AN ELLIPTIC INVERSE PROBLEM ARISING IN GROUNDATER FLOW

Andrew Stuart

Mathematics Institute and
Centre for Scientific Computing
University of Warwick

BIRS Stochastic Multiscale Workshop
Banff, March 28^{th} 2011

Outline

Forward Problem

- Darcy Law. p pressure, k permeability.
- Let $D \subset \mathbb{R}^d$ be a bounded open set.
- Define $X = L^\infty(D)$ and $V = H_0^1(D)$.
- Let $k \in X$ with $\text{essinf}_{x \in D} k(x) > 0$ and $f \in V^*$.
- Consider the (weak) elliptic PDE

$$\int_D k(x) \nabla p(x) \nabla v(x)\, dx = \int_D f(x) v(x)\, dx, \quad \forall v \in V.$$

- Unique solution p by Lax-Milgram.

Observation Operator

- Assume that $k = K(u)$ for $K : U \subseteq \ell^p \to K$.
- Define $G : U \to V$ by $G(u) = p$.
- Let $\mathcal{O} : V \to \mathbb{R}^K$ denote K linear functionals on V.
- Define $\mathcal{G} : X \to \mathbb{R}^K$ by $\mathcal{G} = \mathcal{O} \circ G$.
- We call $\mathcal{G}$ the OBSERVATION OPERATOR.

Inverse Problem

- FIND u GIVEN $y = \mathcal{G}(u) + \eta$.
- Only DENSITY ρ of random variable η is known.
- PRIOR measure μ_0 on u.
- would like BAYES THEOREM in infinite dimensions.
- This would give the POSTERIOR measure μ^y on $u|y$:

$$\frac{d\mu^y}{d\mu_0}(u) \propto \rho\big(y - \mathcal{G}(u)\big).$$

Outline

Log-Normal Prior

- $\{\varphi_j\}_{j\geq 1}$ an orthonormal sequence in $L^2(D)$.
- $\{\lambda_j\}_{j\geq 1}$ a positive ℓ^2 sequence of real numbers.
- $\{u_j\}_{j\geq 1}$ independent random variables with $u_j \sim \mathcal{N}(0, \lambda_j^2)$.
- $k(x) = \exp\left(\sum_{j=1}^{\infty} u_j \varphi_j(x)\right)$.
- Defines a measure μ_0 on $U = \ell^2$, and push forward onto X.

Uniform Prior

- $\{\varphi_j\}_{j\geq 1}$ normalized sequence in X with $\|\psi_j\|_X = \lambda_j$.
- $\{\lambda_j\}_{j\geq 1}$ a positive ℓ^1 sequence of real numbers.
- $\{u_j\}_{j\geq 1}$ independent random variables with $u_j \sim \mathcal{U}(-1,1)$.
- $k(x) = a(x) + \sum_{j=1}^{\infty} u_j \varphi_j(x)$
- $\operatorname{essinf}_{x\in D} a(x) > \sum_{j=1}^{\infty} |\lambda_j|$.
- Defines a measure μ_0 on $U = [-1,1]^{\mathbb{N}} \subset \ell^\infty$, and push forward onto X.

Outline

Concrete Setting

- Study concrete case $y = \mathcal{G}(u) + \eta, \quad \eta \sim \mathcal{N}(0, \Gamma)$.
- Define $\Phi(u) = \frac{1}{2}\|\Gamma^{-\frac{1}{2}}(\mathcal{G}(u) - y)\|^2$.
- BAYES THEOREM gives

$$\frac{d\mu^y}{d\mu_0}(u) \propto \exp\big(-\Phi(u)\big).$$

- UNCERTAINTY QUANTIFICATION consists of evaluating $\mathbb{E}^{\mu^y}\psi(u)$ for certain $\psi : U \to S$.
- We consider $\psi(u) = p$ and $S = V$ or $\psi(u) = p \otimes p$ and $S = \mathcal{L}(V, V)$.

Well-Posed Inverse Problem

Cotter, Dashti, Robinson, Stuart 2009.

