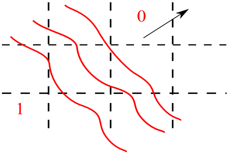
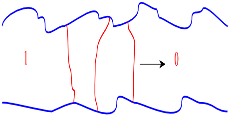


(d) Planar travelling front



(e) Pulsating front (periodic framework)



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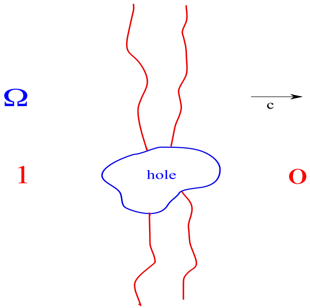
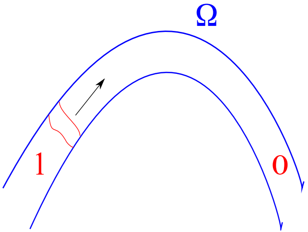
Goal :

To deal with more general situations



(g) Obstacle



(h) Curved cylinder

spirals,... or

$$u_t = u_{xx} + b(x)f(u)$$

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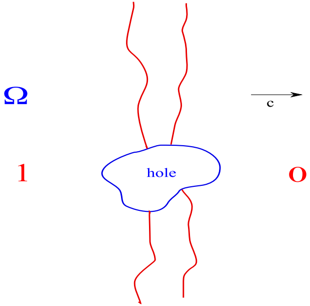

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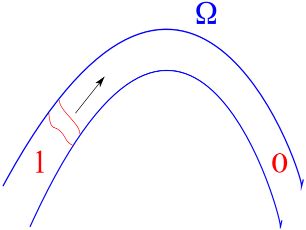

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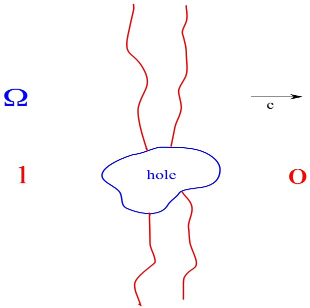

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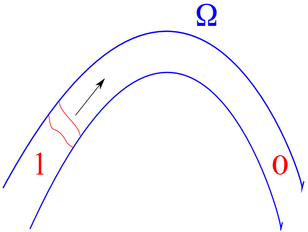

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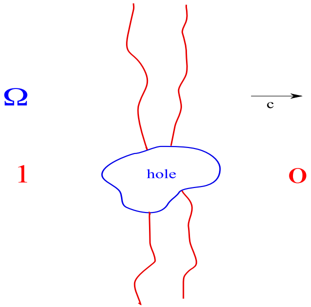

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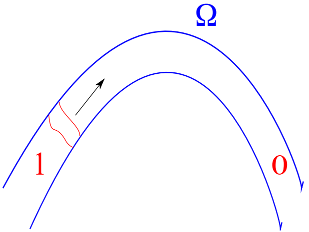

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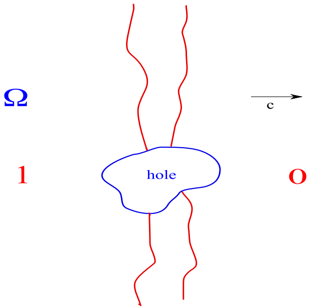

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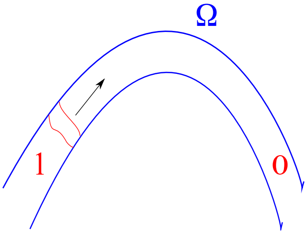

(p) Curved cylinder

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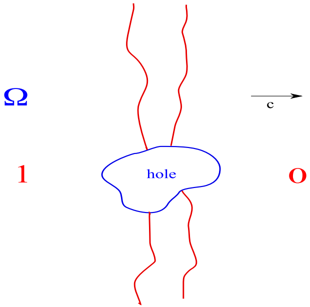

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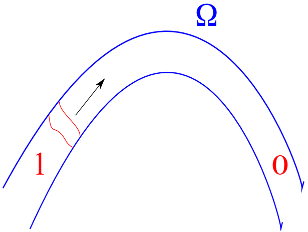

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Generalised waves [HB & François Hamel]

[Eq] + [BC]

