Dual polar graphs and the quantum algebra $U_q(\mathfrak{sl}_2)$

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Overview

1. The quantum algebra \( U_q(\mathfrak{sl}_2) \)
2. Distance-regular graphs
3. Near polygons
4. Dual polar graphs
5. A \( U_q(\mathfrak{sl}_2) \)-module structure for dual polar graphs
Let \( q \in \mathbb{C} \) such that \( q \) is not a root of 1.

**Definition**

Let \( U_q(\mathfrak{sl}_2) \) denote the unital associative \( \mathbb{C} \)-algebra with generators \( k^{\pm1}, e, f \) and the following relations:

\[
\begin{align*}
kk^{-1} &= k^{-1}k = 1 \\
ke &= q^2ek \\
kf &= q^{-2}fk \\
ef - fe &= \frac{k - k^{-1}}{q - q^{-1}}
\end{align*}
\]
Let Γ = (X, R) denote a finite, undirected, connected graph, without loops or multiple edges. Let D denote the diameter of Γ. Γ is called **distance-regular** whenever for all integers h, i, j (0 ≤ h, i, j ≤ D) and for all vertices x, y ∈ X with ∂(x, y) = h, the number

\[ p^h_{ij} = |\{z ∈ X| ∂(x, z) = i, ∂(z, y) = j\}| \]

is independent of x and y. The \( p^h_{ij} \) are called the **intersection numbers** of Γ.
Let $V = \mathbb{C}^X$.
Observe $\text{Mat}_X(\mathbb{C})$ acts on $V$ by left multiplication.
We call $V$ the **standard module**.
For $y \in X$, let $\hat{y}$ denote the element of $V$ with 1 in the $y$-coordinate and 0 in all other coordinates.
For $0 \leq i \leq D$ let $A_i$ denote the $i$th distance matrix of $\Gamma$. We abbreviate $A = A_1$.

Observe

1. $A_0 = I$
2. $\sum_{i=0}^{D} A_i = J$
3. $\tilde{A}_i = A_i \quad (0 \leq i \leq D)$
4. $A_i^t = A_i \quad (0 \leq i \leq D)$
5. $A_i A_j = \sum_{h=0}^{D} p_{ij}^h A_h \quad (0 \leq i, j \leq D)$

Using these facts $\{A_i\}_{i=0}^{D}$ form a basis for a commutative subalgebra $M$ of $\text{Mat}_X(\mathbb{C})$, called the **Bose-Mesner algebra** of $\Gamma$. It turns out $A$ generates $M$. 

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M has a second basis \( \{ E_i \}_{i=0}^D \) such that

1. \( E_0 = |X|^{-1} J \)
2. \( \sum_{i=0}^D E_i = I \)
3. \( \bar{E}_i = E_i \) \( (0 \leq i \leq D) \)
4. \( E_i^t = E_i \) \( (0 \leq i \leq D) \)
5. \( E_i E_j = \delta_{ij} E_i \) \( (0 \leq i, j \leq D) \)

We call \( \{ E_i \}_{i=0}^D \) the \textbf{primitive idempotents} of \( \Gamma \).
Since \( \{ E_i \}_{i=0}^D \) form a basis for \( M \) there exists complex scalars \( \{ \theta_i \}_{i=0}^D \) such that \( A = \sum_{i=0}^D \theta_i E_i \).

Observe \( AE_i = E_i A = \theta_i E_i \) for \( 0 \leq i \leq D \).

The scalars \( \{ \theta_i \}_{i=0}^D \) are mutually distinct since \( A \) generates \( M \).

We call \( \theta_i \) the **eigenvalue** of \( \Gamma \) associated with \( E_i \).
Since $A_i \circ A_j = \delta_{ij} A_i$ for $0 \leq i, j \leq D$, $M$ is closed under $\circ$. There exists complex scalars $q^{h}_{ij}$ ($0 \leq h, i, j \leq D$) such that

$$E_i \circ E_j = |X|^{-1} \sum_{h=0}^{D} q^{h}_{ij} E_h \quad (0 \leq i, j \leq D)$$

