Open Problems Concerning Automorphism Groups of Projective Planes

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A **projective plane** is a point-line incidence structure such that

- every pair of distinct points lies on a common line;
- every pair of distinct lines meets in a common point;
- there exists a quadrangle (four points, no three of which are collinear).

There exists a cardinal number $n$ (finite or infinite), called the **order** of the plane, such that

- every line has $n + 1$ points;
- every point is on $n + 1$ lines;
- there are $n^2 + n + 1$ points and the same number of lines.

An **automorphism** (i.e. collineation) of a projective plane is a permutation of the points which preserves collinearity.
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An automorphism (i.e. collineation) of a projective plane is a permutation of the points which preserves collinearity.
Known planes of small order

Number of planes up to isomorphism (i.e. collineations):

<table>
<thead>
<tr>
<th>$n$</th>
<th>number of planes of order $n$</th>
<th>$n$</th>
<th>number of planes of order $n$</th>
</tr>
</thead>
<tbody>
<tr>
<td>2</td>
<td>1</td>
<td>16</td>
<td>$\geq 22$</td>
</tr>
<tr>
<td>3</td>
<td>1</td>
<td>17</td>
<td>$\geq 1$</td>
</tr>
<tr>
<td>4</td>
<td>1</td>
<td>19</td>
<td>$\geq 1$</td>
</tr>
<tr>
<td>5</td>
<td>1</td>
<td>23</td>
<td>$\geq 1$</td>
</tr>
<tr>
<td>7</td>
<td>1</td>
<td>25</td>
<td>$\geq 193$</td>
</tr>
<tr>
<td>8</td>
<td>1</td>
<td>27</td>
<td>$\geq 13$</td>
</tr>
<tr>
<td>9</td>
<td>4</td>
<td>29</td>
<td>$\geq 1$</td>
</tr>
<tr>
<td>11</td>
<td>$\geq 1$</td>
<td>...</td>
<td>...</td>
</tr>
<tr>
<td>13</td>
<td>$\geq 1$</td>
<td>49</td>
<td>$&gt; 280,000$</td>
</tr>
</tbody>
</table>
**pzip: A compression utility for finite planes**

Storage requirements for a projective plane of order $n$:

<table>
<thead>
<tr>
<th>$n$</th>
<th>size of line sets</th>
<th>size of MOLS</th>
<th>gzipped MOLS</th>
<th>pzip</th>
</tr>
</thead>
<tbody>
<tr>
<td>11</td>
<td>5 KB</td>
<td>1.3 KB</td>
<td>0.2 KB</td>
<td>0.06 KB</td>
</tr>
<tr>
<td>25</td>
<td>63 KB</td>
<td>15 KB</td>
<td>9 KB</td>
<td>0.9 KB</td>
</tr>
<tr>
<td>49</td>
<td>550 KB</td>
<td>110 KB</td>
<td>81 KB</td>
<td>6 KB</td>
</tr>
</tbody>
</table>

See [http://www.uwyo.edu/moorhouse/pzip.html](http://www.uwyo.edu/moorhouse/pzip.html)
The Classical Planes

Let $F$ be a field. Denote by $F^3$ a 3-dimensional vector space over $F$.

The classical projective plane $P^2(F)$ has as its points and lines the subspaces of $F^3$ of dimension 1 and 2, respectively. Incidence is inclusion. The order of the plane is $|F|$, finite or infinite.

The automorphism group of $P^2(F)$ is $PΓL_3(F)$, which acts 2-transitively on points, and transitively on ordered quadrangles. No known planes have as much symmetry as the classical planes.
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Let $\Pi$ be a projective plane, and let $G = \text{Aut}(\Pi)$.

**Theorem (Ostrom-Dembowski-Wagner)**

*In the finite case, $\Pi$ is classical iff $G$ is 2-transitive on points.*

In the infinite case, there exist nonclassical planes whose automorphism group is 2-transitive on points (even transitive on ordered quadrangles).
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Consider a classical projective plane $\Pi = P^2(F)$.

Every quadrangle in $\Pi$ generates a subplane isomorphic to $P^2(K)$ where $K$ is the prime subfield of $F$ (i.e. $\mathbb{F}_p$ or $\mathbb{Q}$, according to the characteristic of $F$).

Such a subplane is proper iff $[F : K] > 1$. 
Open Question

Let $\Pi$ be a finite projective plane in which every quadrangle generates a proper subplane. Must $\Pi$ be classical? (necessarily of order $p^r$ with $r \geq 2$)

The answer is known only in special cases:

If $\Pi$ is a finite projective plane in which every quadrangle generates a subplane of order 2, then $\Pi \cong P^2(\mathbb{F}_{2^r})$ (Gleason, 1956).