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or

$$u_t = \Phi(t,x,u,Du,D^2u,\cdots), \quad x\in\Omega$$

d_Ω : geodesic distance in Ω

- Two *time- global* solutions of [Eq] & [BC]: $p^-(t,x)$ and $p^+(t,x)$ defined on $\mathbb{R}\times\Omega$
- Two families of open disjoint sets in Ω : $(\Omega_t^-)_{t\in\mathbb{R}}$ and $(\Omega_t^+)_{t\in\mathbb{R}}$
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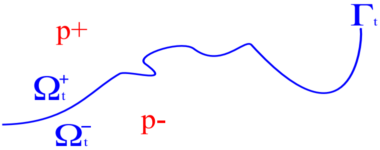
Assume $\Gamma_t \subset \bigcup_{finite} \{\text{graphs}\}$



Definition of (generalised) transition waves

Two given time-global solutions : $p^{\pm}(t,x)$

Definition

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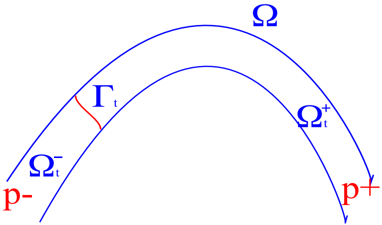

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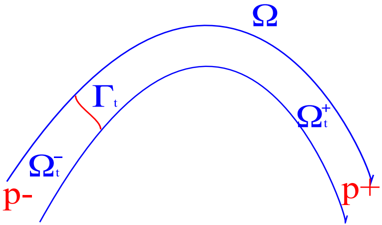

Related notions

- Periodic media: pulsating traveling waves
- Almost periodic media and extensions: H. Matano's definition
- Time periodic problems and random media: W. Shen (adapting Matano's definition)

A definition for general heterogeneous media.

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- Almost periodic media and extensions: H. Matano's definition
- Time periodic problems and random media: W. Shen (adapting Matano's definition)

A definition for general heterogeneous media.

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Definition

A transition wave u between p^- and p^+ has global mean speed *c if*

$$\frac{d_\Omega(\Gamma_t, \Gamma_s)}{|t-s|} \to c \text{ as } |t-s| \to +\infty$$

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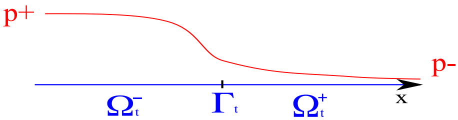
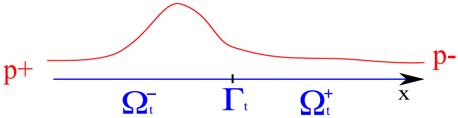
2. Further specifications

- Fronts : $p_i^-(t,x) < p_i^+(t,x)$ for all (t,x,i), or $p_i^-(t,x) > p_i^+(t,x)$



- Spatially extended pulses : $p^-(t,x) \equiv p^+(t,x)$



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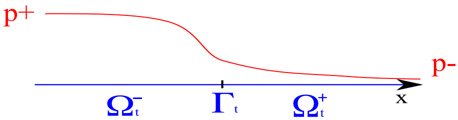

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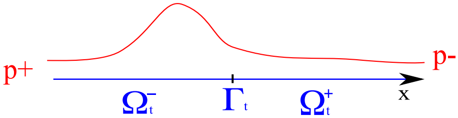

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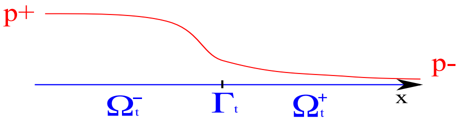

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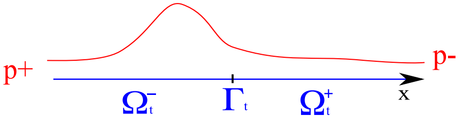

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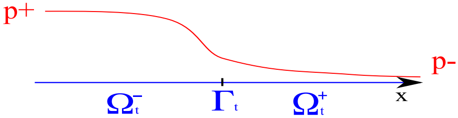

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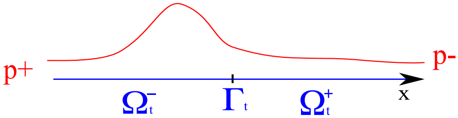

- Invasions (of p^- by p^+) :

