Bridging of the Scales, Multiscale Modeling & Simulation of Uncertain Archetype Motion

Wing Kam Liu[a], M Steven Greene[b], Khalil Elkhodary[a]

[a] Department of Mechanical Engineering, Northwestern University
[b] Theoretical & Applied Mechanics, Northwestern University

Stochastic Multiscale Analysis Workshop

Banff, Canada

March 27 – April 1, 2011

Outline

- Overarching Theme

- Current research projects
  - Digital 3D, design of high strength alloys. Transportation application.
  - Polymer nanocomposite design. Transportation and military applications.
  - Microsystem gyroscope design. Deep space application.
  - Post-yield bone mechanics. Clinical application.
  - Nanomedicine delivery and diagnostic mechanics. Medical application.

- Mathematical machinery
  - Physical postulates of material motion ← Why.
  - Governing equations of multiresolution continua ← Where.
  - Constitutive modeling ← How.

UNCERTAINTY

- Discussion

Outline

- Overarching Theme

- Current research projects
  - Digital 3D, design of high strength alloys. Transportation applications.
  - Polymer nanocomposite design. Transportation and military applications.
  - Microsystem gyroscope design. Deep space application.
  - Post-yield bone mechanics. Clinical application.
  - Nanodiamond delivery and diagnostic mechanics. Medical applications.

- Mathematical machinery
  - Physical postulates of material motion ← Why.
  - Governing equations of multiresolution continua ← Where.
  - Constitutive modeling ← How.

UNCERTAINTY

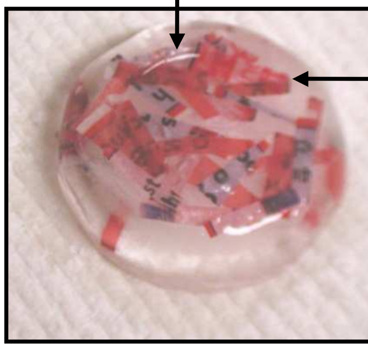
- Conclusions

A Macroscale Demonstration

Ice

“Reinforced” Ice

t_0



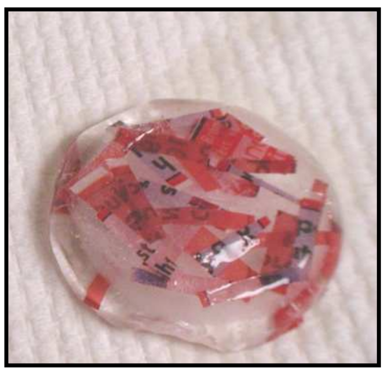
Newspaper ⟷ Fine resolution material morphology

Structure dictates the bulk properties

Dropped

Dropped

t_f



Properties dictate the bulk performance

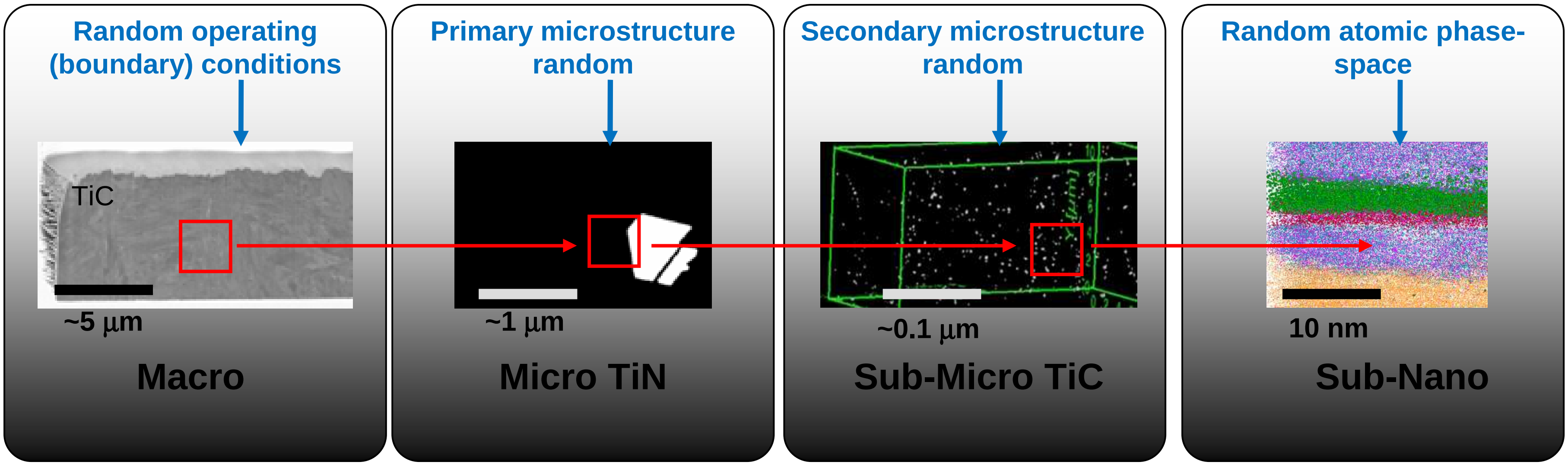
Outline

- Overarching Theme
- Current research projects
  - Digital 3D, design of high strength alloys. Transportation application.
  - Polymer nanocomposite design. Transportation and military applications.
  - Microsystem gyroscope design. Deep space application.
  - Post-yield bone mechanics. Clinical application.
  - Nanomedicine delivery and diagnostic mechanics. Medical application.
- Mathematical machinery
  - Physical postulates of material motion ← Why.
  - Governing equations of multiresolution continua ← Where.
  - Constitutive modeling ← How.

  UNCERTAINTY
- Conclusions

High Strength Steel Alloy Design

- **Goal**. Deliver high strength steels for transportation applications
- **Schedule**. 2005--2010 and beyond
- **Collaborators**. *Multiple* faculty at *multiple* universities with *multiple* industry/public entities.

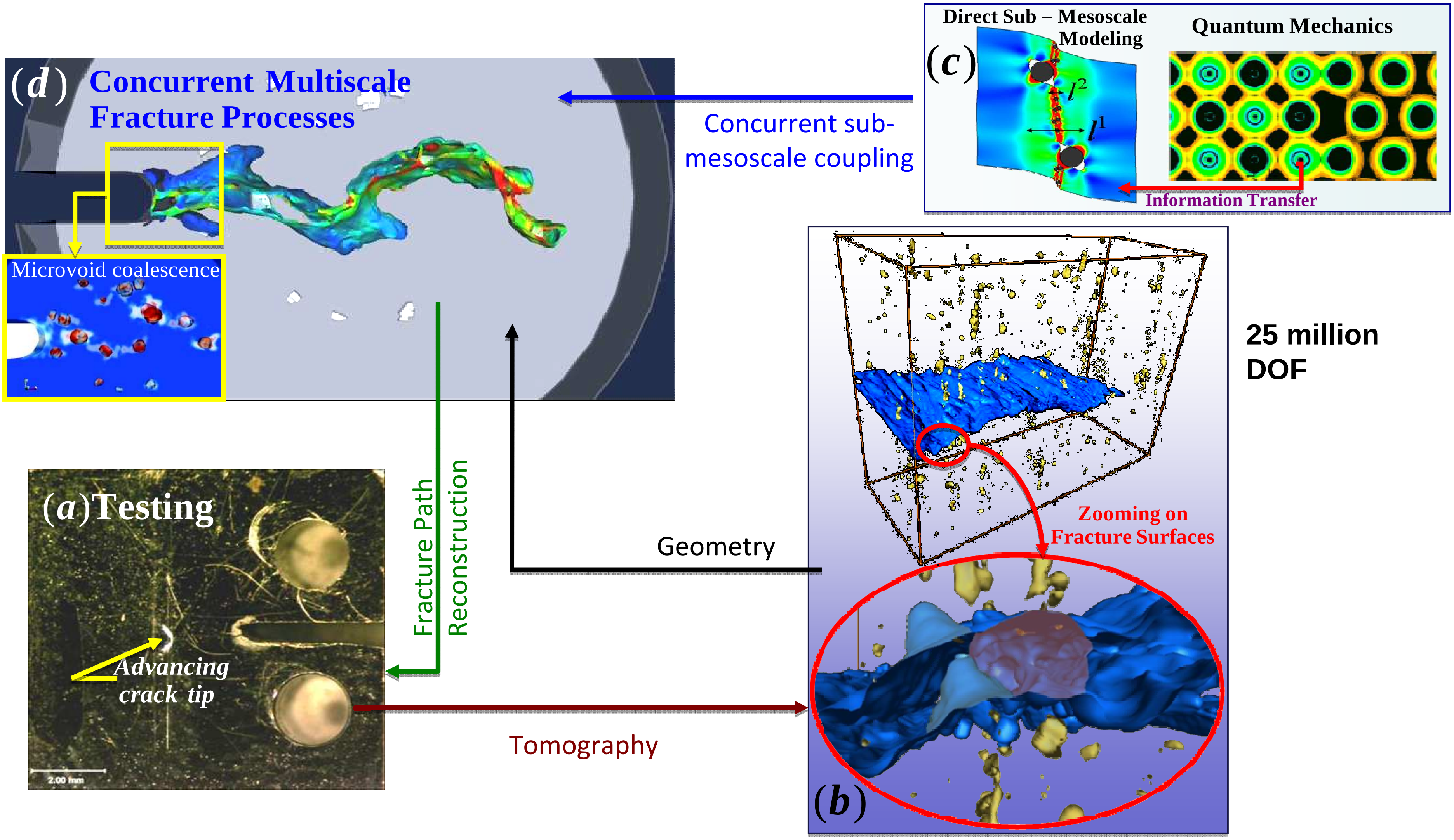


Increasing resolution, decreasing spatial scale, increasing uncertainty

Small Number of degrees of freedom – suitable for average behavior	Discrete behavior of TiN primary inclusions begins to be observed	Discrete behavior of TiC secondary particles is observed	Very fine resolution – material interface is observed

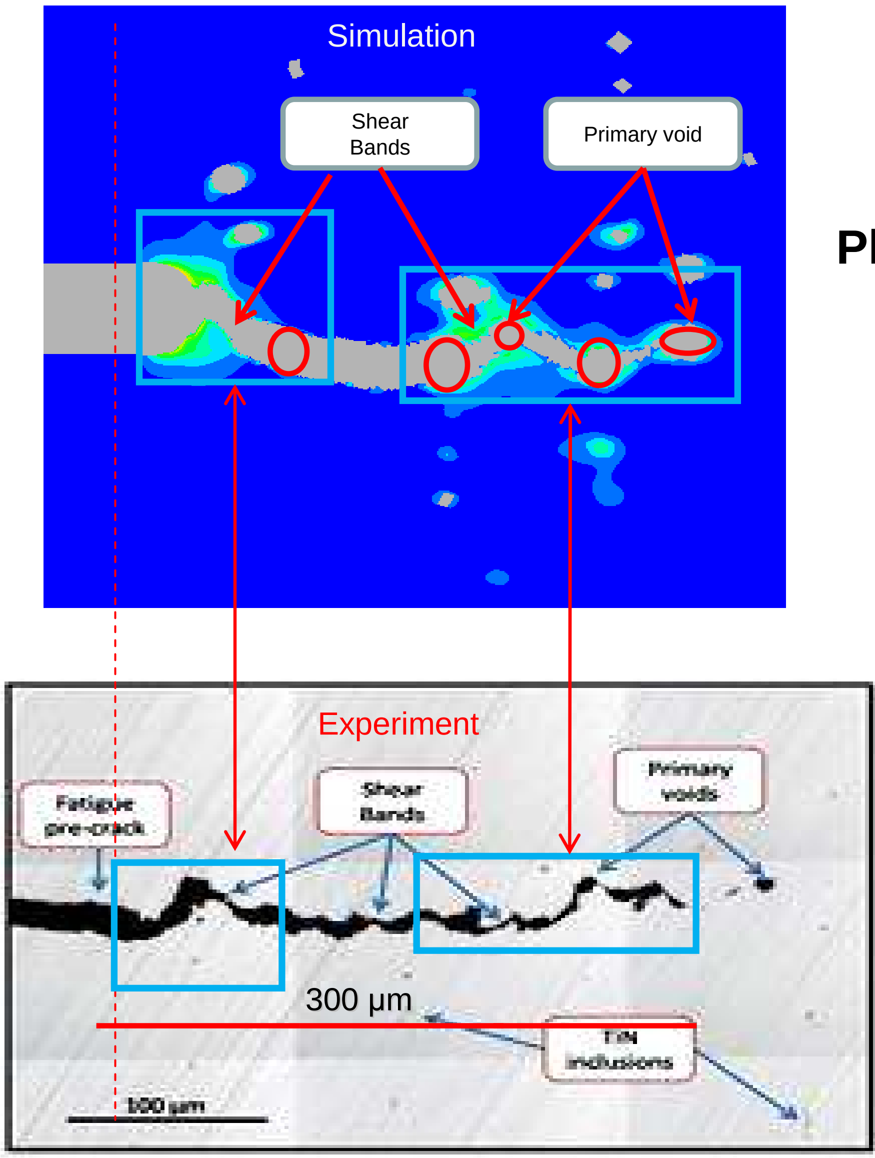
- True system size: 10,000 secondary particles, 1,000 primary particles, >**1 billion DOF**
- Feasible system size: 115 primary particles, >**25 million DOF**

Predictive multiscale modeling of microstructured alloy aims to increase toughness without sacrificing performance (strength and weight) for transportation applications



