

# New Recursion Formulae and Integrability for Calabi-Yau Spaces

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## 1 Overview of the Field, Recent Developments, and Open Problems

The workshop is devoted to an emerging, new, field of study. As such, the participants consisted of young group of researchers with their background ranging widely in mathematics and physics. Indeed many participants were first time visitors of the BIRS. Young energies were felt throughout the workshop, during lectures, informal discussions, and excursions.

The main focus of the workshop was to establish a mathematical understanding of the central theme of this meeting, the **topological recursion**. This recursion idea was presented in 2007 by Eynard and Orantin [39], based on many earlier works, including [1, 34], on computing the expectation values of products of the resolvents in random matrix theory (or matrix models). It's importance in topological string theory was immediately noticed by Mariño [65], Bouchard, Klemm, Mariño, Pasquetti [8], Dijkgraaf, Vafa [24], Ooguri, Sułkowski, Yamazaki [87], and many others.

The theory starts with a set of input data, called the **spectral curve**. The data consist of an open Riemann surface  $\Sigma$  and two holomorphic functions  $x$  and  $y$  defined on it, much like the case of the Bloch group. The topological recursion then produces a tower of infinitely many *invariants* in the form of numbers  $F_g$  and meromorphic symmetric differential forms  $W_{g,n}$  defined on  $\Sigma^n$ . But what are these calculating?

The topological recursion in question is a *quantization mechanism*. In mathematical terms, the quantization here means determining quantum invariants for all genera from the one- and two-point functions in genus 0. For example, the Witten conjecture on the  $\psi$ -class intersection numbers on the moduli space  $\overline{\mathcal{M}}_{g,n}$  of  $n$ -pointed stable curves in the form of the Virasoro constraint condition is equivalent to an inductive (recursion) formula with respect to  $2g - 2 + n$  [25]. In general, the Virasoro condition for Gromov-Witten

invariants gives a non-trivial constraint for Fano type target varieties, yet it bears no information for the Calabi-Yau target case. The **remodeling conjecture** of [65, 8] is therefore received with a surprise in mathematics community because it predicts the existence of a recursion formula, based on  $2g - 2 + n$ , for both closed and open Gromov-Witten invariants for an arbitrary toric Calabi-Yau threefold. The key discovery of [8, 65] is the identification of the spectral curve in this context with the **mirror curve** of the toric Calabi-Yau threefold, which is indeed a polynomial curve in  $\mathbb{C}^* \times \mathbb{C}^*$ . Thus the topological recursion is identified as a universal B-model theory to the Gromov-Witten theory for a wide variety of toric target spaces.

In the beginning of 2008 the remodeling conjecture [8] was considered to be intractable. Then Bouchard and Mariño [9] proposed a conceptually simpler conjecture on single Hurwitz numbers. The conjecture states that a particular choice of the generating function of single Hurwitz numbers, that counts the number of *connected* Hurwitz covers of genus  $g$  with  $n$  poles, satisfies the Eynard-Orantin topological recursion. They derived this conjectural formula as the limit of the simplest case of the remodeling conjecture when the integer parameter in the theory, the *framing number*, tends to infinity. In this limit the Calabi-Yau geometries are lost, and the recursion formula becomes simpler. Yet the polynomial curve in the remodeling conjecture becomes a transcendental *Lambert curve*  $x = ye^{-y}$ , which is the spectral curve in the Hurwitz context.

Mainly due to the simplicity of the conjectural formula, and since Hurwitz numbers are far easier objects to understand than open Gromov-Witten invariants, the Bouchard-Mariño conjecture on Hurwitz numbers attracted the attention of many mathematicians. It was solved in two different ways. One method [5] is to find a matrix model that counts the single Hurwitz numbers, and then apply the matrix integral techniques to identify the spectral curve. Then the topological recursion is an automatic consequence of the general theory of matrix models. The matrix model technique to identify the spectral curve also appears in a more recent work of [13].

Another, more geometrically illuminating, solution to the Bouchard-Mariño conjecture was obtained in [38, 74]. The surprising discovery of these papers is that the topological recursion in this context is simply the **Laplace transform** of the well-known *cut-and-join* equation of Hurwitz numbers [50, 91]. Thus the B-model recursion formula is the Laplace transform of the natural A-model equation, and the mirror symmetry is therefore understood as the Laplace transform.