We call $q^{h}_{ij}$ the **Krein parameters** of $\Gamma$. 
The graph $\Gamma$ is said to be $Q$-polynomial (with respect to given ordering $\{E_i\}_{i=0}^D$ of the primitive idempotents) whenever for $0 \leq h, i, j \leq D$, $q_{ij}^h = 0$ (resp. $q_{ij}^h \neq 0$) whenever one of $h, i, j$ is greater than (resp. equal to) the sum of the other two.
Assume $\Gamma$ is $Q$-polynomial with respect to $\{E_i\}_{i=0}^D$.
Fix a vertex $x \in X$.
For $0 \leq i \leq D$ let $E_i^* = E_i^*(x)$ denote the diagonal matrix in $\text{Mat}_X(\mathbb{C})$ with $(y,y)$-entry

$$(E_i^*)_{yy} = \begin{cases} 
1, & \text{if } \partial(x,y) = i \\
0, & \text{if } \partial(x,y) \neq i
\end{cases} \quad (y \in X).$$

We call $\{E_i^*\}_{i=0}^D$ the dual primitive idempotents of $\Gamma$ with respect to $x$.

Observe $E_i^* V = \mathbb{C}\text{-span}\{\hat{z} | \partial(x,z) = i\}$. 
Observe

1. $\sum_{i=0}^{D} E_i^* = 1$
2. $\bar{E}_i^* = E_i^*$ (0 ≤ i ≤ D)
3. $E_i^{*t} = E_i^*$ (0 ≤ i ≤ D)
4. $E_i^* E_j^* = \delta_{ij} E_i^*$ (0 ≤ i, j ≤ D)

From these facts $\{E_i^*\}_{i=0}^{D}$ form a basis for a commutative subalgebra $M^* = M^*(x)$ of $Mat_X(\mathbb{C})$. Call $M^*$ the **dual Bose-Mesner algebra** of $\Gamma$ with respect to $x$. 
For $0 \leq i \leq D$ let $A_i^* = A_i^*(x)$ denote the diagonal matrix in $Mat_X(\mathbb{C})$ with $(y, y)$-entry

$$(A_i^*)_{yy} = |X|(E_i)_{xy}, \quad y \in X.$$ 

Then $\{A_i^*\}_{i=0}^D$ is a basis for $M^*$ such that

1. $A_0^* = I$
2. $\overline{A}_i^* = A_i^*$ \hspace{1cm} ($0 \leq i \leq D$)
3. $A_i^{*t} = A_i^*$ \hspace{1cm} ($0 \leq i \leq D$)
4. $A_i^*A_j^* = \sum_{h=0}^{D} q_{ij}^h A_h^*$ \hspace{1cm} ($0 \leq i, j \leq D$)

Abbreviate $A^* = A_1^*$ and call it the **dual adjacency matrix** of $\Gamma$ with respect to $x$.

It turns out $A^*$ generates $M^*$. 
Since \( \{ E_i^* \}_{i=0}^D \) form a basis for \( M^* \), there exists complex scalars \( \{ \theta_i^* \}_{i=0}^D \) such that \( A^* = \sum_{i=0}^D \theta_i^* E_i^* \).

Observe \( A^* E_i^* = E_i^* A^* = \theta_i^* E_i^* \) for \( 0 \leq i \leq D \).

The scalars \( \{ \theta_i^* \}_{i=0}^D \) are mutually distinct since \( A^* \) generates \( M^* \).

We call \( \theta_i^* \) the **dual eigenvalue** of \( \Gamma \) associated with \( E_i^* \).
Let $T = T(x)$ denote the subalgebra of $\text{Mat}_X(\mathbb{C})$ generated by $M$ and $M^*$. We call $T$ the subconstituent algebra or Terwilliger algebra of $\Gamma$ with respect to $x$. Observe that $A, A^*$ generate $T$.

Fact: $V$ is a direct sum of irreducible $T$-modules.
Let $W$ denote an irreducible $T$-module.
Observe $W = \sum_{i=0}^{D} E_i^* W = \sum_{i=0}^{D} E_i W$ (d.s.).
Define
- $r = \min\{i | 0 \leq i \leq D, E_i^* W \neq 0\}$, endpoint of $W$
- $t = \min\{i | 0 \leq i \leq D, E_i W \neq 0\}$, dual endpoint of $W$
- $d = |\{i | 0 \leq i \leq D, E_i^* W \neq 0\}| - 1$, diameter of $W$
- $d' = |\{i | 0 \leq i \leq D, E_i W \neq 0\}| - 1$

It turns out that $d = d'$. 
A connected graph $\Gamma = (X, R)$ of diameter $D \geq 2$ is called a near polygon if the following two axioms hold.