If $\Pi$ is a finite projective plane of order $n^2$ in which every quadrangle generates a subplane of order $n$, then $n = p$ and $\Pi \cong P^2(\mathbb{F}_{p^2})$ (Blokhuis and Sziklai, 2001 for $n$ prime; Kantor and Penttila, 2010 in general).
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Consider a projective plane $\Pi$ with automorphism group $G = \text{Aut}(\Pi)$.

**Theorem (Brauer, 1941)**

*In the finite case, $G$ has equally many orbits on points and on lines.*

**Open Problem (attributed to Kantor)**

*In the general case, must $G$ have equally many orbits on points and on lines?*
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Orbits on $n$-tuples of Points

In the classical case $\Pi = P^2(F)$, $G$ has:
- 1 orbit on points;
- 1 orbit on ordered pairs of distinct points;
- 2 orbits on ordered triples of distinct points;
- $O(|F|)$ orbits on ordered 4-tuples of distinct points. (In the case of collinear 4-tuples, consider the cross-ratio.)

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Does there exist an infinite plane with only finitely many orbits on $k$-tuples of distinct points for every $k \geq 1$?

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A permutation group $G$ on $X$ is **oligomorphic** if $G$ has finitely many orbits on $X^k$ for each $k \geq 1$. See Cameron (1990).

(Taking $k$-tuples of points in $X$, or $k$-tuples of distinct points, doesn’t matter.)

**Open Question**

Does there exist an infinite projective plane $\Pi$ admitting a group $G \leq \text{Aut}(\Pi)$ which is oligomorphic on points? (equivalently, on lines).

If such a plane exists, we may assume (by the Löwenheim-Skolem Theorem) that its order is $\aleph_0$ (countably infinite). Such a plane is called $\aleph_0$-categorical.
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From now on, assume $\Pi$ is an $\aleph_0$-categorical projective plane, and let $G \leq Aut(\Pi)$ be oligomorphic on points.

Useful fact: In an oligomorphic group $G$, the stabilizer of any finite point set is also oligomorphic.

**Lemma**

*Every finite substructure $S \subset \Pi$ lies in a finite subplane.*

**Proof.**

Let $G_S \leq G$ be the pointwise stabilizer of $S$. Then $G_S(S)$ fixes pointwise the substructure $\langle S \rangle$ generated by $S$. This substructure must be finite, otherwise $G_S(S)$ has infinitely many fixed points, hence infinitely many orbits.
\(\aleph_0\)-categorical planes

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Π an $\aleph_0$-categorical projective plane, $G \leq \text{Aut}(\Pi)$ oligomorphic

Without loss of generality, $G$ fixes pointwise a finite subplane $\Pi_0 \subset \Pi$. (Otherwise replace $G$ by the oligomorphic subgroup $G(S)$ where $S$ is a quadrangle.)

Consider a point $P \in \Pi$. We say
- $P$ is of type I if $P \in \Pi_0$;
- $P$ is of type II if $P \not\in \Pi_0$ but $P$ lies on a line of $\Pi_0$;
- $P$ is of type III if $P$ lies on no line of $\Pi_0$.

Dually classify lines of $\Pi$ as type I, II or III.
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The Burnside Ring $\mathcal{B}(G)$

Two $G$-sets $X$ and $Y$ are equivalent if there exists a $G$-equivariant bijection $\theta : X \rightarrow Y$, i.e. $\theta(x^g) = \theta(x)^g$ for all $x \in X$, $g \in G$.

The equivalence class of a $G$-set $X$ is denoted $[X]$.

Given $G$-sets $X$ and $Y$, the disjoint union $X \uplus Y$ and Cartesian product $X \times Y$ are $G$-sets.

The Burnside ring $\mathcal{B}(G)$ is the $\mathbb{Z}$-algebra consisting of formal sums $\sum [X] c[X][X]$, $c[X] \in \mathbb{Z}$ (almost all zero), where

$$[X] + [Y] = [X \uplus Y], \quad [X][Y] = [X \times Y].$$
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Π an ℵ₀-categorical projective plane,  
\( G \leq \text{Aut}(\Pi) \) oligomorphic

Let \( P \) and \( \ell \) be a point and line of \( \Pi_0 \).

The set \( II_\ell \) of type II points of \( \ell \) is a \( G \)-set; as is the set \( II_P \) of type II lines through \( P \).