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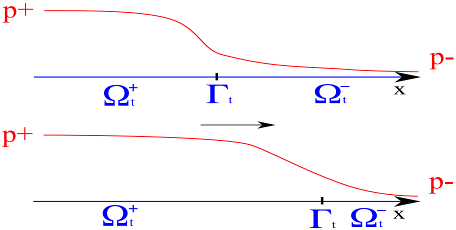

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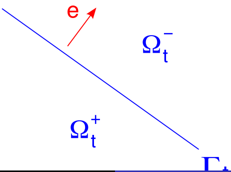

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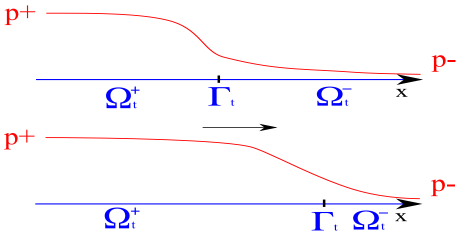

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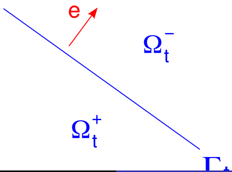

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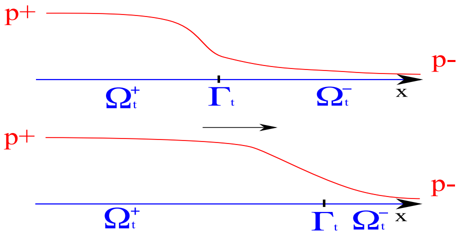

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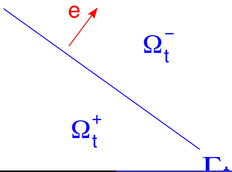

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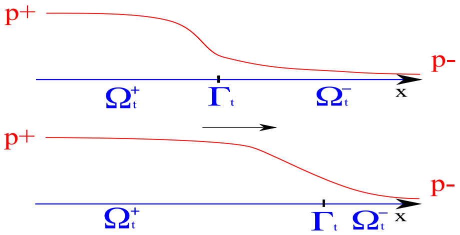

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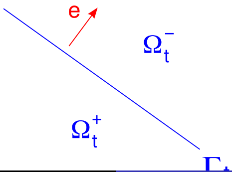

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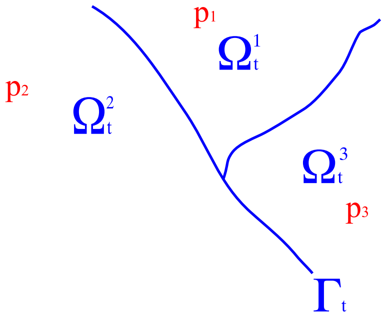
- Multiple transitions between $p_1, \ldots, p_k$:



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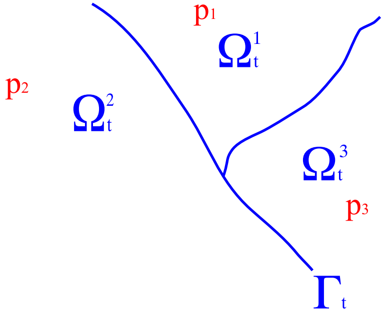

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3. Classical notions

Homogeneous case: $\frac{\partial u}{\partial t} - \Delta u = f(u) \quad \text{in} \quad \mathbb{R}^N$
Planar travelling waves:

$$u(t,x) = \phi(x \cdot e + ct)$$

with $\phi(\xi) \to 1$ and $\phi(\xi) \to 0$ when $\xi \to \pm\infty$,
Then, $u(t,x)$ is a GTW - a generalised invasion planar front of speed c,

$$p^- \equiv 0, \quad p^+ = 1$$

and

$$\Omega_t^+ = \{x; x \cdot e > -ct\}, \quad \Omega_t^- = \{x; x \cdot e < -ct\}$$

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Solutions of Hamel - Nadirashvili type

Homogeneous case
$\partial_t u - \partial_{xx} u = f(u)$ in $\mathbb{R}$ KPP case (example $f > 0$ concave, $f(0) = f(1) = 0$).
For $c_2 > c_1 \geq 2\sqrt{f'(0)}$, front, with speed c_1 when $t \to -\infty$ and with speed c_2 when $t \to \infty$