25 million DOF

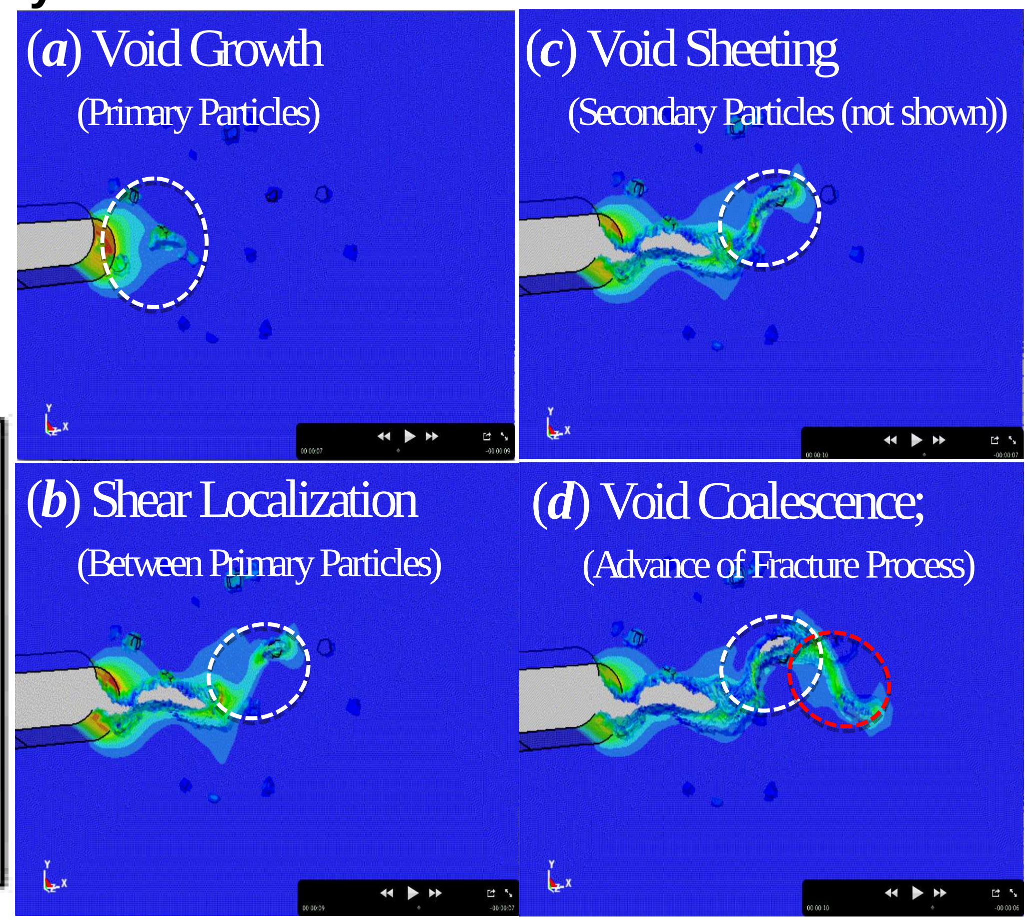
High Strength Steel Alloy. Validation with Experiments



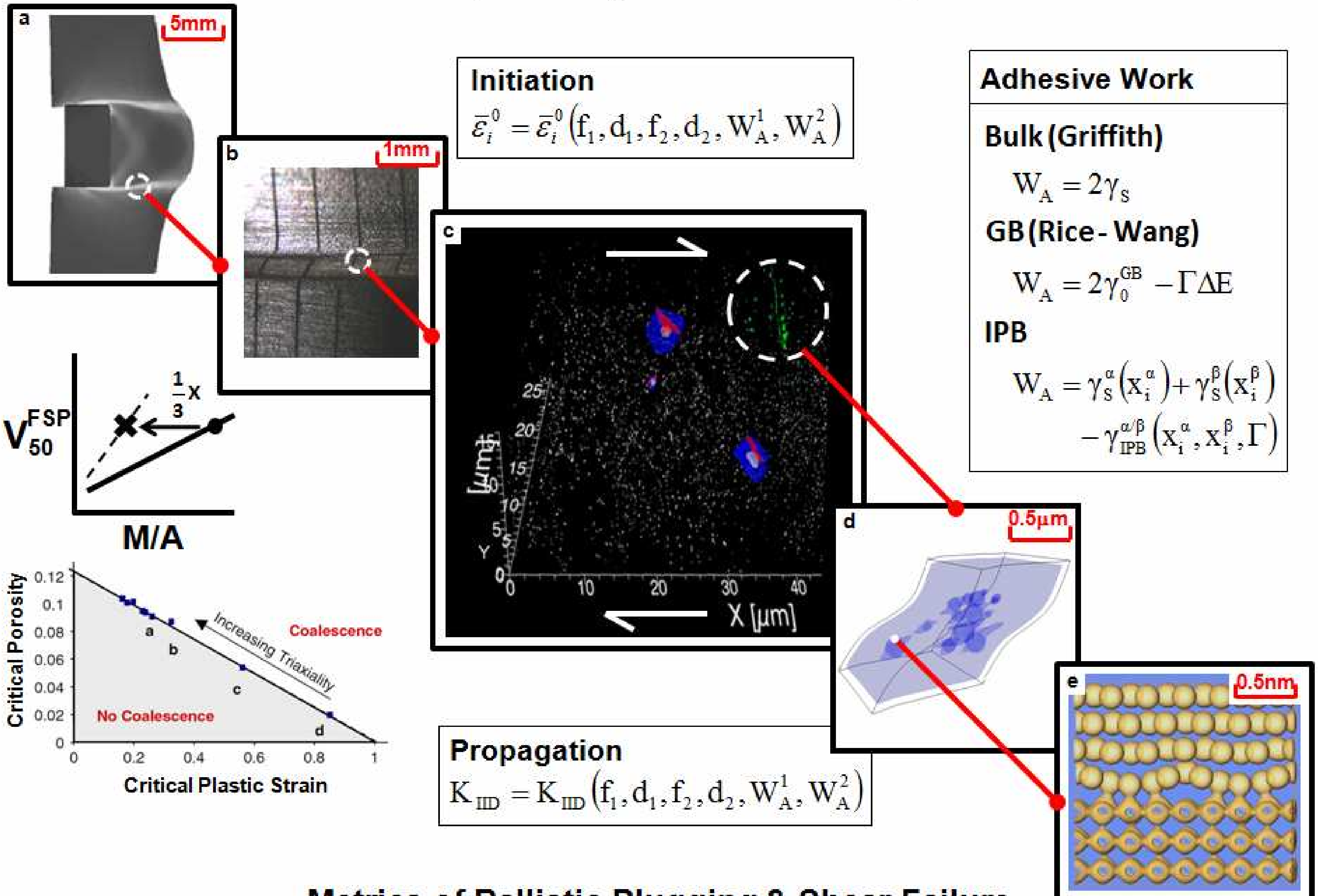
115 primary particles

- Notch size
- Features of zig-zag pattern are captured
- Experiments has more zig-zags

Physics of Reconstruction of Fracture Process



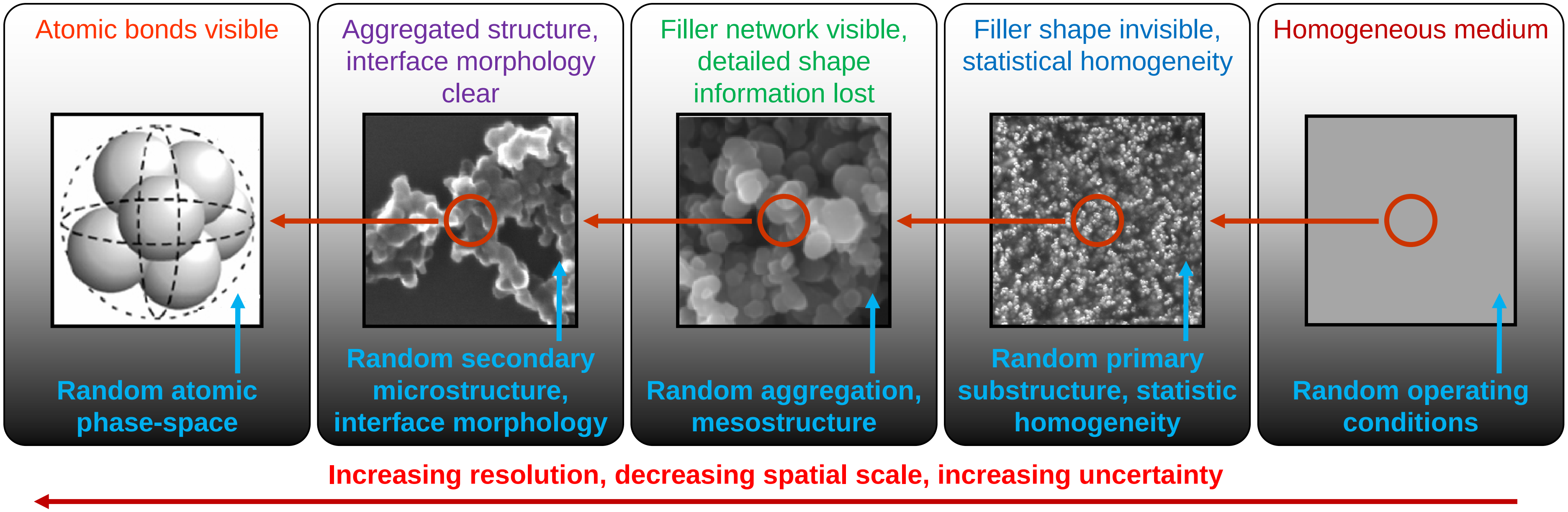
Performance ⇒ Properties ⇒ Structure



Metrics of Ballistic Plugging & Shear Failure

Polymer Nanocomposite Material Design

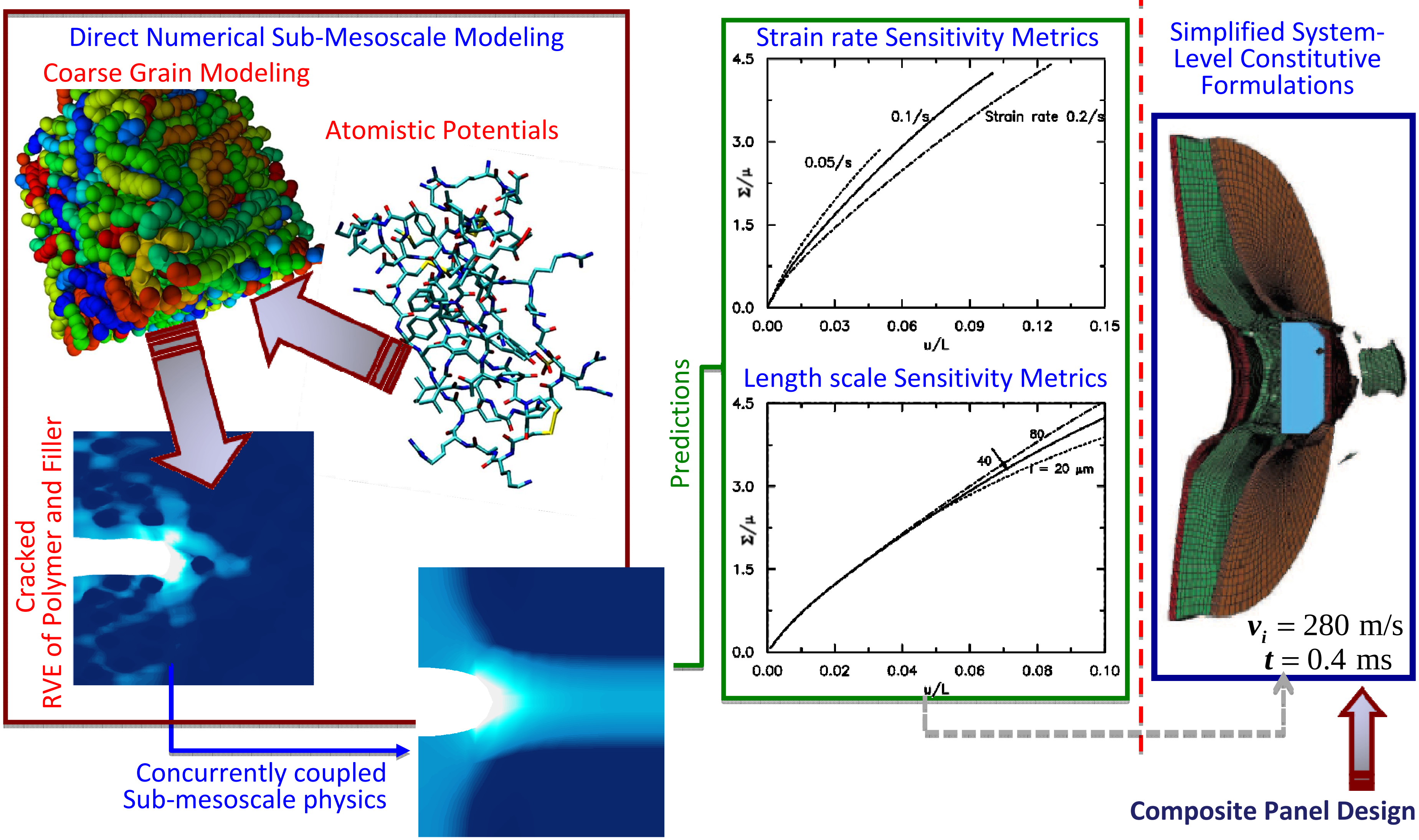
- **Goal**. Increase transportation materials performance (1), ballistic protection (2), and fracture toughness (3)
- **Schedule**. 2008-current
- **Collaborators**. 5 faculty at NU, Prof. Kruger ETH Zurich, Goodyear Tire & Rubber Co, ARO, NSF



- *True* system size: ~1000 particles, >**1.2 billion DOF**
- *Feasible* system size: 40 particles, >**35 thousand DOF**
- *Multiresolution* system size: 0 particles ~**1700 DOF**

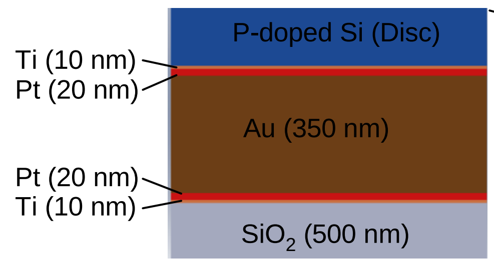
Polymer Nanocomposite Design. Energy Desipation & Dynamic Fracture

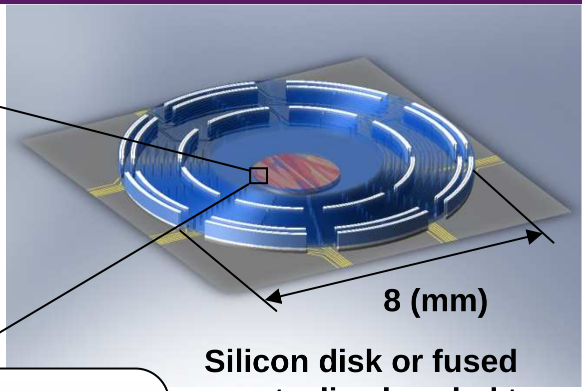
Predictive multiscale modeling of microstructured elastomer aims to guide our parallel development of system-level constitutive models to design composite sandwich plates