Theorem

For both priors:

• *the posterior* μ^y *is absolutely continuous with respect to the prior measure* μ_0 *with density proportional to* $\exp\big(-\Phi(u)\big)$*;*

• *posterior expectation of* ψ *is then*

$$\mathbb{E}^{\mu^y}\psi(u) = \frac{1}{Z}\mathbb{E}^{\mu_0}\exp\big(-\Phi(u)\big)\psi(u)$$

$$Z = \mathbb{E}^{\mu_0}\exp\big(-\Phi(u)\big);$$

• *there is* $C = C(r) > 0$ *such that, for all* y_1, y_2 *with* $\max\{\|y_1\|, \|y_2\|\} \le r$,

$$\|\mathbb{E}^{\mu^{y_1}}\psi(u) - \mathbb{E}^{\mu^{y_2}}\psi(u)\| \le C\Big(\mathbb{E}^{\mu}\|\psi(u)\|^2 + \mathbb{E}^{\nu}\|\psi(u)\|^2\Big)^{\frac{1}{2}}\|y_1 - y_2\|.$$

Outline

Karhunen-Loeve Approximation

- Let $D = [0,1]^d$ and $\varphi_{\mathbf{k}} = \exp(2\pi i \mathbf{k} \cdot x)$. Fourier basis.
- Let $\lambda_{\mathbf{k}} = \left(4\pi^2 |\mathbf{k}|^2\right)^{-s/2}$. decay of standard deviations.
- Let $P^N : U = \ell^2 \to \{u_{\mathbf{k}}, |\mathbf{k}| \le N\}$.
- Let $U^N = P^N U$ and $\sharp = |U^N| \asymp N^d$

Measure on $\mathbb{R}^\sharp$

- $C^N : U^N \to U^N$ with $C^N = \mathrm{diag}(\lambda_{\mathbf{k}})$ and $\mu_0^N = \mathcal{N}(0, C^N)$.
- Let $\mu^{y,N}$ denote the measure on U^N

$$\frac{d\mu^{y,N}}{d\mu_0^N} \propto \exp\Big(-\Phi(P^N u)\rangle\Big).$$

- In coordinates have Lebesgue density

$$\exp\Big(-\Phi(P^N u) - \frac{1}{2}\langle u, (C^N)^{-1} u\rangle\Big)$$

on $\mathbb{R}^\sharp$, amenable to MCMC.

- New MCMC: number of steps K independent of $\sharp(N)$.

Approximation Theorem

Dashti and Stuart 2010.

Theorem

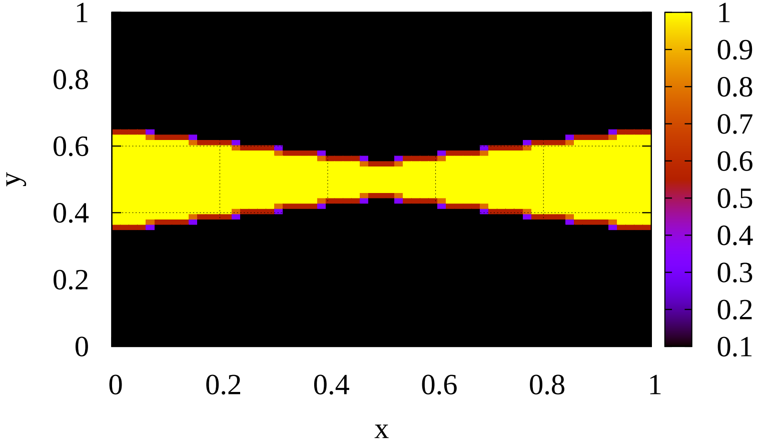
Let $\psi(u) = p$ and $S = V$ or $\psi(u) = p \otimes p$ and $S = \mathcal{L}(V, V)$. Assume that $s > \frac{d}{2}$ and let $t < s - \frac{d}{2}$. There is $C > 0$ such that,

$$\|\mathbb{E}^{\mu^y}\psi(u) - \mathbb{E}^{\mu^{y,N}}\psi(P^N u)\|_S \leq CN^{-t}.$$

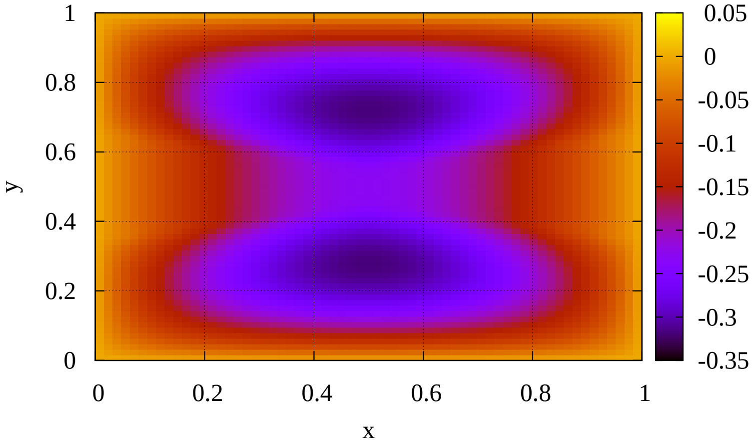
This error must be traded against MCMC error $M^{-\frac{1}{2}}$.