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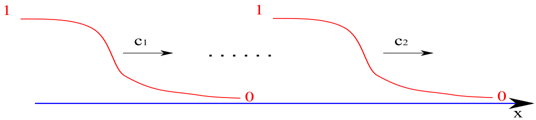

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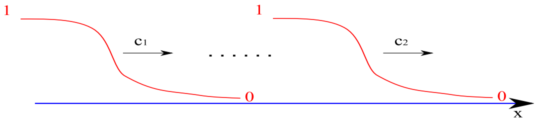

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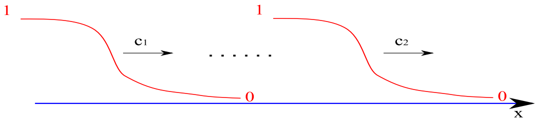

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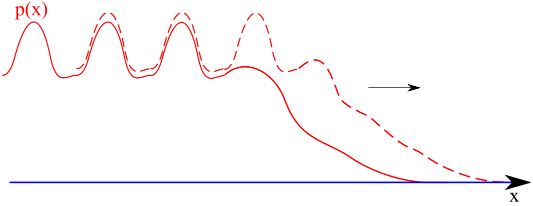
Examples of non-constant limiting states $p^{\pm}$



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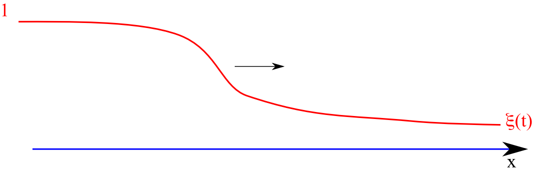

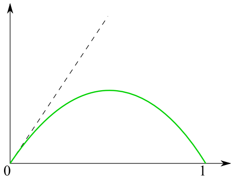

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(t) with KPP type f

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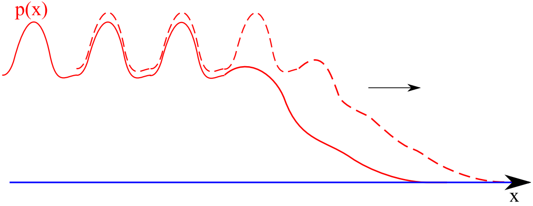

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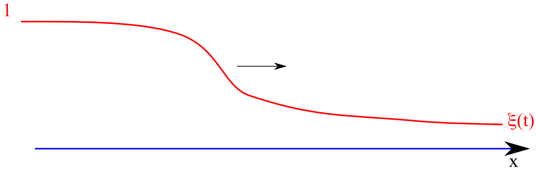

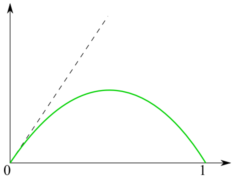

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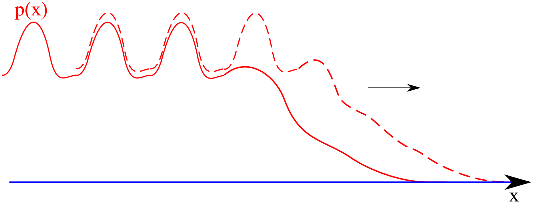

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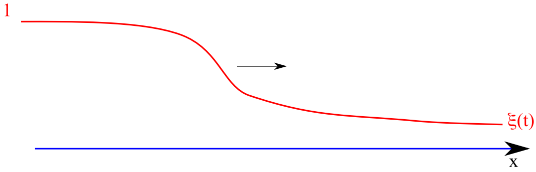

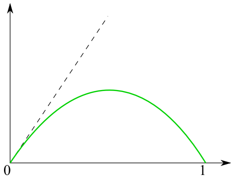

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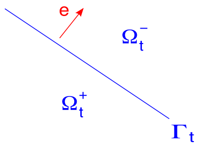
4. Properties

Example of a classification result

$$u_t = \Delta u + f(u)$$



Transition wave u between 0 and 1, $f'(0) < 0$, $f'(1) < 0$, almost-planar ($\Gamma_t = \{x \cdot e = \xi_t\}$) and

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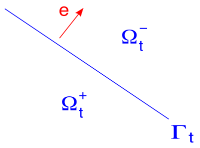

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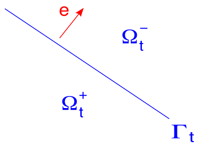