- **Goal**. Design Boeing microsystem for harsh deep space applications.
- **Schedule**. 2005-2008
- **Collaborators**. Northwestern, Boeing

System Description

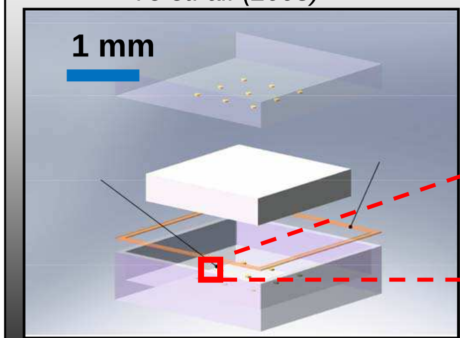




Silicon disk or fused quartz disc bonded to silicon substrate

Uncertain geometry from manufacturing

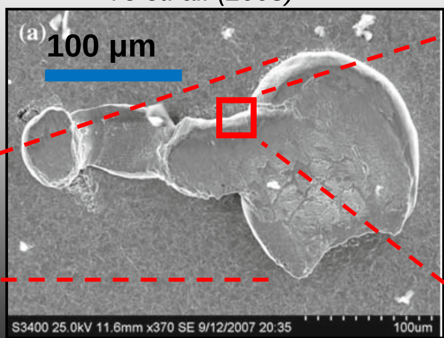
To et. al. (2008)



microsystem casing

Uncertain defect population and location

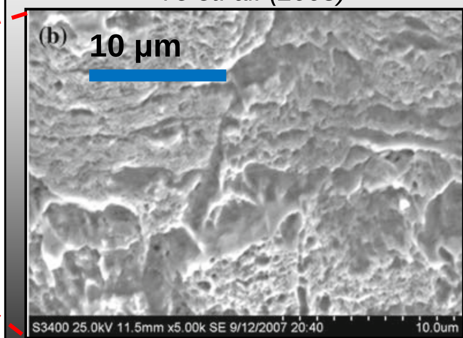
To et. al. (2008)



gold bump global fracture

Uncertain primary microstructure of adhesives

To et. al. (2008)



local fracture pattern

Microsystem subjected to numerous **uncertain boundary conditions**

Aging, thermal, radiation, mechanical loading

Multiphase problem in which material structure governs phase interaction

mm **µm** **nm**

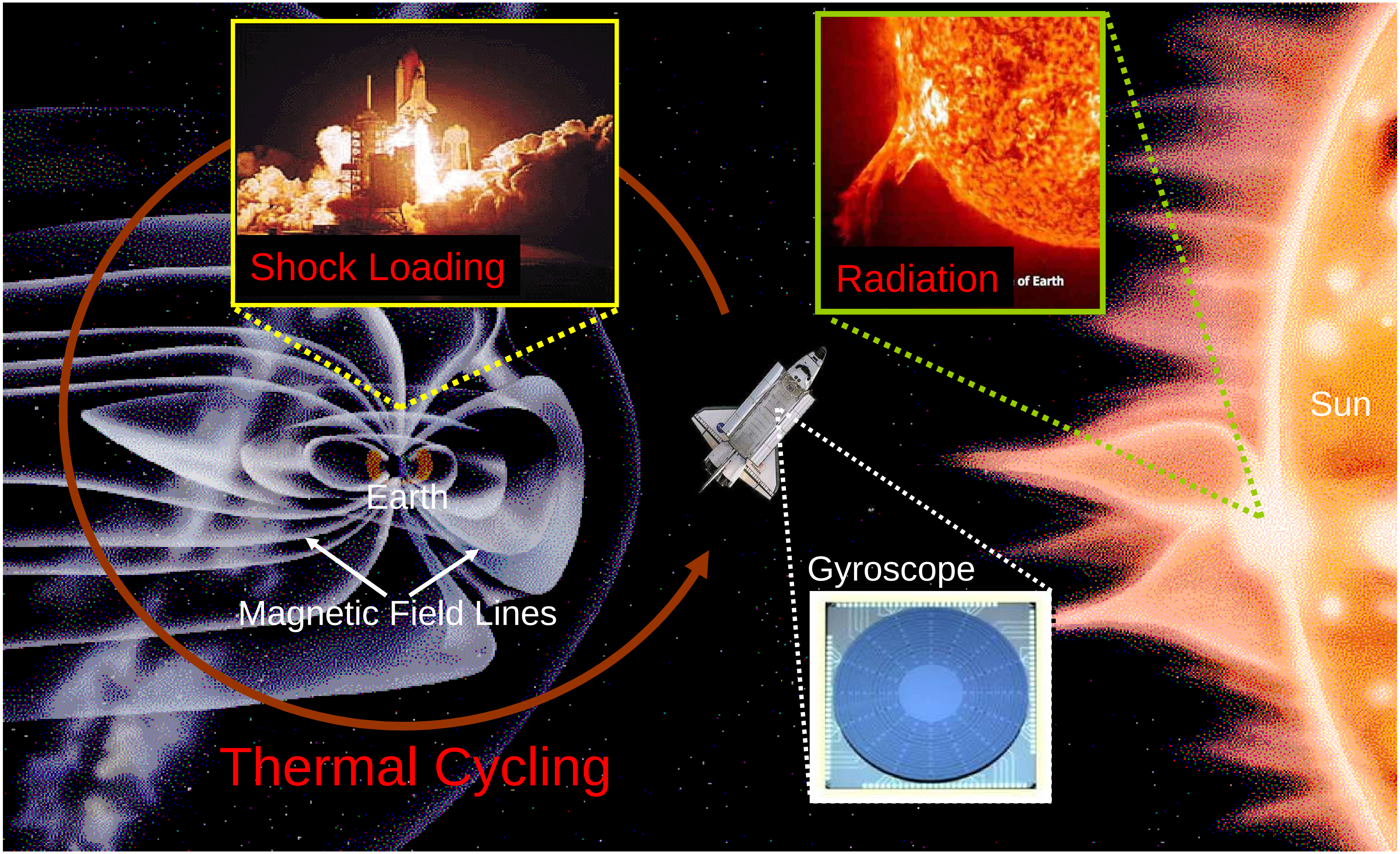
Increasing resolution, decreasing spatial scale, increasing uncertainty

Delamination in thermal cycling and shock **Creep** **Grain boundary migration** **Material interdiffusion**

Mechanisms & Phenomena

To et al (2008)

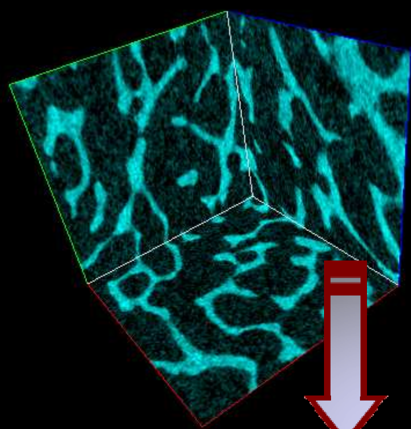
Microsystem. Uncertain Harsh Environment!



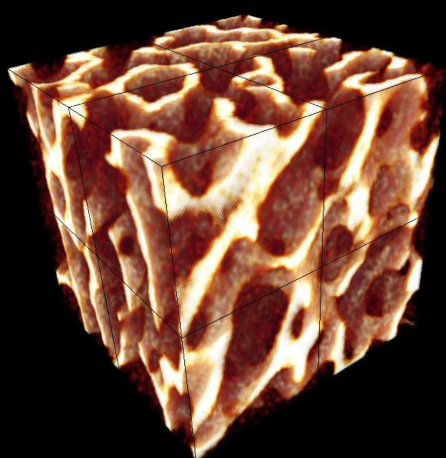
Post Yield Bone Mechanics. Fracture Path Prediction

- *True* system size: Big bone is *O* (cm)
- *Feasible* system size: 3 x 3 (mm), 8 micron resolution ~20 million voxels ~22 million DOF

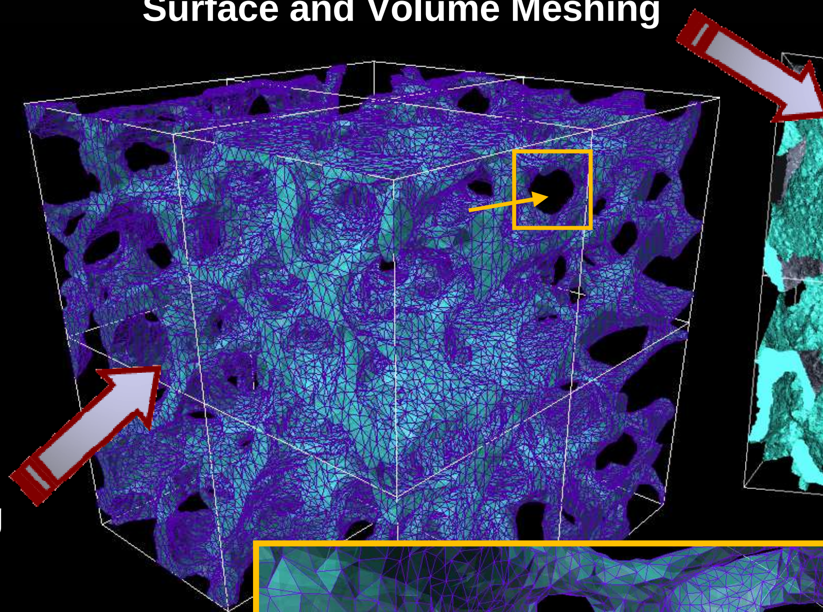
Binary image transformation



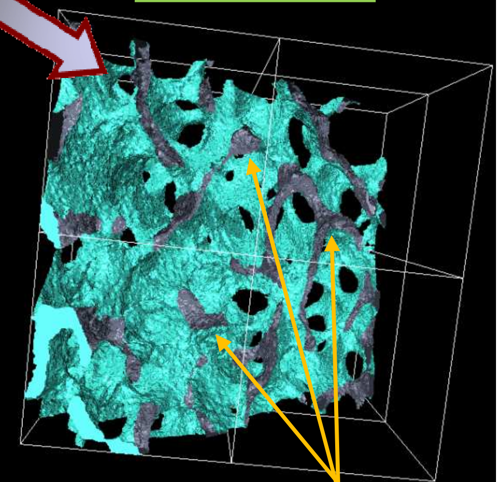
Surface identification and Rendering



Surface and Volume Meshing

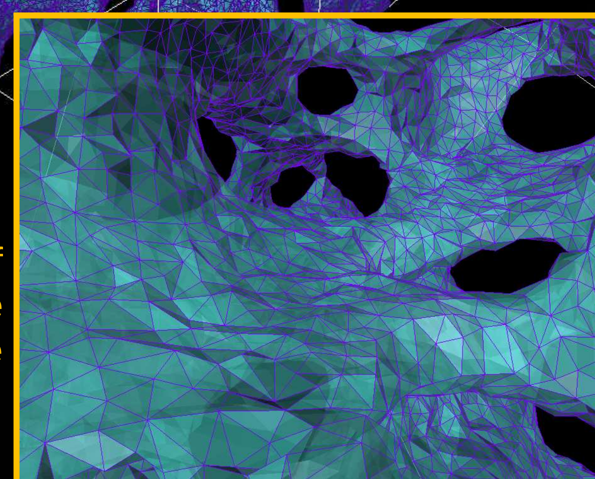


Simulation



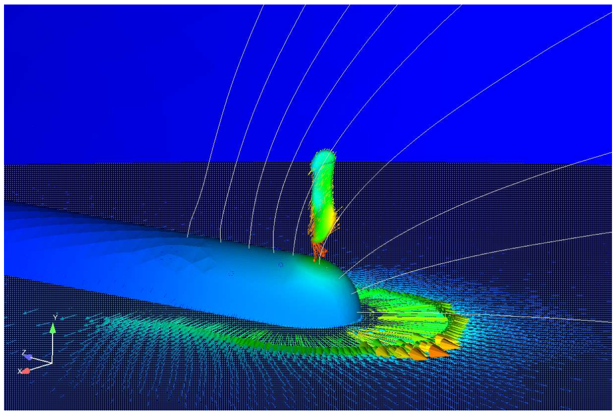
Global fracture path

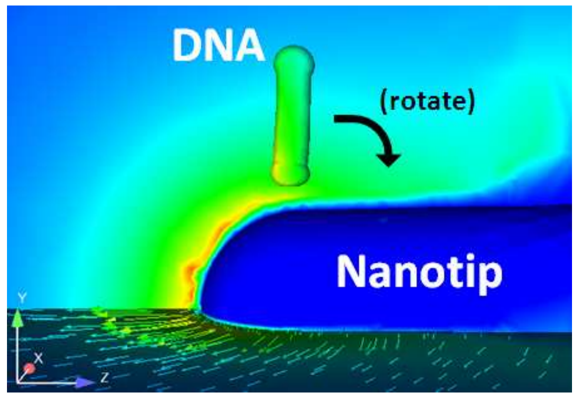
Meshed interior of connected bone structure