Outline

True Diffusion Coefficient



True Pressure



MCMC Sampling

MOVIE

(Stuart and White 2011, in prepartion)

Outline

Polynomial Chaos

- Legendre polynomials $\int_{-1}^{1}(L_k(t))^2\frac{dt}{2} = 1, \quad k = 0, 1, 2, \ldots.$
- $\mathcal{F} = \{\nu \in \mathbb{Z}^{\mathbb{N}} : |\nu|_1 < \infty\}.$
- Such multiindices have compact support.
- $L_\nu(z) = \prod_{j\in\mathbb{N}} L_{\nu_j}(z_j), \quad z \in \mathbb{C}^{\mathbb{N}}, \ \nu \in \mathcal{F}$.
- $\{L_\nu : \nu \in \mathcal{F}\}$ is orthonormal basis for $L^2\big(U, \mu_0(du)\big)$.
- If $\Theta \in L^2\big(U, \mu_0(du); S\big)$ then $\Theta(u) = \sum_{\nu\in\mathcal{F}} \theta_\nu L_\nu(u)$.

Approximating Posterior Expectations

- We consider $\psi(u) = p$ and $S = V$ or $\psi(u) = p \otimes p$ and $S = \mathcal{L}(V, V)$.
- Recall that posterior expectation of ψ requires calculation of two integrals in infinite dimensions:

$$\mathbb{E}^{\mu^y} \psi(u) = \frac{1}{Z} \mathbb{E}^{\mu_0} \exp\big(-\Phi(u)\big)\psi(u)$$
$$Z = \mathbb{E}^{\mu_0} \exp\big(-\Phi(u)\big);$$

- Both $\Theta_1(u) := \exp\big(-\Phi(u)\big)$ and $\Theta_2(u) := \exp\big(-\Phi(u)\big)\psi(u)$ are in $L^2\big(U, \mu_0(du)\big)$.
- Suggests approximating integrals by $\Theta^M(u) = \sum_{\nu \in \Lambda_M} \theta_\nu L_\nu(u)$ for some index set $\Lambda_M \subset \mathcal{F}$ of cardinality M.

Approximation Theorem

Let $\mathbb{E}^{\mu^{y,M}}\psi(u)$ denote the resulting approximation.

Schwab and Stuart 2011

Theorem

Let $\psi(u) = p$ and $S = V$ or $\psi(u) = p \otimes p$ and $S = \mathcal{L}(V, V)$. Assume that, for some $\sigma \in (0, 1]$, $\sum_{j=1}^{\infty} \lambda_j^{\sigma} < \infty$. There is an index set Λ_N of cardinality M such that,

$$\|\mathbb{E}^{\mu^y}\psi(u) - \mathbb{E}^{\mu^{y,M}}\psi(P^M u)\|_S \leq C M^{\frac{1}{2}-\frac{1}{\sigma}}.$$

- If $\sigma < 1$ then this rate beats $M^{-\frac{1}{2}}$ from MCMC.
- Challenge is to realize this approximation so that cost/per unit error trade-off beats MCMC.

Outline

- BAYESIAN approach to inverse problems allows for a natural approach to quantify uncertainty in the presence of data.
- KARHUNEN-LOEVE truncation leads to tractable MCMC methods with quantifiable error.
- POLYNOMIAL CHAOS representation of the posterior density affords the possibility of beating MCMC cost/unit error.

Outline

Two Main Papers

- " Uncertainty quantification and weak approximation of an elliptic inverse problem", M. Dashti and A.M. Stuart, submitted 2010. `http://arxiv.org/abs/1102.0143`
- "Sparse deterministic approximation of Bayesian inverse problems", Ch. Schwab and A.M. Stuart, submitted 2011. `http://arxiv.org/abs/1103.4522`

Other Related Papers

All papers can be found at:

http://www.warwick.ac.uk/~masdr/

16c "Inverse Problems: A Bayesian Perspective", A.M. Stuart, Acta Numerica **19**(2010).

80 "Bayesian inverse problems for functions and applications to fluid mechanics", S.L. Cotter, M. Dashti, J.C. Robinson, A.M. Stuart, Inverse Problems, **25**(2009), 115008.

81 "Approximation of Bayesian inverse problems for PDEs", S.L. Cotter, M. Dashti and A.M. Stuart, SIAM J. Num. Anal, **48**(2010), 322–345.

68 "A Bayesian approach to data assimilation", A. Apte, M. Hairer. A.M. Stuart, J. Voss, PhysicaD, **230**(2007), 50–64.