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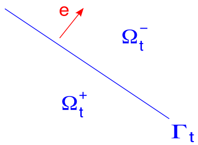

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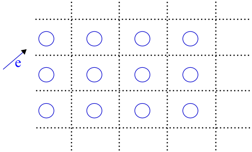
$\Rightarrow$

Robustness of the definitions

Periodic bistable-type medium



$$u_t = \mathrm{div}(A(x)\nabla u) + q(x)\cdot\nabla u + f(x,u)$$

Assumptions :

- $p^-(x) < p^+(x)$
- u is an almost-planar invasion in the direction e, with mean speed c,

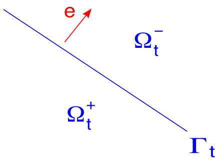
$$-M \;\leq\; d_\Omega(\Gamma_t,\Gamma_s) - c|t-s| \;\leq\; M$$

- $s \mapsto f(x,s)$ is nonincreasing in $[p^-(x), p^-(x)+\delta]$ and $[p^+(x)-\delta, p^+(x)]$

Conclusion : u is a pulsating front

Invariance in a moving frame

$$u_t = \mathrm{div}(A(x')\nabla u) + q(x') \cdot \nabla u + f(x', u)$$



Assumptions : Straight cylinder in the direction e, variables $(x \cdot e, x')$, $p^-(x') < p^+(x')$, almost-planar $\Gamma_t = \{x \cdot e = \xi_t\}$, bistable-type profile around $p^\pm$, and

$$-M \;\leq\; |\xi_t - \xi_s| - c|t - s| \;\leq\; M$$

Conclusion : u is invariant in the moving frame

Further monotonicity and qualitative properties for bistable-type fronts...

Monostable waves which are trapped between two planar fronts

$$u_t = \Delta u + f(u) \text{ in } \mathbb{R}^N$$

Positive function f in $(0,1)$ with $f(0) = f(1) = 0$ and $f'(0) > 0$.
Planar front $\varphi_c(s)$

$$\varphi_c'' - c\varphi_c' + f(\varphi_c) = 0,\ \varphi_c(-\infty) = 0,\ \varphi_c(+\infty) = 1,\ c \geq c^*$$

Assumption :

$$\varphi_c(x \cdot e + ct) \leq u(t,x) \leq \varphi_c(x \cdot e + ct + a)$$

Conclusion : u is a planar front

Properties

- The sets Γ_t reflect the positions of level sets.
- Intrinsic character of the speed
- Monotonicity in time

Heterogeneous media: Existence of generalised fronts

The bistable case: B.-Hamel-Matano

Ignition temperature case: $u_t - u_{xx} = f(x,u)$ where, for all x, $f(x,.)$ has an ignition temperature $\theta(x)$.
Simplest case: $f(x,u) = a(x)g(u)$, g ignition-like.

Theorem (Nolen-Ryzhik, Mellet-Roquejoffre-Sire)

There is (up to time shift) a unique generalised travelling wave connecting 0 to 1.

Result extended by Zlatos for more general f. Main requirement: for all x, $f_u(x,u) < 0$ in a uniform neigbourhood of 1.

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KPP case

Simplest case: $f(x,u) = a(x)u(1-u)$
Assume $a \equiv 1$ outside a compact subset, $a > 1$ inside.
What matters here is the value of λ, the bottom of the spectrum of the operator

$$A = -\frac{d^2}{dx^2} - a(x).$$

in $L^2(\mathbb{R})$.

Theorem (Nolen-Ryzhik-Roquejoffre-Zlatos)

If $\lambda > 2$, there is no generalised transition front.

In fact, the only time global solution is a bump-like solution, i.e. a solution such that

$$u(t,x) \sim e^{\lambda t}\phi_\lambda(x)$$

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KPP case continued

Theorem (Nolen-Ryzhik-Roquejoffre-Zlatos)

Assume $\lambda < 2$. Then,
(i) There are generalised transition fronts.
(ii) Under some (possibly non-optimal) assumptions on the spectrum of A, a travelling front with speed c satisfies $c \leq \lambda/\sqrt{\lambda - 1}$.