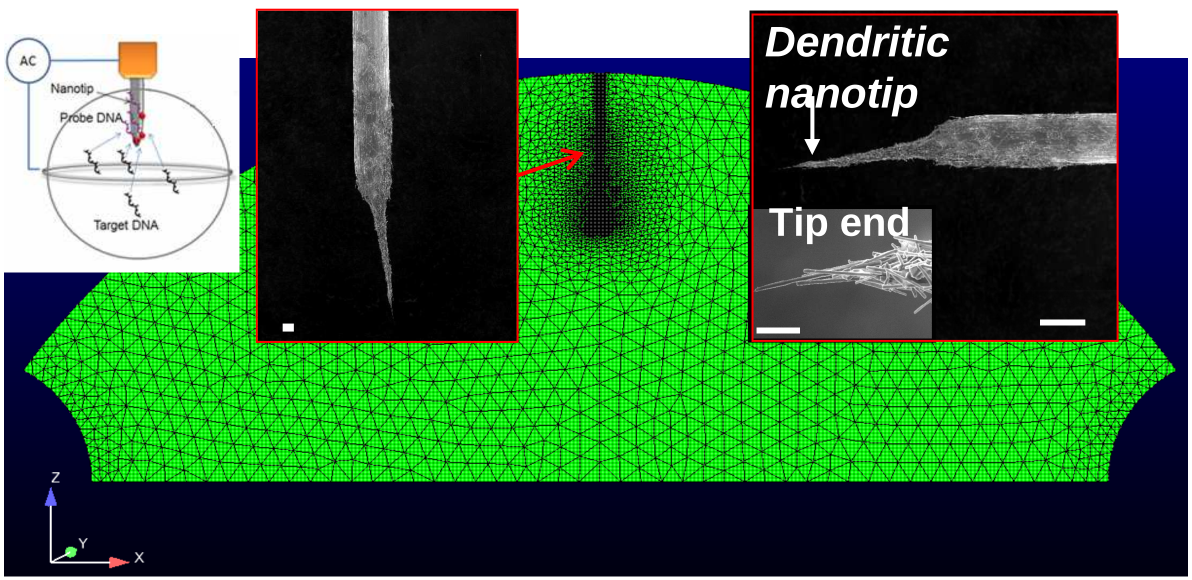


Nanotip Enrichment System for Sensing and Diagnostic

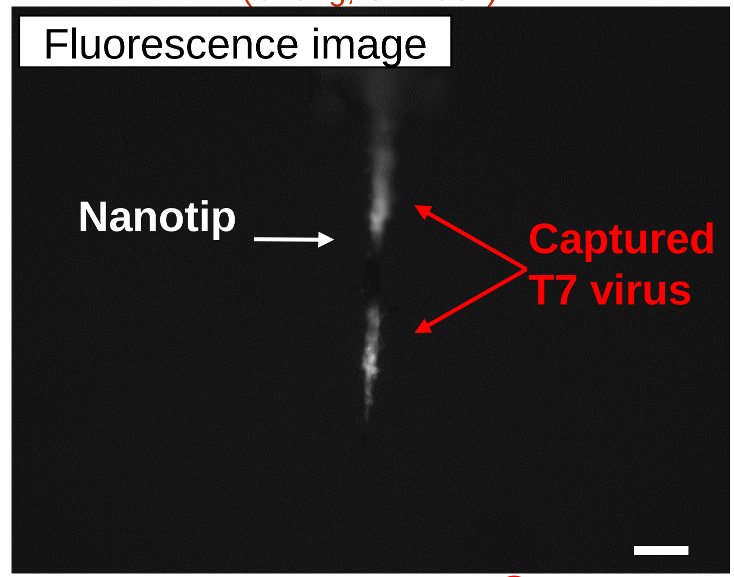
Immersed molecular electrokinetic finite element method is utilized to optimize design and determine experimental parameters for nanotip based sensor

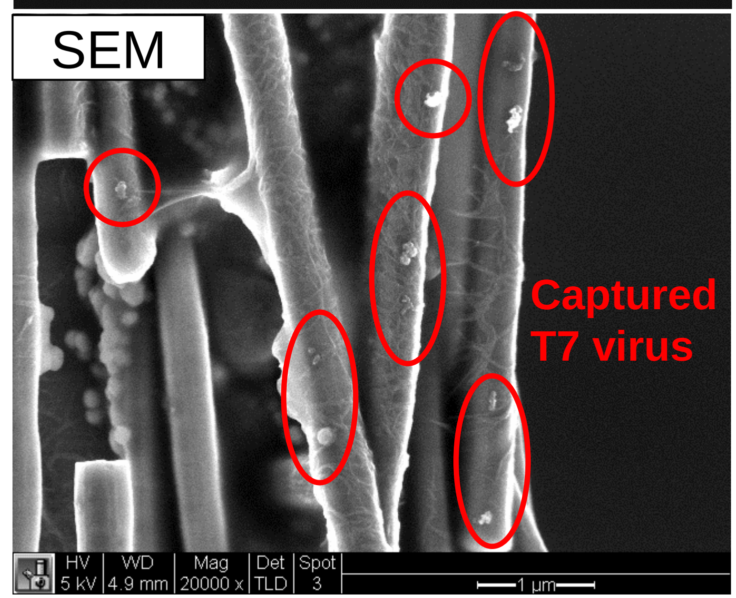




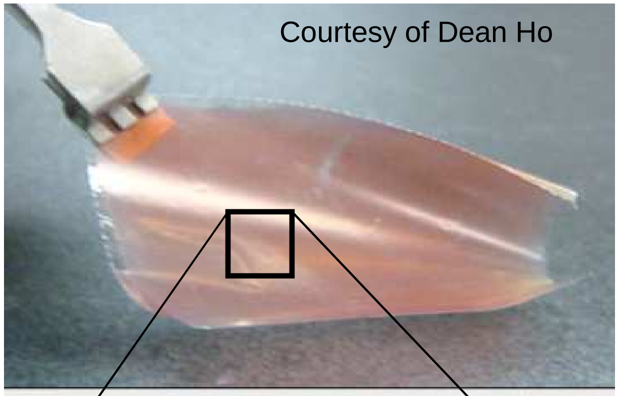


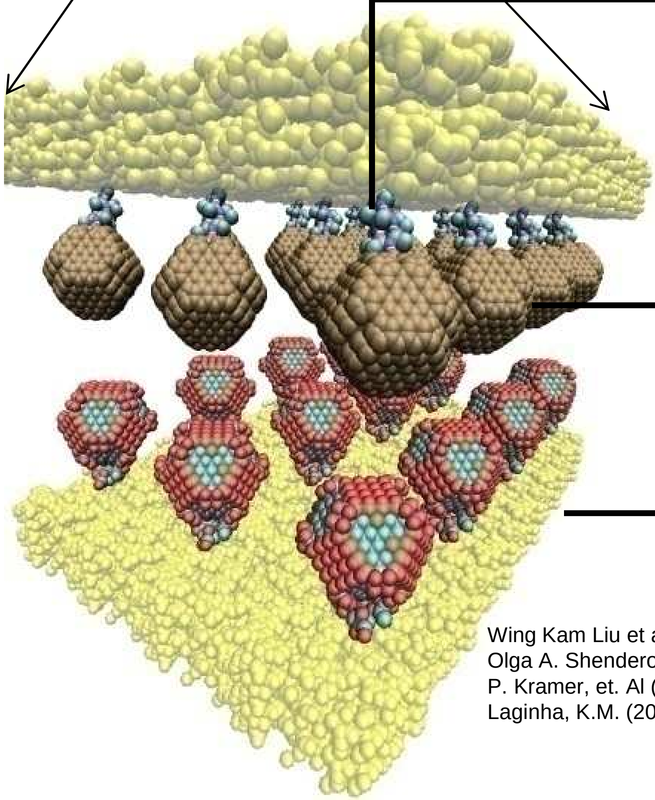
Captured T7 virus on a nanotip (Chung, U. Wash)





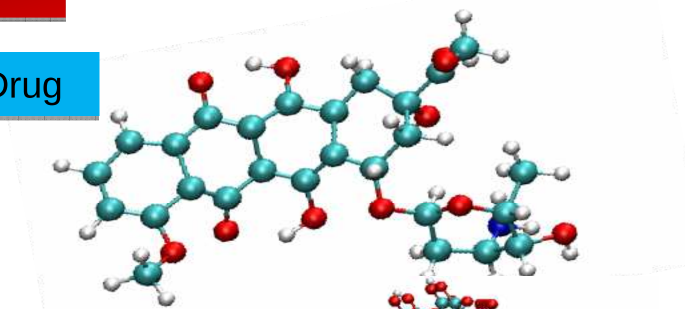
Materials



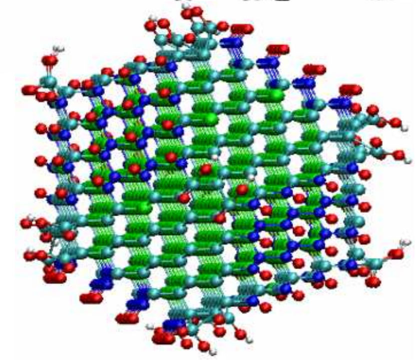


Doxorubicin

• Anticancer Drug



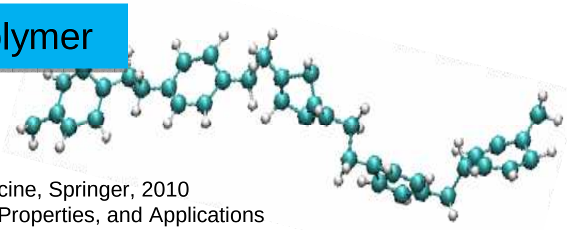
Nanodiamond



- **Drug Carrier**
- **4-5 nm in dia.**
- **Truncated Octahedral Shape**
- **Graphitized Surface**
- **Diamond Core**

Parylene

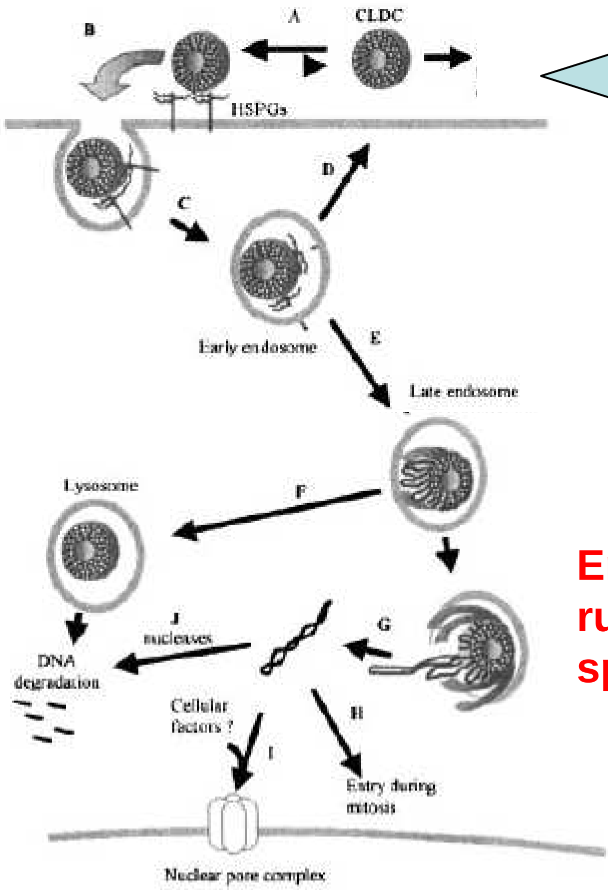
• Biocompatible Polymer



Wing Kam Liu et al. in "Nanodiamonds: Applications in Biology and Nanoscale Medicine, Springer, 2010
Olga A. Shenderova and Dieter M. Hab, Ultrananocrystalline Diamond: Synthesis, Properties, and Applications
P. Kramer, et. Al (2003), Journal of Polymer Science: Polymer Chemistry Edition 22 (2): 475–491.
Laginha, K.M. (2007) Clinical Cancer Research. Vol. 11 (19).

Extracellular and Intracellular behaviors of ND complex

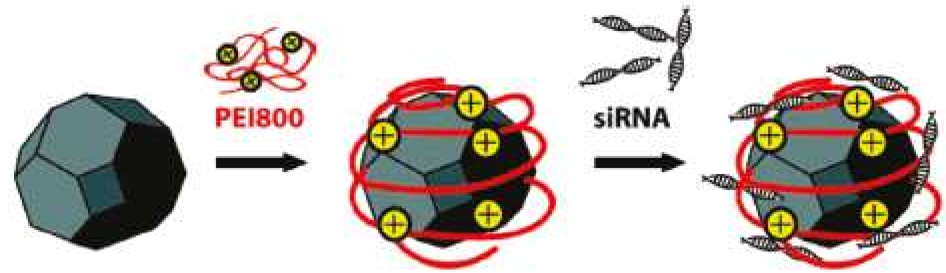
Cell adhesion



Endosomal rupture by proton sponge effect

Christopher M. Wiethoff, Russel Middaugh, “Barrier to Nonviral Gene Delivery”, J of Pharmaceutical sciences, vol 92, No2, 2003

Synthesis of ND complex



ND+PEI800+siRNA

3 important phenomena in gene delivery

1) Synthesis of ND complex

2) Cell adhesion of ND complex (extracellular behavior)

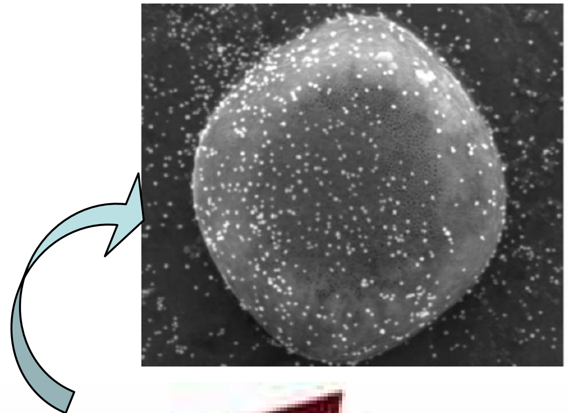
3) Endosomal rupture : H+ entry into the endosome leads into swelling and rupture of endosome (intracellular behavior)