**Lemma**

\([II_P] = [II_\ell], \) independent of the choice of point \( P \) and line \( \ell \) of \( \Pi_0 \).
Let $P$ and $\ell$ be a point and line of $\Pi_0$. The set $\mathcal{II}_\ell$ of type II points of $\ell$ is a $G$-set; as is the set $\mathcal{II}_P$ of type II lines through $P$.

**Lemma**

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Π an $\aleph_0$-categorical projective plane, $G \leq \text{Aut}(\Pi)$ oligomorphic

Denote by $III$ the $G$-set consisting of all type III points. Dually, $\tilde{III}$ is the $G$-set consisting of all type III lines.

**Lemma**

Let $\ell$ be a line of $\Pi_0$. Then $[II_{\ell}]^2 = [III] + c[II_{\ell}]$

where $c = n_0(n_0 - 1)$, $n_0 = \text{order of } \Pi_0$.

\[(R, S) \mapsto RS\]

\[II_\ell \times II_{\ell'} \rightarrow \tilde{III} \uplus \left( \bigcup_{O \in \Pi_0; \ O \notin \ell \cup \ell'} II_O \right)\]
Π an ℵ₀-categorical projective plane, $G \leq \text{Aut}(\Pi)$ oligomorphic

**Lemma**

Let $\ell$ be a line of $\Pi_0$. Then $[\text{II}_\ell]^2 = [\text{III}] + c[\text{II}_\ell]$ where $c = n_0(n_0 - 1)$, $n_0 = \text{order of } \Pi_0$.

**Corollary**

$[\text{III}] = [\text{III}]$ and $[\text{II}_\ell]^2 = [\text{III}] + c[\text{II}_\ell]$.

**Proof.**

Dualising the previous lemma,

$[\text{III}] + c[\text{II}_\ell] = [\text{II}_\ell]^2 = [\text{III}] + c[\text{II}_\ell]$.

Cancellation of the $c[\text{II}_\ell]$ terms is justified in $\mathcal{B}(G)$.
Let $\nu_{m,n} = \text{number of } G\text{-orbits on } II^m_\ell \times III^n$.

**Lemma**

*For all $m, n \geq 0$, we have $\nu_{m+2,n} = \nu_{m,n+1} + c\nu_{m+1,n}$.*

**Proof.**

$$[III]^{m+2} [III]^n = [II\ell]^m ([III] + c[II\ell]) [III]^n$$

$$= [II\ell]^m [III]^{n+1} + c[II\ell]^{m+1} [III]^n.$$
\( \Pi \) an \( \aleph_0 \)-categorical projective plane, 
\( G \leq \text{Aut}(\Pi) \) oligomorphic

The previous recurrence for

\[
\nu_{m,n} = \text{number of } G\text{-orbits on } II^m_\ell \times III^n
\]

is rephrased in terms of the generating function

\[
F(s, t) = \sum_{m,n \geq 0} \nu_{m,n} s^m t^n
\]

as follows.

**Lemma**

\[
F(s, t) = \sum_{k \geq 0} (a_k + b_k s) F_k(s, t) \text{ where}
\]

\[
F_k(s, t) = \frac{1}{(1 - cs)t - s^2} \left[ t^{k+1} - \frac{s^{2(k+1)}}{(1 - cs)^{k+1}} \right].
\]
\[ \Pi \text{ an } \aleph_0\text{-categorical projective plane, } \]
\[ G \leq \text{Aut}(\Pi) \text{ oligomorphic} \]

**Theorem**

*Under our assumption (existence of an \( \aleph_0\text{-categorical projective plane}, \)* there exist (infinitely many) finite nonclassical projective planes, in which every quadrangle generates a proper subplane.

**Proof (Sketch).**

Without loss of generality, the subplane \( \Pi_0 \subset \Pi \) is nonclassical. Let \( M \) be the maximum order of a subplane of the form \( \langle \Pi_0, P, Q, R, S \rangle \) where \( (P, Q, R, S) \) is a quadrangle of \( \Pi \). Any subplane of \( \Pi \) containing \( \Pi_0 \) of order exceeding \( M \), has the required property.
In all known cases of a finite projective plane of order \( n \) with a subplane of order \( n_0 \), we have

- \( n = n_0^r \) for some \( r \geq 1 \); or
- \( n_0 \in \{2, 3\} \).

Moreover, subplanes of order 3 are rare unless \( n = 3^r \).

Hopes for an \( \aleph_0 \)-categorical plane do not look bright!
Subplanes of known planes

In all known cases of a finite projective plane of order $n$ with a subplane of order $n_0$, we have

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Thank You!

Questions?