Time dependent equations

Results of

- W. Shen
- Nadin – Rossi
- B - Hamel

Random environment

Random ergodic stationary case

- Ignition reaction term
  - Existence: J. Nolen, L. Ryzhik
  - Uniqueness: J. Nolen, J-M. Roquejoffre, L. Ryzhik, A. Zlatos
- KPP reaction term : Additional assumptions, A. Zlatos

Fronts passing an obstacle K - bistable case

Joint work with
François Hamel and Hiroshi Matano
Comm. Pure Applied Math (2009)

Goal: Travelling front approaching an obstacle and passing it in bistable framework

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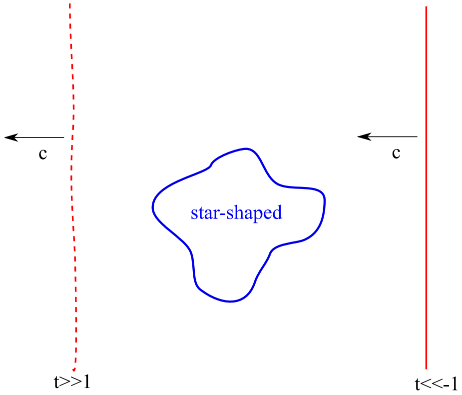

Equation

$$\begin{cases} u_t = \Delta u + f(u) & \text{in } \Omega = \mathbb{R}^N \backslash K, \\ \nu \cdot \nabla u = 0 & \text{on } \partial\Omega = \partial K \end{cases}$$

Assumptions on f

Bistable nonlinearity f:

- f of class $C^{1,1}([0,1])$
- $$f(0) = f(1) = 0, \qquad f'(0) < 0,\ f'(1) < 0,$$
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with $c > 0$.

ϕ unique up to shifts, and $\phi' > 0$ in $\mathbb{R}$.
Existence of (c, ϕ) with $c > 0$ implies

$$\int_s^1 f(\tau)d\tau > 0 \ \text{ for all } 0 \le s < 1,$$

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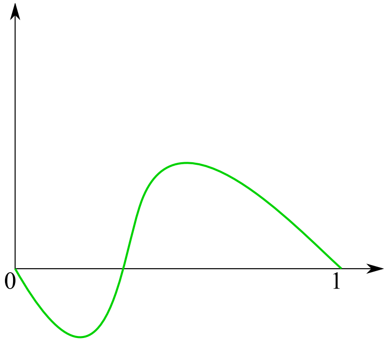
Example of bistable f

Example



Planar traveling front

(c, ϕ) is a *travelling front* for Reaction – diffusion equation in the whole space (homogeneous case):

$$\frac{\partial u}{\partial t} - \Delta u = f(u) \quad \text{in} \quad \mathbb{R}^N$$

Planar front:

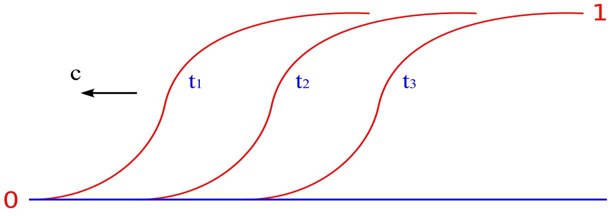
$$u(t, x) = \phi(x \cdot e + ct)$$

with $\phi(\xi) \to 1$ and $\phi(\xi) \to 0$ when $\xi \to \pm\infty$,
e, $|e| = 1$, direction of propagation

$$u_t = \Delta u + f(u)$$

$$\boxed{u(t,x) = \phi(x \cdot e + ct)}$$

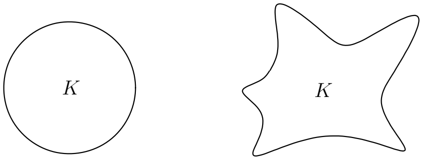
$$\phi : \mathbb{R} \to \mathbb{R}$$



Star-shaped obstacle

Definition

A *star-shaped* obstacle is such that either $K = \emptyset$ or $\exists x \in \mathring{K}$ such that, for all $y \in \partial K$ and $t \in [0, 1)$, $x + t(y - x) \in \mathring{K}$ and $\nu_K(y) \cdot (y - x) \geq 0$, where $\nu_K(y) =$ outward unit normal to K at y.