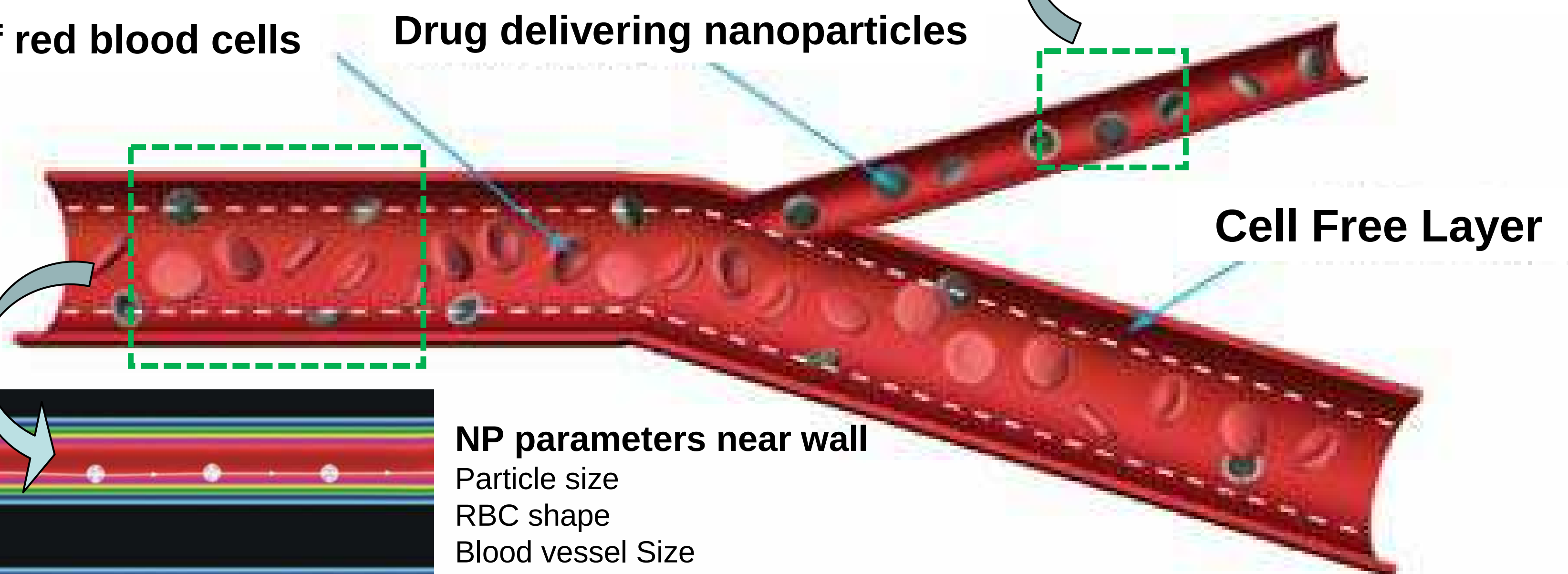
Cell-Nanoparticle in blood vessel

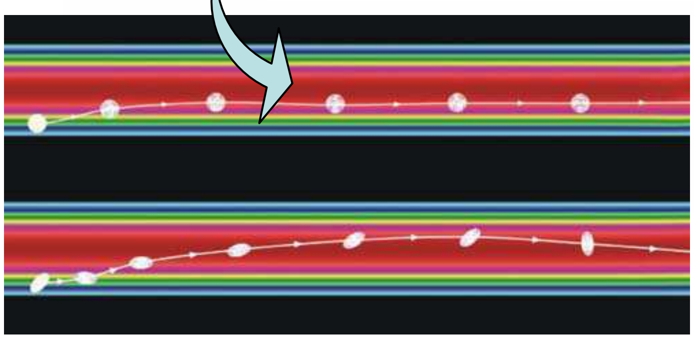
Particles (rejected by the red blood cells)

- Tend to accumulate in close proximity to the walls (as platelets)
- Easy to leave larger vessels in favor of the smaller.
- High particle accumulation in the micro capillary bed

→ Nanoparticles to release drugs or therapeutic agents fast and accurately arrive at target cells.







NP parameters near wall
Particle size
RBC shape
Blood vessel Size
Blood velocity
Cell free layer thickness
…

Decuzzi et al., Pharmaceutical Research, 2009

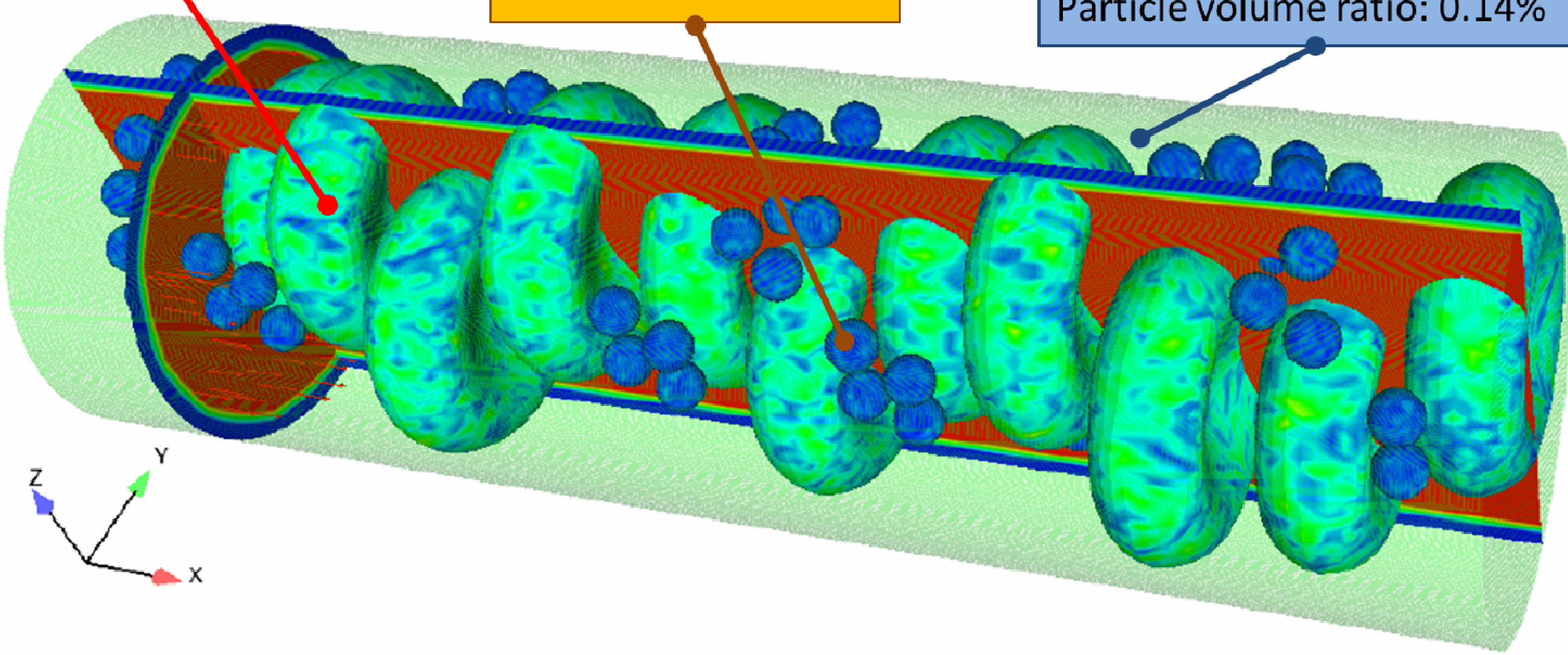
Cell-Nanoparticle in blood vessel

Basic Properties for Simulation

RBCs
Radius: 3µm
Width: 2µm
Volume: ~56.55µm^3
Mesh size: 0.2µm
Elements: 2304
Nodes: 4612

Particles
Radius: 0.5µm
Volume: 0.5249µm^3
Mesh size: 0.1µm
Elements: 2816
Nodes: 3045

Channel
Radius: 6µm
Length: 36µm
Volume: 4071.50µm^3
Mesh size: 0.6µm
Elements: 21360
Nodes: 23729
RBC volume ratio: 15.28%
Particle volume ratio: 0.14%



Outline

- Overarching Theme
- Current research projects
  - Digital 3D, design of high strength alloys. Transportation applications.
  - Polymer nanocomposite design. Transportation and military applications.
  - Microsystem gyroscope design. Deep space application.
  - Post-yield bone mechanics. Clinical application.
  - Nanodiamond delivery and diagnostic mechanics. Medical applications.
- Mathematical machinery
  - Physical postulates of material motion ← **Why.** — **UNCERTAINTY**
  - Governing equations of multiresolution continua ← **Where.** — **UNCERTAINTY**
  - Constitutive modeling ← **How.** — **UNCERTAINTY**
- Conclusions

Overarching Theme

Structure

Property

Performance

UNCERTAINTY

Porthos

Athos

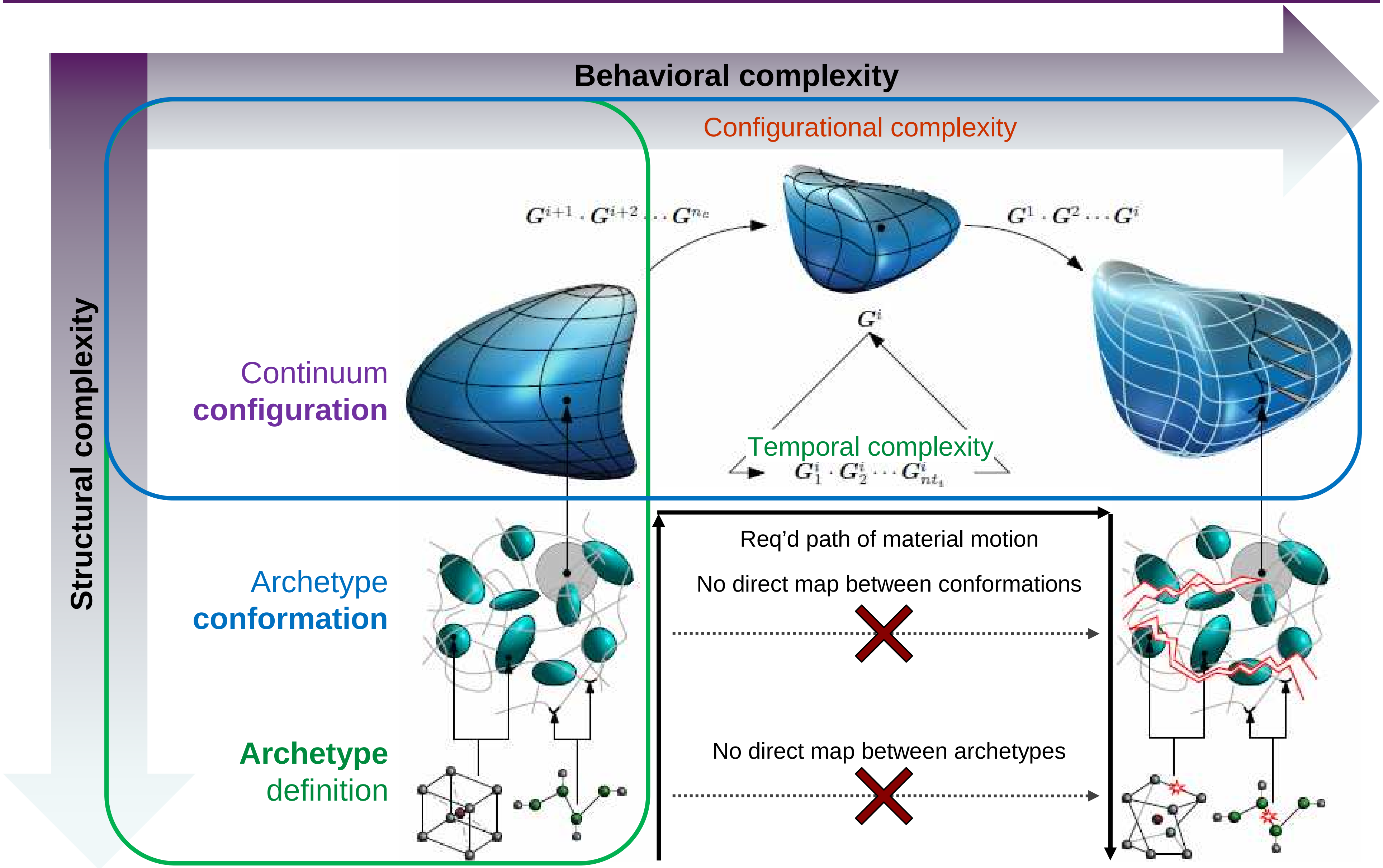
Aramis

D'ARTAGNAN

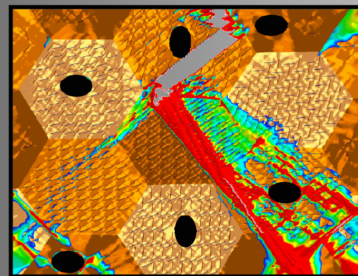
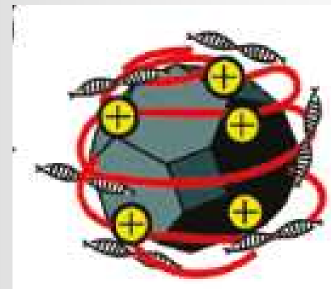
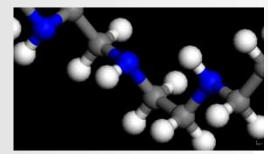
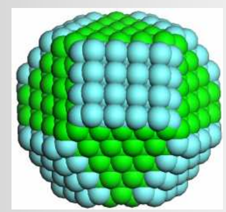
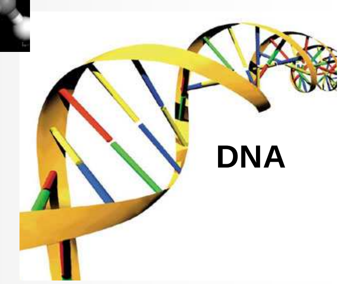
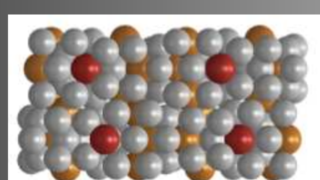
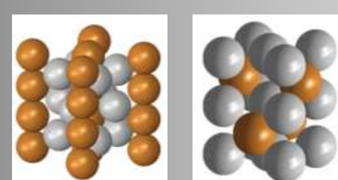
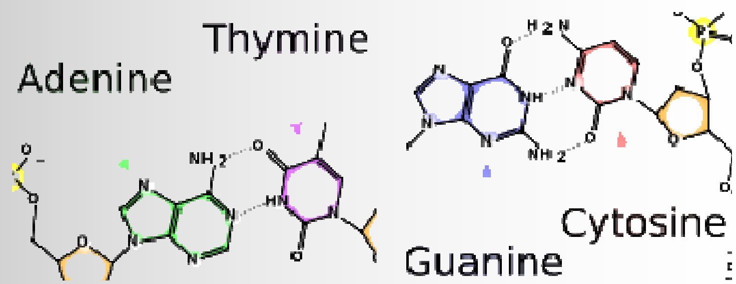
Notwithstanding all the pains he took, D'Artagnan was unable to learn any more concerning his three new-made friends. He formed, therefore, the resolution of believing for the present that all was said of their past, hoping for more certain and extended revelations in the future.