(NP1) There are no induced subgraphs of shape $K_{1,2,1}$.

(NP2) If $y \in X$ and $M$ is a maximal clique of $\Gamma$ with $\partial(y, M) < D$, then there exists a unique vertex in $M$ nearest to $y$. 
A distance-regular graph $\Gamma$ is a near polygon if and only if the axiom (NP1) holds and $a_i = a_1 c_i$ for $1 \leq i \leq D$.

In this case we call $\Gamma$ a regular near polygon.
Definition

Let $\Gamma = (X, R)$ denote a regular near polygon. A subgraph $G$ of $\Gamma$ is called **weak-geodetically closed** whenever for all vertices $x, y$ in $G$ and for all vertices $z$ in $X$

$$\partial(x, z) + \partial(z, y) \leq \partial(x, y) + 1 \quad \rightarrow \quad z \in G.$$
Let $\Gamma$ denote a regular near polygon. A subgraph $Q$ of $\Gamma$ is called a **quad** whenever $Q$ has diameter 2 and $Q$ is weak-geodetically closed.
Let $b$ denote a prime power.

Let $\mathbb{F}_b$ denote a finite field of order $b$.

Let $U$ denote a finite dimensional vector space over $\mathbb{F}_b$ endowed with a symplectic form, a quadratic form, or a Hermitean form.

A subspace $W$ of $U$ is called **isotropic** whenever the form vanishes completely on $W$.

Each maximal isotropic subspace of $U$ has same dimension, say $D$. 
We define a graph $\Gamma = (X, R)$ where

- $X$ is the set of all maximal isotropic subspaces of $U$
- $R = \{yz \in X^2 | \dim(y \cap z) = D - 1\}$

$\Gamma$ is distance-transitive so $\Gamma$ is distance-regular.

For $y, z \in X$, $\partial(y, z) = i$ if and only if $\dim(y \cap z) = D - i$.

We call $\Gamma$ a **dual polar graph**.
From now on, fix a dual polar graph $\Gamma = (X, R)$. Fix a vertex $x$ and $T = T(x)$. 
\( \Gamma \) is a Q-polynomial with respect to the ordering \( \theta_0 > \theta_1 > \ldots > \theta_D \) of eigenvalues. Moreover, the dual eigenvalues are given by

\[
\theta_i^* = \zeta + \xi b^{-i} \quad \text{for } 0 \leq i \leq D,
\]

where

\[
\zeta = \frac{-b(b^{D+e-2} + 1)}{b - 1},
\]

\[
\xi = \frac{b^2(b^{D+e-2} + 1)(b^{D+e-1} + 1)}{(b - 1)(b^e + b)}.
\]
Raising, flattening and lowering maps

**Definition**

\[ R = \sum_{i=0}^{D-1} E_{i+1}^* A E_i^* \quad \text{raising map} \]

\[ F = \sum_{i=0}^{D} E_i^* A E_i^* \quad \text{flattening map} \]

\[ L = \sum_{i=1}^{D} E_{i-1}^* A E_i^* \quad \text{lowering map} \]

Observe \( F^t = F \) and \( R^t = L \).
Let $y \in X$ such that $\partial(x, y) = i$. 

\[
R \hat{y} = \sum_{z \in \Gamma_{i+1}(x) \cap \Gamma(y)} \hat{z}
\]

\[
F \hat{y} = \sum_{z \in \Gamma_{i}(x) \cap \Gamma(y)} \hat{z}
\]

\[
L \hat{y} = \sum_{z \in \Gamma_{i-1}(x) \cap \Gamma(y)} \hat{z}
\]

Observe that $A = R + F + L$. 
The map $K$

Pick $q \in \mathbb{C}$ such that $b = q^2$.

**Definition**

$$K = \sum_{i=0}^{D} q^{-2i} E_i^*.$$ 

Observe $K$ is invertible and

$$A^* = \zeta I + \xi K.$$ 

$R, F, L, K$ together generate $T$. 

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Lemma

1. $KR = q^{-2}RK$.
2. $KF = FK$.
3. $KL = q^2LK$.