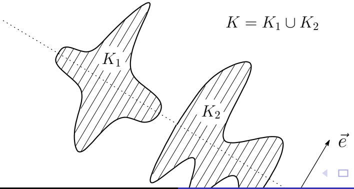


Directionally convex" obstacle

Definition

The obstacle K is said to be *directionally convex* if there exists a hyperplane $P = \{x \in \mathbb{R}^N,\ x \cdot e = a\}$ where $|e| = 1$ and $a \in \mathbb{R}$, such that

- $K \cap P = \pi(K)$, where $\pi(K)$ is the orthogonal projection of K onto P,
- for every line Σ parallel to e, the set $K \cap \Sigma$ either is a single line segment or is empty.



Equation

$$\begin{cases} u_t = \Delta u + f(u) & \text{in } \Omega = \mathbb{R}^N \backslash K, \\ \nu \cdot \nabla u = 0 & \text{on } \partial\Omega = \partial K \end{cases}$$

ν: the outward unit normal on $\partial\Omega$

Take (wlog) propagation direction $\vec{e} = \vec{e_1}$

Generalised fronts

Star-shaped or directionnally convex obstacles

Theorem

Assume

- *f is of "bistable" type,*
- *K bounded and smooth, either star-shaped or directionally convex.*

Then, there exists an entire solution $u(t,x)$ of the equation in $\Omega = \mathbb{R}^N \setminus K$, such that
$0 < u(t,x) < 1$ and $u_t(t,x) > 0, \quad \forall (t,x) \in \mathbb{R} \times \overline{\Omega}$ and

$$u(t,x) - \phi(x_1 + ct) \to 0$$

as $t \to \pm\infty$ uniformly in $x \in \overline{\Omega}$,
and as $|x| \to +\infty$ uniformly in $t \in \mathbb{R}$.
Furthermore, this solution is unique.

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General case

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Assume f is of "bistable" type. For a general compact obstacle K, there exists an entire solution $u(t,x)$ in $\Omega = \mathbb{R}^N \setminus K$, such that: $0 < u(t,x) < 1$ and $u_t(t,x) > 0$, $\forall (t,x) \in \mathbb{R} \times \overline{\Omega}$ and

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A Nonlinear Liouville type theorem

Stationary problem in exterior domain

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Suppose $v = v(x)$ is a bounded solution of the exterior problem

$$\begin{cases} -\Delta v = f(v) & \text{in} \quad \mathbb{R}^N \setminus K \\ \frac{\partial v}{\partial \nu} = 0 & \text{on} \quad \partial K, \\ \lim_{|x| \to \infty} v(x) = 1. \end{cases}$$

If K is either star-shaped or directionally convex w.r.t. some hyerplane, then $v \equiv 1$.

Thus, $u_\infty = v \equiv 1$.

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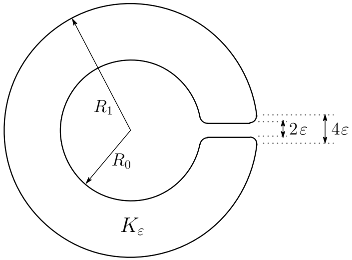

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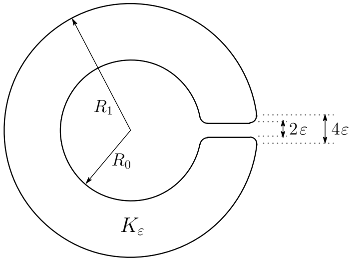

Travelling waves in an exterior domain

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$$\begin{cases} u_t = \Delta u + f(u) & \text{in } \Omega = \mathbb{R}^N \backslash K, \\ \nu \cdot \nabla u = 0 & \text{on } \partial\Omega = \partial K \end{cases}$$

Theorem

Assume

- f *is of "bistable" type,*
- K *bounded and smooth, either star-shaped or directionally convex.*

Then, there exists an entire solution $u(t,x)$ *of the equation in* $\Omega = \mathbb{R}^N \setminus K$*, such that*
$0 < u(t,x) < 1$ *and* $u_t(t,x) > 0$, $\forall (t,x) \in \mathbb{R} \times \overline{\Omega}$ *and*

$$u(t,x) - \phi(x_1 + ct) \to 0$$

as $t \to +\infty$ *uniformly in* $x \in \overline{\Omega}$

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Sketch of the proof of the existence result : 5 main steps **Step 1 : Before reaching the obstacle ($t << -1$)**