Alexander Dumas
The Three Musketeers
1844

Multiscale Machinery. Physical Postulates



Archetypes and Conformation. Examples

	POLYCRYSTALLINE METAL	GENE DELIVERY SYSTEM
Hierarchy of conformational structures	 **Granular structure**	
Conformation of archetypes into mesostructure	 **Subgranular structure**	 **PEI**  **Nanodiamond**  **DNA**
Archetypes Building blocks Space filling units	 **Dispersoids**  **Precipitate crystals**	

Post Yield Bone Mechanics

- **Goal**. Predict post yield behavior, fracture path likelihood in spongy bone
- **Schedule**. 2010-current
- **Collaborators**. Iwona Jasiuk U-Illinois, Elise Morgan Boston University

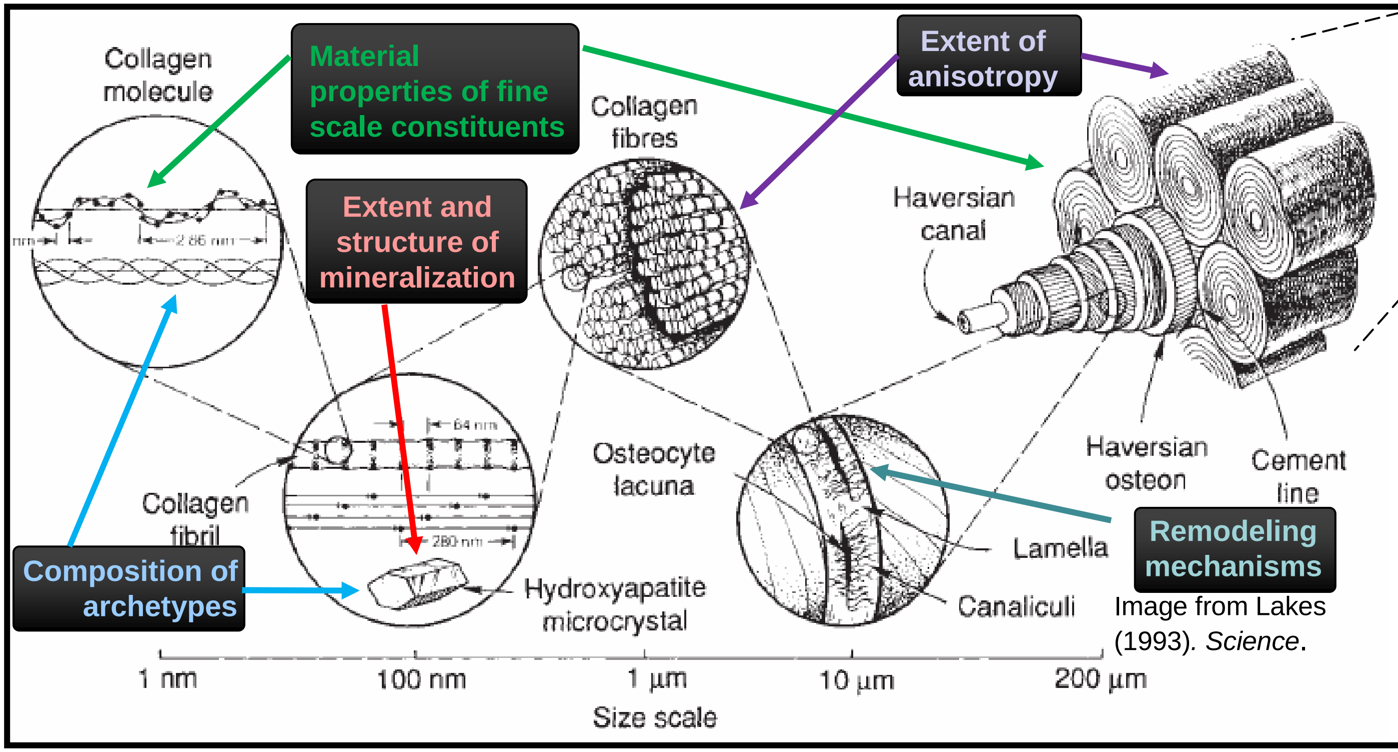
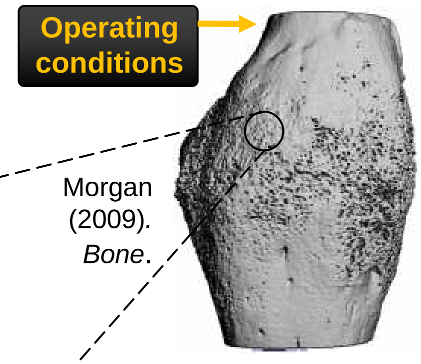
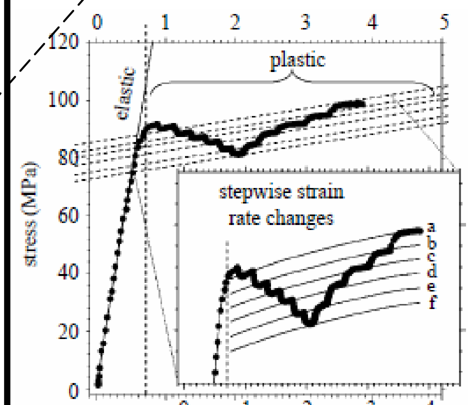


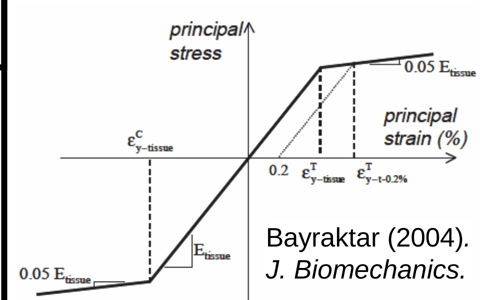
Image from Lakes (1993). *Science*.



Morgan (2009). *Bone*.



Gupta (2006). *J. Royal Society*.



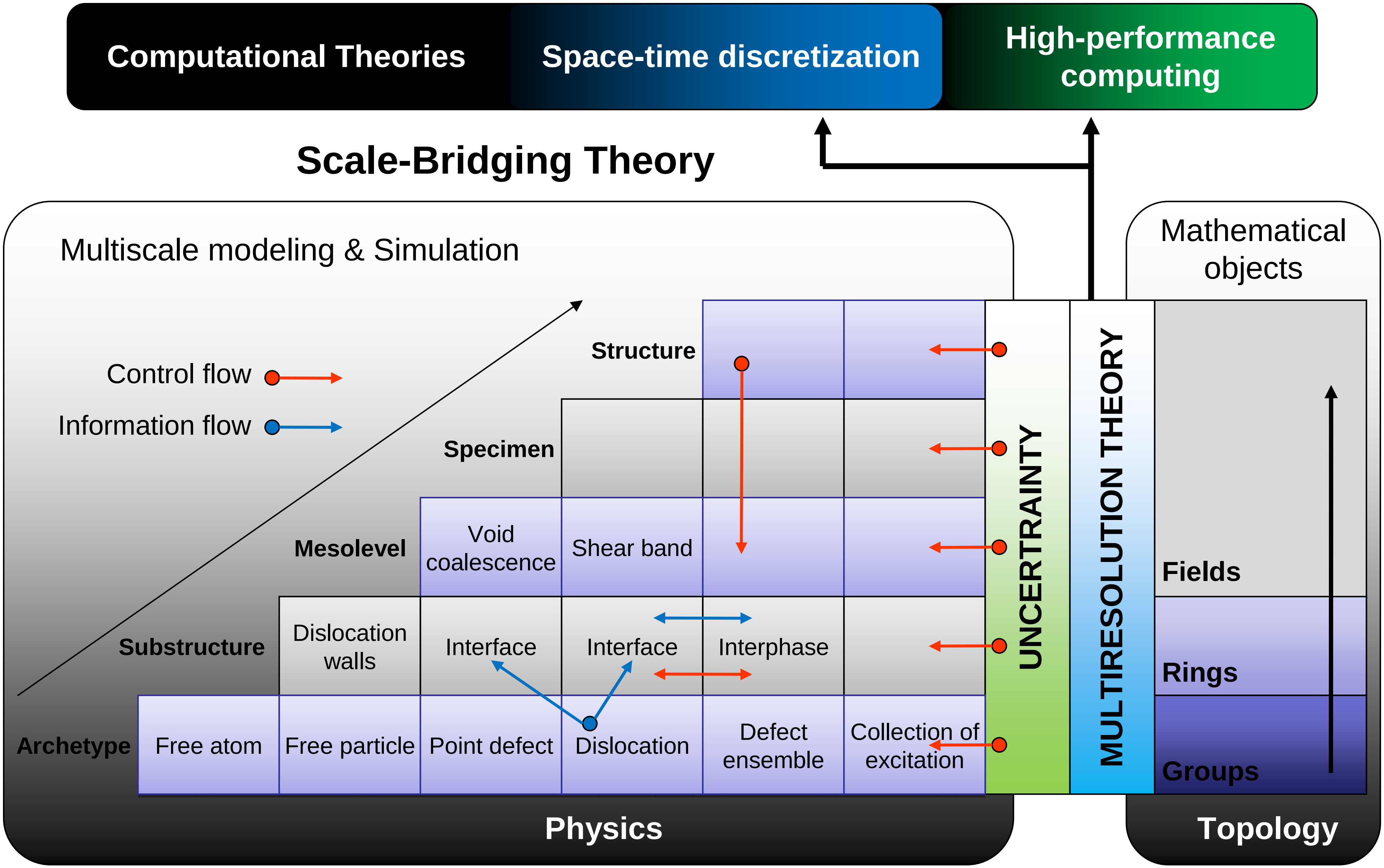
Bayraktar (2004). *J. Biomechanics*.

Increasing resolution, decreasing spatial scale, increasing uncertainty

Microscopic Mechanisms

Macroscopic rate dependent plasticity

Multiscale Machinery. Physical Postulates



Uncertainty Types

- Random multiscale conditions
  - Inherent structural variation (batch to batch randomness, random field)
  - Uncertain multiscale boundary conditions (chaotic environments)
- Lack of knowledge, lack of information
  - Misunderstood physics
  - Insufficient physical experiments
  - Model inadequacy
- Limited computational resource

STOCHASTIC PDE's

Greene, M.S., Y. Liu, et al. (2011).
Computational uncertainty analysis in multiresolution materials via stochastic constitutive theory.
CMAME **200(1-4): 309-325.**

General Stochastic Formulation

Write general partial differential equation governing evolution of some system

$(\Omega, \Im, P)$ Probability space

Ω Could be infinite dimensional

Ghanem (1999) Xiu (2009)
Ghanem & Spanos (1991)
Xiu & Karniadakis (2002)
Xiu (2006)
Acharjee (2006)

Governing Equation

RANDOM SOLUTION

$$[\mathbf{T}]\,\mathbf{v}(\mathbf{x},t,\omega)+[\bar{\mathbf{S}}+\mathbf{S}_{\omega}]\,\mathbf{v}(\mathbf{x},t,\omega)=\mathbf{0}$$

Temporal operator

Random spatial operator

Mean spatial operator

$$[\bar{\mathbf{B}}+\mathbf{B}_{\omega}]\mathbf{v}(\mathbf{x},t,\omega)=\mathbf{0}$$

Boundary condition

$$\mathbf{v}(\mathbf{x},t=0,\omega)=\mathbf{v}_0(\mathbf{x},\omega)$$

Initial condition

Equation Domain

Time domain

$$\text{in } D(\omega)\times(0,T]\times\Omega$$

Spatial domain

Random domain

$$\text{in } \partial D(\omega)\times[0,T]\times\Omega$$

Spatial boundary

$$\text{in } D(\omega)\times\{t=0\}\times\Omega$$

Uncertainty

(1)
Stochastic differential operators.

- Random field coefficients
- Material properties
- Dimension reduction req'd

(2)
Uncertain environment, i.e. random boundary or initial conditions.

- Mechanical loads unknown
- Temperature field unknown
- Current state of system uncertain

(3)
Uncertain domain.

- Rough edges from processing
- Micro/tele scope finite resolution

Stochastic Formulation for Solid Mechanics

General

$$\left[\boldsymbol{T}+\bar{\boldsymbol{S}}+\boldsymbol{S}_Z\right]\mathbf{v}(\mathbf{x},t,\mathbf{Z})=\mathbf{0}$$

in $D(\omega)\times(0,T]\times\Omega$

- Uncertainty parameterization technique required
- KL transformation, Polynomial Chaos Expansion
- Dimension reduction methodology

Solid Mechanics

Composition operation

$$\left[\boldsymbol{T}+\bar{\boldsymbol{S}}\right](\mathbf{v})=\left[-\underline{\boldsymbol{T}}+\underline{\boldsymbol{N}}\circ\underline{\boldsymbol{f}}\circ\underline{\boldsymbol{g}}\circ\underline{\boldsymbol{h}}\right](\mathbf{v})$$

- $[\boldsymbol{T}](\bullet)=\text{acceleration}\triangleq\partial_t(\bullet)$
- $[\boldsymbol{N}](\bullet)=\text{balance law}\triangleq\vec{\nabla}\cdot(\bullet)$
- $\boxed{[\boldsymbol{f}](\bullet)\triangleq\text{constitutive law}}$
- $[\boldsymbol{g}](\bullet)=\text{objectivity}\triangleq\text{sym}(\bullet)$
- $[\boldsymbol{h}](\bullet)=\text{compatibility}\triangleq\vec{\nabla}(\bullet)$