Reminiscent of the defining relations of $U_q(\mathfrak{sl}_2)$.
It’s almost as $k \approx K, e \approx L, f \approx R$ but not quite.
Relations involving $R, F, L$

Lemma

1. \[ LF - q^2 FL = (q^{2e} - 1)L. \]
2. \[ FR - q^2 RF = (q^{2e} - 1)R. \]

\textit{Pf:} $\Gamma$ is regular near polygon, its quads are classical, has constant line size $a_1 + 2$. 

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Dual polar graphs and the quantum algebra $U_q(\mathfrak{sl}_2)$
Relations involving $R, L$

Lemma

1. \[-\frac{q^4}{q^2 + 1} R L^2 + L R L - \frac{q^{-2}}{q^2 + 1} L^2 R = q^{2e+2D-2} L,\]

2. \[-\frac{q^4}{q^2 + 1} R^2 L + R L R - \frac{q^{-2}}{q^2 + 1} L R^2 = q^{2e+2D-2} R.\]

**Pf:** $A, A^*$ satisfy the tridiagonal relation

\[[A, A^2 A^* - \beta AA^* A + A^* A^2 - \gamma (AA^* + A^* A) - \rho A^*] = 0.\]
Summary of relations in $T$

1. $KK^{-1} = K^{-1}K = 1$
2. $KR = q^{-2}RK$
3. $KF = FK$
4. $KL = q^2LK$
5. $LF - q^2FL = (q^{2e} - 1)L$
6. $FR - q^2RF = (q^{2e} - 1)R$
7. $-\frac{q^4}{q^2 + 1}RL^2 + LRL - \frac{q^{-2}}{q^2 + 1}L^2R = q^{2e+2D-2}L$
8. $-\frac{q^4}{q^2 + 1}R^2L + RLR - \frac{q^{-2}}{q^2 + 1}LR^2 = q^{2e+2D-2}R$
The central elements $C_0, C_1, C_2$ of $T$

**Definition**

1. $C_0 = KF + \frac{q^{2e} - 1}{q^2 - 1} K,$

2. $C_1 = -\frac{q^{-2}}{q^2 + 1} KLR + \frac{q^2}{q^2 + 1} KRL + \frac{q^{2e+2D-2}}{q^2 - 1} K,$

3. $C_2 = -\frac{q^{-2}}{q^2 + 1} K^2 LR + \frac{1}{q^2 + 1} K^2 RL + \frac{q^{2e+2D-2}}{q^4 - 1} K^2.$

**Theorem**

$C_0, C_1, C_2$ generate the center of $T.$
Lemma

Let \( W \) denote an irreducible \( T \)-module with diameter \( d \), endpoint \( r \) and dual endpoint \( t \). Then on \( W \)

1. \( C_0 \) acts as the scalar 
\[
\frac{1}{q^2 - 1} \left( q^{2e+2D-2d-2r-2t} - q^{2t-2r} \right),
\]

2. \( C_1 \) acts as the scalar 
\[
\frac{q^{2e+2D-1}}{q^4 - 1} q^{-d-2r} (q^{d+1} + q^{-d-1}),
\]

3. \( C_2 \) acts as the scalar 
\[
\frac{q^{2e+2D-2}}{q^4 - 1} q^{-2d-4r}.
\]
Lemma

There exist central elements $\Phi, \Psi$ of $T$ with the following property. For all irreducible $T$-module $W$ with endpoint $r$, dual endpoint $t$ and diameter $d$, $\Phi, \Psi$ act on $W$ as follows:

$$\Phi = q^{r+t+d-D}1$$
$$\Psi = q^{r-t}1$$
Lemma

\[ C_2 = \frac{q^{2e-2}}{q^4 - 1} (\Phi \Psi)^{-2} \]
Recall the standard $T$-module $V = \mathbb{C}^X$.

**Theorem**

There exists a unique $U_q(\mathfrak{sl}_2)$-module structure on $V$ such that on $V$

\[
\begin{align*}
    k &= q^D\Phi\Psi K, \\
    k^{-1} &= q^{-D}(\Phi\Psi K)^{-1}, \\
    e &= \Phi\Psi KL, \\
    f &= q^{1-2e-D}R.
\end{align*}
\]