- Ideas of Guo - Morita, Fife - McLeod
- Two sub- and super-solutions

$$\simeq \phi(x_1 + ct) + \text{ exponentially small terms}$$

- Solutions $u_n(t,x)$, $t \geq -n$, increasing in n, increasing in t
- $u_n(t,x) \to u(t,x)$ as $n \to +\infty$
- $0 < u(t,x) < 1$ and $u_t > 0$ for all $(t,x) \in \mathbb{R} \times \overline{\Omega}$

$$\boxed{u(t,x) - \phi(x_1 + ct) \to 0 \text{ as } t \to -\infty, \text{ uniformly in } x \in \overline{\Omega}}$$

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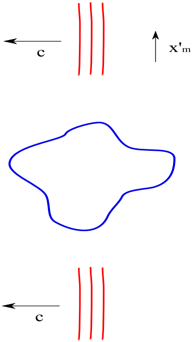
Theorem

The time global solution satisfying this property is unique

Step 2 : Intermediate time, behavior near the horizon

Variables (x_1, x'), $x' \in \mathbb{R}^{N-1}$



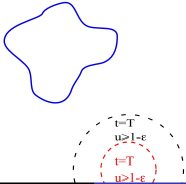
$$u(t, x_1, x' + x'_m) \underset{|x'_m| \to +\infty}{\longrightarrow} \phi(x_1 + ct) \text{ locally in } t,\ x_1,\ x'$$

Step 3 : Large time, the state 1 surrounds the obstacle

$$u(t,x) \to u_\infty(x) \text{ as } t \to +\infty \text{ locally in } x \in \overline{\Omega},$$

$$\begin{cases} \Delta u_\infty + f(u_\infty) &= 0 & \text{in } \overline{\Omega}, \\ 0 < u_\infty(x) &\leq 1 & \text{in } \overline{\Omega}, \\ \nu \cdot \nabla u_\infty &= 0 & \text{on } \partial\Omega \end{cases}$$

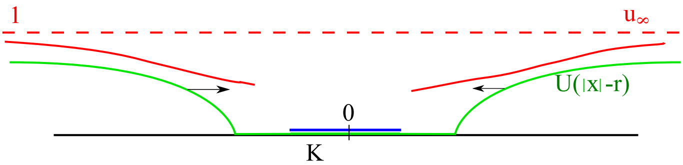
$$\boxed{u_\infty(x) \to 1 \text{ as } |x| \to +\infty}$$



Step 4 : Large time, the state 1 invades the domain

Liouville type theorem

$$\boxed{K \text{ star-shaped} \implies u_\infty \equiv 1 \text{ in } \overline{\Omega}}$$



where

$$\begin{cases} U'' + f(U) = 0 & \text{in } [0,+\infty), \\ U(0) = 0, \; U(+\infty) = 1 & \\ U' > 0 & \text{in } [0,+\infty) \end{cases}$$

(possible because of the profile of f)

Step 5 : Large time, the front recovers its flat shape behind the obstacle as $t \longrightarrow +\infty$

$$\boxed{u(t,x) - \phi(x_1 + ct) \underset{t \to +\infty}{\longrightarrow} 0 \text{ uniformly in } x \in \overline{\Omega}}$$

- Sub-solution of the type

$$u(t,x) = \phi\left(x_1 + ct - \beta t^{-\alpha} e^{-\frac{|x'|^2}{\gamma t}} + \varphi(t) - \varphi(t_0)\right) - \psi(t)\zeta(x)$$

with $\gamma > 0$ large, $\alpha > 0$ small : strong diffusion in the variables x' with weak relaxation in time

- Similar super-solution

Asymptotic speed of spreading

Ê Spreading properties in heterogeneous media:

1. Homogeneous equations in general geometries

HB, F. Hamel and N. Nadirashvili:

- The speed of propagation for KPP type problems. I – Periodic domains, JEMS (2005)
- - The speed of propagation for KPP type problems. II – General domains, JAMS (2009, to appear)

2. Non homogeneous equations

- Freidlin , Freidlin - Gärtner (probabilistic approach)
- Majda - Souganidis ; Lions - Souganidis (Homogenization)
- Nolen - Xin (probabilistic)
- W. Shen (dynamical systems)
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