Solution (vector) space

$\mathbf{v}(\mathbf{x},t,\mathbf{Z})$

Tensor space

$\nabla\mathbf{v}(\mathbf{x},t,\mathbf{Z})$

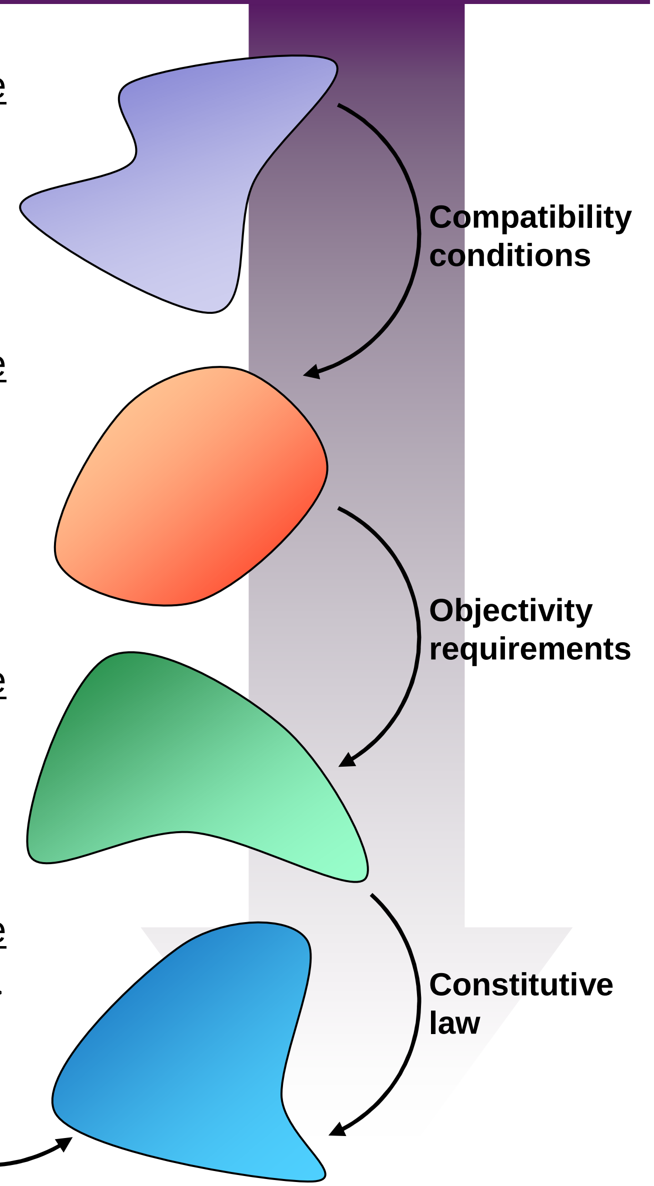
Symmetric tensor space

$\text{sym}\left[\nabla\mathbf{v}(\mathbf{x},t,\mathbf{Z})\right]$

Stress tensor space

$\boldsymbol{\sigma}\left\{\text{sym}\left[\nabla\mathbf{v}(\mathbf{x},t,\mathbf{Z})\right]\right\}$

Balance Law
acts on stress space



Stochastic Formulation for Multiresolution Continua

Solid Mechanics

$$\left[T + \bar{S} + S_Z\right](\mathbf{v}) = \left[-T + N \circ \left(\bar{f} + f_Z\right) \circ g \circ h\right](\mathbf{v})$$

Macroscopic stress

Relative stress

density

body force

$$-\rho\dot{\mathbf{v}} + \left[\boldsymbol{\sigma} - \sum_{n=1}^{N-1}\boldsymbol{\beta}^n\right]\cdot\overleftarrow{\nabla} + \mathbf{b} = \mathbf{0}$$

Temporal operator

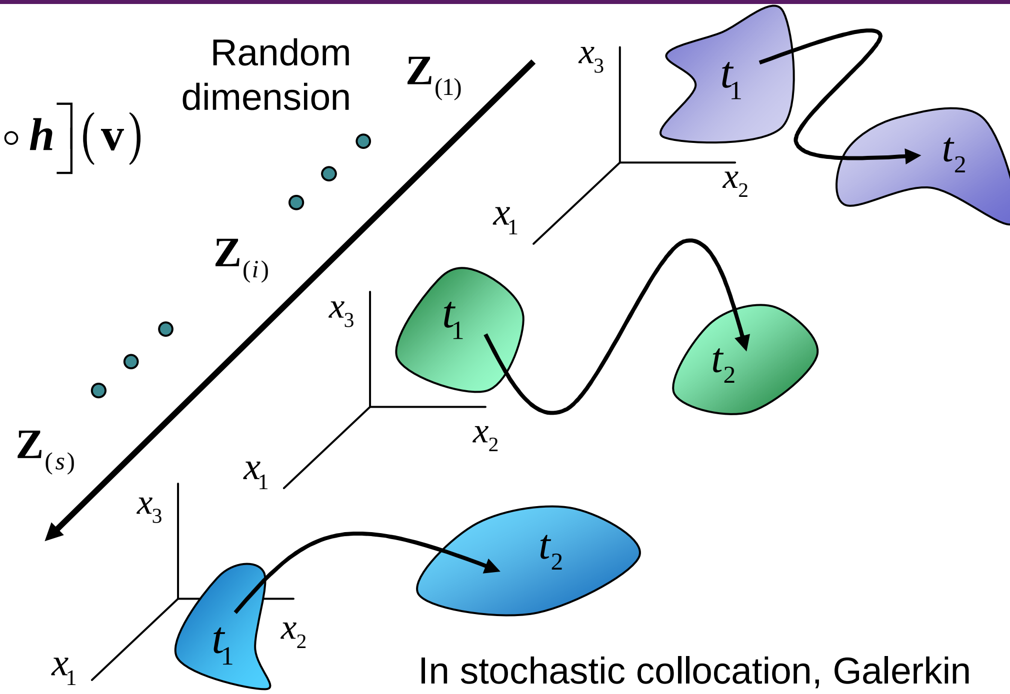
Spatial operator

$$-\dot{\boldsymbol{\gamma}}\cdot\mathbf{I}^n + \bar{\boldsymbol{\beta}}^n\cdot\overleftarrow{\nabla} - \boldsymbol{\beta}^n + \mathbf{B}^n = \mathbf{0},\quad n = 1, 2, \dots, N$$

Boundary condition

$$\left[\boldsymbol{\sigma} - \sum_{n=1}^{N-1}\boldsymbol{\beta}^n\right]\cdot\mathbf{n} = \mathbf{t}$$

$$\bar{\boldsymbol{\beta}}^n\cdot\mathbf{n} = \mathbf{T}^n,\quad n = 1, 2, \dots, N$$



In stochastic collocation, Galerkin projection, **random dimension discretized** like space and time

RANDOM SOLUTIONS

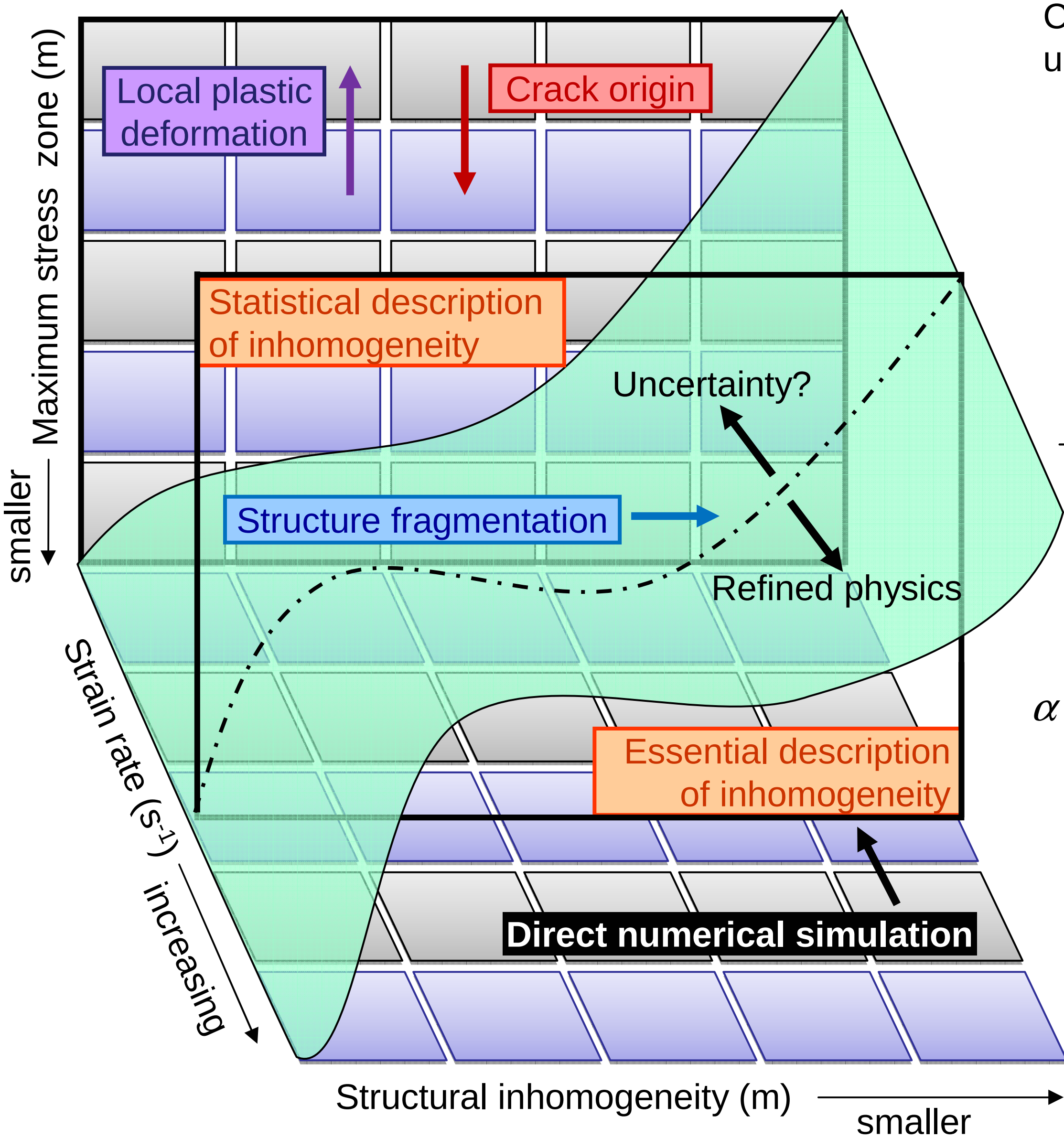
VELOCITY

MICRO-VELOCITY GRADIENT

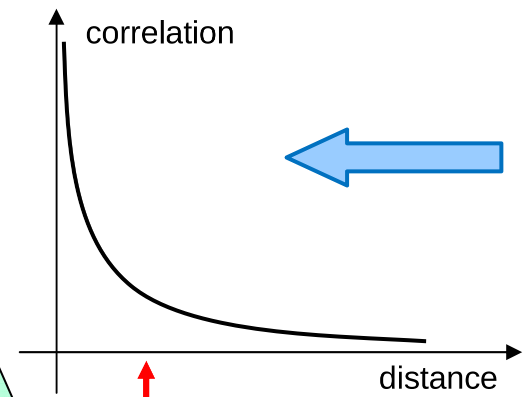
$$\boldsymbol{\sigma},\boldsymbol{\beta}^n,\bar{\boldsymbol{\beta}}^n = \mathbf{f}_\sigma,\mathbf{f}_{\boldsymbol{\beta}},\mathbf{f}_{\bar{\boldsymbol{\beta}}}\left(\mathrm{sym}\nabla\mathbf{v}(\mathbf{x},t,\mathbf{Z}),\underline{\mathbf{L}}^n(\underline{\mathbf{x}},t,\mathbf{Z}) - \mathbf{L},\nabla\mathbf{L}^n(\mathbf{x},t,\mathbf{Z})\right)$$

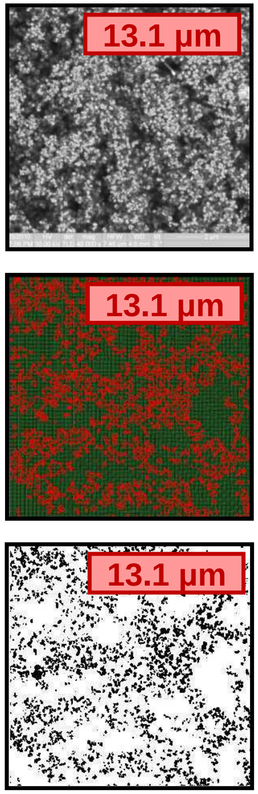
Constitutive law an implicit function of a denumerable set of random variables

Stochastic Formulation and Length Scales



Consideration of length scale effects and uncertainty intimately related





KL Expansion

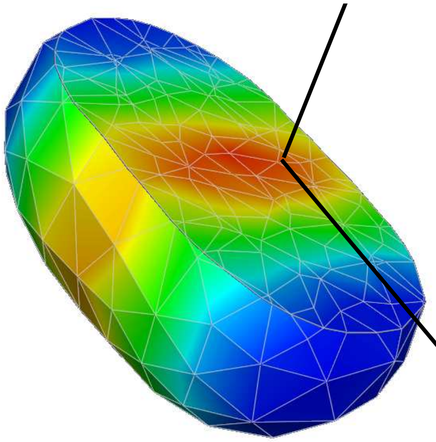
$$\alpha(\mathbf{x},t,\omega)=\sum_{i=1}^{P}\sqrt{\lambda_i}\,\xi_i(\omega)\,\psi_i(\mathbf{x},t)$$

Number of terms is intractable if correlation length is small relative to macroscopic domain

Stochastic Constitutive Theory

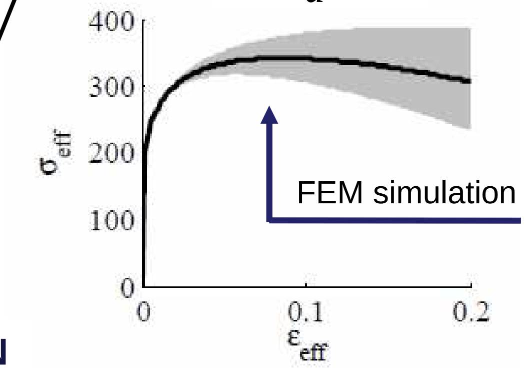
- Denumerable set of random variables are structural descriptors of microstructure
- Unique stress distributions averaged to provide constitutive behavior