Around the time Eynard and Orantin were developing the topological recursion, in geometry, independently and simultaneously a similar recursion formula was discovered in a very different context by Mirzakhani [68, 69]. Her formula is a recursion that calculates the Weil-Petersson volume of the moduli space of bordered hyperbolic surfaces of genus  $g$  and  $n$  boundaries inductively on  $2g - 2 + n$ . It was then proved in Mulase and Safnuk [73] that the integral transformation formula of Mirzakhani was equivalent to the Virasoro constraint condition for the mixed intersection numbers of the tautological  $\psi$ -classes and Mumford-Miller-Morita  $\kappa$ -classes. In [73] it is noticed that the sine function plays an essential role in identifying the Mirzakhani formula and the Virasoro constraint. The work is further generalized in [62] for the fully mixed intersection numbers of  $\psi$  and higher  $\kappa$  classes. Eynard and Orantin [40] then realized that this sine curve is the spectral curve of the topological recursion formula for this case, and the B-model recursion is exactly the Laplace transform of the Mirzakhani recursion.

Once the idea of the Laplace transform was understood, the first case of the remodeling conjecture, the case for the open Gromov-Witten invariants of  $\mathbb{C}^3$  as a target, was established by Chen [19] and Zhou [95, 96], using the technique of [38, 74].

The BIRS Workshop was proposed around that time. The main objective was to understand the mathematical structure of the Eynard-Orantin topological recursion. In particular, understanding the relation to integrable systems, and further achievements for the remodeling conjecture, were conceived to be the most important goals.

## 2 Presentation Highlights

The Workshop gathered four quite different groups of experts:

- Mathematics experts in Gromov-Witten theory and related topics (Bryan, Cavalieri, Chiodo, Coates, Kimura, Ross, Rossi, Shadrin, and Xu).
- The driving force of the theory in the physics front (Alim, Borot, Bouchard, Brini, Eynard, Fuji, Kashani-Poor, Klemm, Mariño, Orantin, Sułkowski, and Yamazaki).
- Mathematicians working on the topological recursion (Fang, Liu, Marchal, Marks, Mulase, Norbury, Penkava, and Safnuk).
- Experts in algebraic geometry of integrable systems (Hernández Serrano, Plaza Martín, and Previato).

The Eynard-Orantin topological recursion has modular invariance properties. At the Workshop several talks reported current developments of Gromov-Witten theory, topological string theory, and the crepant resolution conjecture, around the (quasi) modular forms. These talks were given by Brini [11], Bryan [88], Coates [21], Kashani-Poor, and Klemm [53]. The talks of Bryan and Coates were about purely mathematical appearances of quasi modular forms, while Kashani-Poor and Klemm talked about the  $\Omega$ -deformations of the B-model in connection to the holomorphic anomaly equations [3]. Brini talked about bridging the physics insight and mathematics ideas around the crepant resolution conjecture [14, 22].

The crepant resolution conjecture connects the usual Gromov-Witten theory and orbifold Gromov-Witten theory. Many new results on orbifold theories were also presented at the Workshop, including Ross [16, 89] on the idea of the orbifold vertex, and Kimura [32] on orbifold K-theory.

We still do not have a good mathematical definition of the Eynard-Orantin differentials. Mathematical theories that are related to the topological recursion include Hurwitz theory, the FJRW Laudau-Ginzburg theory [46], symplectic field theory, and intersection theory. Recent developments in these areas were reviewed by the experts of each field: Cavarieli [17], Chiodo [20], Rossi [42], and Xu [63].

A survey of the relation between Gromov-Witten theory and integrable systems, from the point of view of the Givental formalism, Frobenius manifolds, and cohomological field theory, was given by Shadrin [26].

The remodeling conjecture is still open in its full generality. Liu reported recent mathematical developments on this subject due to Fang, Liu and Zhou. Her idea is based on her earlier work on topological vertex [61]. Eynard presented his new work [36].

The validity of the topological recursion in geometry goes beyond the scope of the remodeling conjecture. An interesting example is the Norbury-Scott conjecture [79], which was presented by Norbury.

From the physics point of view of the Gopakumar-Vafa duality between the Chern-Simons gauge theory on  $S^3$  and the Gromov-Witten theory on the resolved conifold  $\mathcal{O}_{\mathbb{P}^1}(-1) \oplus \mathcal{O}_{\mathbb{P}^1}(-1)$ , any effective tool such as the topological recursion on the Gromov-Witten side should be applicable to the theory of knot invariants [13, 31, 64]. Although no formal presentation was made during the Workshop, there were energetic discussions between participants and the three authors of [13] present at the Workshop. In the context of [13], every knot would give rise to a spectral curve.