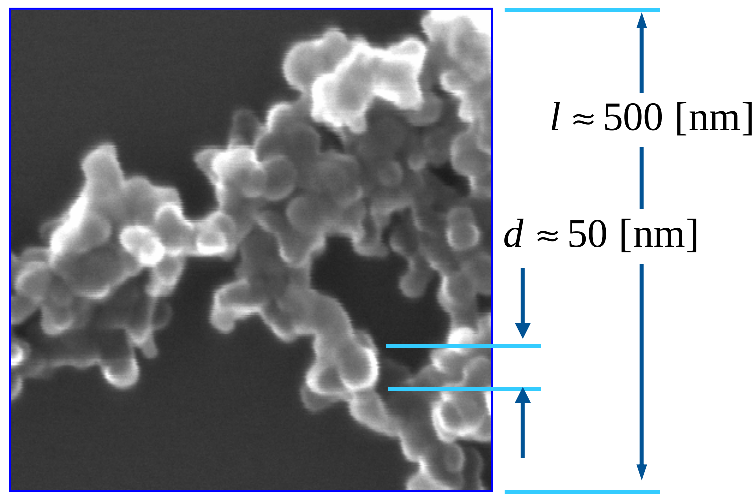
MACRO-SCALE SIMULATION



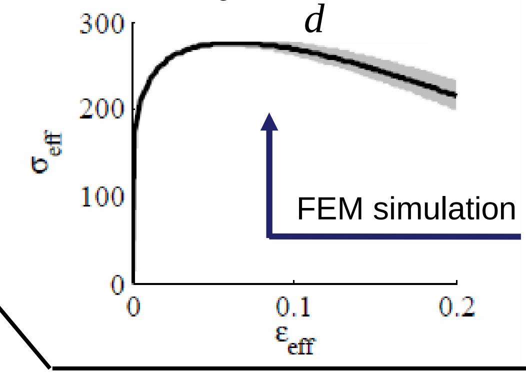
MICRO-SCALE VOLUME ELEMENTS

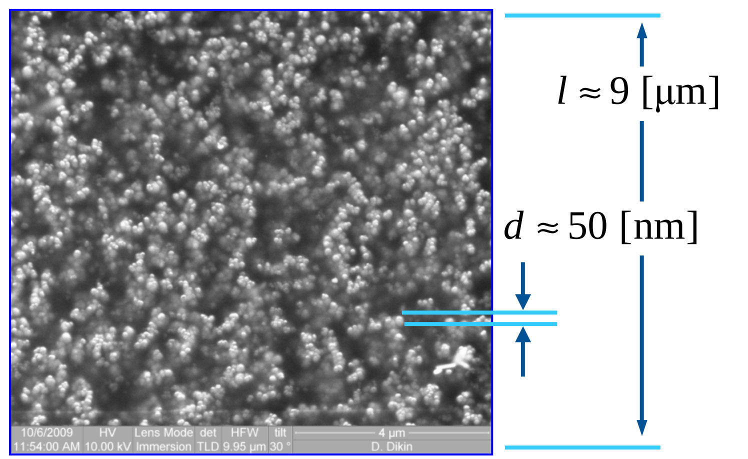
$$\delta = \frac{l^{SVE}}{d} < \infty$$





$$\delta = \frac{l^{RVE}}{d} \approx \infty$$

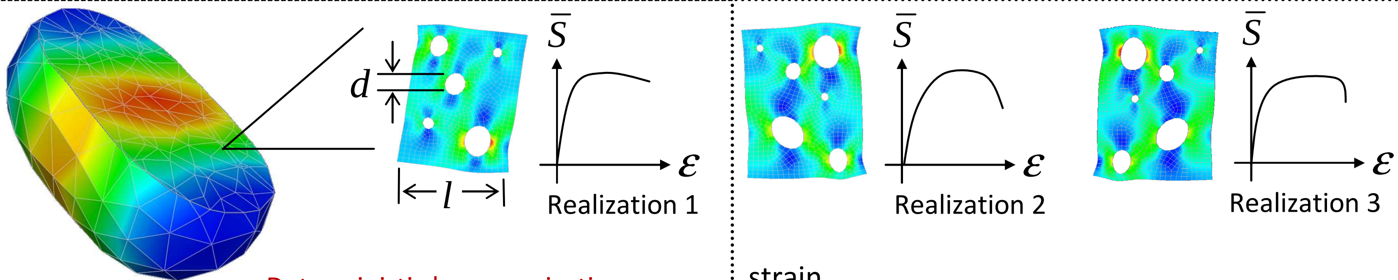




SEM images provided by D. Dikin (2010)

For multiscale analysis, deterministic fine scale simulations create randomness

Macroscale, e.g. — Fine resolution statistical volume element realizations



Deterministic homogenization
(standard constitutive modeling)

$$\langle \mathbf{S} \rangle = f\left(\boldsymbol{\kappa}, \langle \boldsymbol{\varepsilon} \rangle, \langle \dot{\boldsymbol{\varepsilon}} \rangle, \langle T \rangle, t, \ldots, SA\right)$$

strain
strain rate
temperature
time
structural approx.

Fit set of phenomenological constitutive law parameters to SVE simulation results

consider statistics of constitutive behavior other than mean

$$\mathbf{S}_\omega = f\left(\boldsymbol{\kappa}, \langle \boldsymbol{\varepsilon} \rangle, \langle \dot{\boldsymbol{\varepsilon}} \rangle, \langle T \rangle, t, \ldots, SA, \omega\right)$$

General form of stochastic constitutive theory

Treat parameters as multivariate statistical distribution whose data are each realization

Transfer uncertainty from general constitutive law to phenomenological parameters

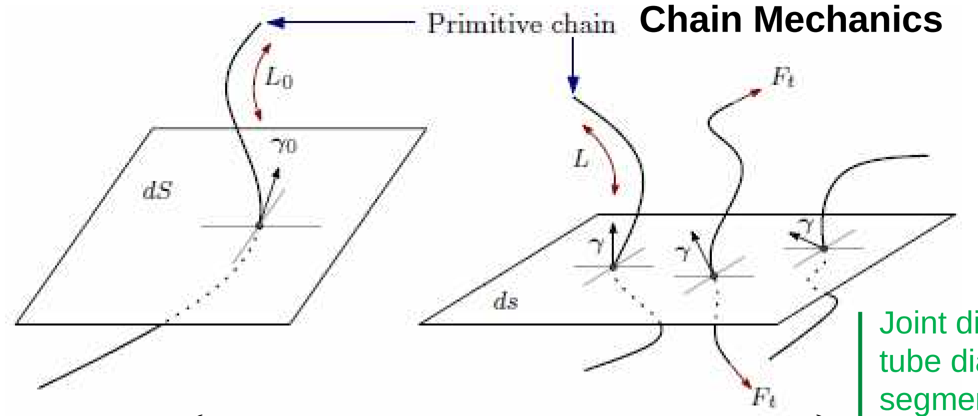
$$\mathbf{S}_\omega = f\left[\mathbf{K}(\omega), \langle \boldsymbol{\varepsilon} \rangle, \langle \dot{\boldsymbol{\varepsilon}} \rangle, \langle T \rangle, t, \ldots, SA\right]$$

Specific form of stochastic constitutive theory

$$\mathbf{K}(\theta) \sim f_K(\boldsymbol{\kappa}, \theta)$$

Greene (2010)

Chain Mechanics



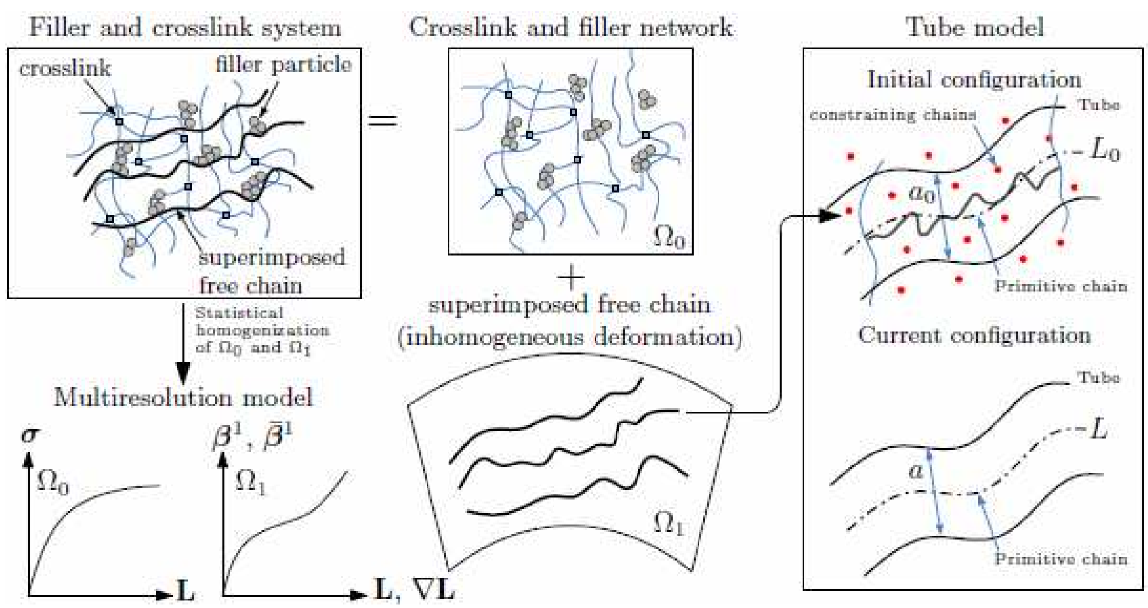
Joint distribution of random tube diameter, number of segments per chain

$$\boldsymbol{\beta}^1 = \int_0^\infty \int_0^\infty \left\{ G \left\langle L^2 \right\rangle \sum_{k=1}^{M_c} \sum_{J=1,\mathrm{odd}}^{\infty} w_{(k)} \mathbf{Y}\left[J, t', t, \tau_d \left(\boldsymbol{\gamma}_{0(k)} \right) \right] \right\} \boldsymbol{\Phi}\left(a_0, n_\nu\right) da_0 \, dn_\nu$$

Microstress

Contour length

Tangent vector



Operating conditions

T	TEMPERATURE
ω	FREQUENCY

Polymer chemistry (real chains)

N	CHAINS PER UNIT VOLUME
ρ	POLYMER DENSITY
b	MONOMER-MONOMER DISTANCE
n	MONOMERS PER CHAIN

Polymer physics

D_c	DIFFUSION CONSTANT
τ_d	RELAXATION TIME

Predicted from MD simulations

$$\tau_d = \frac{\left(\dfrac{nb^2}{a}\right)^2}{D_c \pi^2}$$

Outline

- Overarching Theme
- Current research projects
  - D3D, design of high strength steels. Aerospace application.
  - Polymer nanocomposite design. Automotive application.
  - Post-yield bone mechanics. Clinical application.
  - Microsystem gyroscope design. Deep space application.
  - Nanotip enrichment. Medical application
- Mathematical machinery
  - Physical postulates of material motion ← Why. — UNCERTAINTY
  - Governing equations of multiresolution continua ← Where. — UNCERTAINTY
  - Constitutive modeling ← How. — UNCERTAINTY
- **Conclusions**

Discussion

- Engineering systems are complex and large, reduction of uncertain dimension may not always be possible
- Is there a tradeoff in analysis of uncertainty and higher fidelity models?
- When is computational uncertainty analysis worthwhile?
- Link between material microstructure and material property critical to understanding random property fields.
- Thumbs up for archetypes and multiresolution theory

References

1 Acharjee, S. and N. Zabaras (2006). Uncertainty propagation in finite deformations--A spectral stochastic Lagrangian approach. *CMAME.* **195(19-22): 2289-2312.**

2 Bayraktar, H. H., E. F. Morgan, et al. (2004). Comparison of the elastic and yield properties of human femoral trabecular and cortical bone tissue. *J. Biomechanics.* **37(1): 27-35.**

3 Ghanem, R. (1999). Ingredients for a general purpose stochastic finite elements implementation. *CMAME* **168(1-4): 19-34.**

4 Ghanem, R. G. and P. D. Spanos (1991*). Stochastic Finite Elements A Spectral Approach.* New York, Springer-Verlag.

5 Gupta, H. S., P. Fratzl, et al. (2007). Evidence for an elementary process in bone plasticity with an activation enthalpy of 1eV. *J. Roy. Soc.* **4(13): 277-282.**

6 Lakes, R. (1993). "Materials with structural hierarchy." *Nature* **361: 511-515.**

7 McVeigh, C. and W. K. Liu (2008). Linking microstructure and properties through a predictive multiresolution continuum. *CMAME* **197(41-42): 3268-3290.**

8 McVeigh, C., F. Vernerey, et al. (2006). Multiresolution analysis for material design. *CMAME* **195(37-40): 5053-5076.**

9 Morgan, E. F., Z. D. Mason, et al. (2009). Micro-computed tomography assessment of fracture healing: Relationships among callus structure, composition, and mechanical function. *Bone* **44(2): 335-344.**

10 Greene, M.S., Y. Liu, et al. (2011). Computational uncertainty analysis in multiresolution materials via stochastic constitutive theory. *CMAME* **200(1-4): 309-325.**

11 Tian, R., S. Chan, et al. (2010). A multiresolution continuum simulation of the ductile fracture process. *JMPS* **58(10): 1681-1700.**

12 To, A., W. Liu, et al. (2008). Materials integrity in microsystems: a framework for a petascale predictive-science-based multiscale modeling and simulation system. *Comp. Mech.* **42(4): 485-510.**

13 Vernerey, F., W. Liu, et al. (2009). Multi-length scale micromorphic process zone model. *Comp. Mech.* **44(3): 433-445.**

14 Vernerey, F. J., W. K. Liu, et al. (2007). Multi-scale micromorphic theory for hierarchical materials. *JMPS* **55: 2603-2651.**

15 Vernerey, F. J., W. K. Liu, et al. (2008). A micromorphic model for the multiple scale failure of heterogeneous materials. *JMPS* **56(4): 1320-1347.**

16 Xiu, D. (2009). *Numerical methods for stochastic computations: a spectral method approach*. New Jersey, Princeton University Press.

17 Xiu, D. and G. E. Karniadakis (2002). The Wiener-Askey Polynomial Chaos for Stochastic Differential Equations. *SIAM J. Sci. Comp.* **24(2): 619-644.**

18 Xiu, D. and D. M. Tartakovsky (2006). Numerical Methods for Differential Equations in Random Domains. *SIAM J. Sci. Comp.* **28(3): 1167-1185.**

19 Yin, X., W. Chen, et al. (2008). Statistical Volume Element Method for Predicting Microstructure-Constitutive Property Relations. *CMAME* **197: 3516-3529.**

NORTHWESTERN UNIVERSITY

QUESTIONS

MANY THANKS!