Another, more mysterious, relation between knot theory and the topological recursion is via the A-polynomial. In this scenario the spectral curve is given as the  $SL(2, \mathbb{C})$ -character variety of the fundamental group of the knot complement in  $S^3$  in  $\mathbb{C}^* \times \mathbb{C}^*$ , the latter appears as the character variety of the torus that forms the boundary of the tubular neighborhood of the knot in question. Here the topological recursion changes the representation of  $SL(2, \mathbb{C})$  and determines the generating functions of tensor irreducible representations. Two talks by Fuji and Manabe were based on [23]. A more refined idea was presented by Sułkowski, based on [51, 49].

Although no direct connection to the topic of the Workshop seems to be present, Mariño gave an impressive talk on strong string coupling based on his work [66], proposing a new conjecture.

### 3 Scientific Progress Made and Outcome of the Meeting

The Workshop had been proposed two years prior to the actual meeting at BIRS. In two years physics fashions can change, and mathematical interest may diminish if a main open problem of the field is solved. The concern of the organizers at the time of proposal had been to forecast the future of developments of the newly emerging field of research.

The Workshop has shown us a far more beautiful and fertile land in front of us, despite our concerns, and beyond our wildest dreams.

The momentum built for [38, 74, 95, 96] in 2009 had created an impression that the same techniques of these papers may be applicable for the general remodeling conjecture, at least for the case of open Gromov-Witten invariants. Apparently the combinatorics is more difficult than we expected, and the conjecture is still open. Yet various works by Eynard, Fang, Liu, Orantin, and Zhou suggest that the day that we see a final solution may be coming close.

The closed Gromov-Witten part of the conjecture has also shown an interesting development [6, 97], in connection to [43, 44, 15].

Since the remodeling conjecture is already solved for the case of topological vertex [6, 95, 96, 97], if one can come up with the relation between the topological recursion formula for a general toric Calabi-Yau threefold and that of its component topological vertices, then by assembling this combinatorics, one may be able to solve the remodeling conjecture. Eynard's work that identifies *any* solution to the topological recursion with Hodge integrals on  $\overline{\mathcal{M}}_{g,n}$  [36] suggests this possibility. His idea may also lead to a generalization of the ELSV formula [33].

We note that the topological recursion provides a mechanism to change the representation of  $SL(2, \mathbb{C})$  appearing in the definition of colored Jones polynomials, [49] makes a clear application of the recursion to the Kashaev-Murakami-Murakami hyperbolic volume conjecture. This is an impressive work, and points to a new development of mathematics around the topological recursion.

The direction [13] suggests requires mathematicians to think knot invariants a little differently. Instead of fixing a group and representation and consider invariants of different knots, the topological recursion requires that we start with a fixed knot. The knot defines a spectral curve, and the topological recursion, as a mechanism of quantization, gives an expansion of knot invariants in terms of  $N$ , which determines the group  $U(N)$  as the gauge group of the Chern-Simons theory. The tensor irreducible representations of  $U(N)$  also changes along the recursion, therefore, we expect colored HOMFLY polynomials to appear as the outcome. So far this idea is tested only for torus knots. It would be desirable to find a direct method, possibly combinatorial one, to construct a spectral curve from a given knot. The current construction does not suggest even existence of such a relation, except for difficult matrix model construction, so there may not be such a thing, though.

The Norbury-Scott conjecture states that the Laplace transform of the stationary Gromov-Witten invariants of  $\mathbb{P}^1$ , considered as a function in the Laplace dual coordinates of the descendant parameters, satisfies the topological recursion. Definitely  $\mathbb{P}^1$  is very different from the Calabi-Yau examples of the topological recursion. Here the spectral curve is the mirror dual of  $\mathbb{P}^1$ , but the analysis becomes more complicated than the single Hurwitz numbers and the conjecture is not proven as of now. Most likely a similar story should exist for  $\mathbb{P}^2$ , though the spectral curve will be a family of elliptic curves and at this moment no clear conjecture has been made.

It was also a concern of many mathematicians to construct a simple, concrete, mathematical example of the topological recursion, with which one can calculate everything and see the nature of the recursion, spectral curve, and the mathematical meaning of the Eynard-Orantin differentials. This has seen a fruitful outcome [18, 30, 72]. The appearance of the higher-genus Catalan numbers and the virtual Poincaré polynomial of  $\mathcal{M}_{g,n}$  suggests that there should be a rather simple, universal, mathematical definition for the differential forms  $W_{g,n}$ .

The Workshop provided an opportunity for first-time meetings, person-to-person discussions, and new collaborations. The ever expanding applicability of the topological recursion, though many of them are still conjectures and speculations, was impressive to all participants. Every participant left the BIRS with a strong feeling that there would be a lot more interesting things to come on the central topic we had chosen for this Workshop.

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