Research Interests of Participants at the

BIRS Meeting on Cluster Algebras

September 2011

The following participants have submitted a description of their research interests:

Frédéric Chapoton
Laurent Demonet
Philippe Di Francesco
Michael Gekhtman
David Hernandez
Rinat Kedem
Allen Knutson
Daniel Labardini-Fragoso
Philipp Lampe
Bernard Leclerc
Greg Muller
Gregg Musiker
Kentaro Nagao
Hiraku Nakajima
Tomoki Nakanishi
Yann Palu
Pierre-Guy Plamondon
Fan Qin
Ralf Schiffler
Hugh Thomas
Gordana Todorov
Pavel Tumarkin
Lauren Williams
Milen Yakimov
Jie Zhang
Bin Zhu
Sebastian Zwicknagl

INTERESTS

FRÉDÉRIC CHAPOTON

Points over finite fields and cohomology

Cluster algebras of geometric type define algebraic varieties together with a morphism to an affine space or a complex torus. I am interested in counting points over finite fields in these algebraic varieties, as a first step towards computing their cohomology rings. One is naturally led to consider the fibers of this morphism and in particular the generic fiber. I have done some work on finite type already, obtaining nice and simple formulas, but much remains to be done.
See http://fr.arxiv.org/abs/0912.2342

Operad structure on type A cluster algebras

The collection of all moduli spaces of stable genus 0 curves with marked points $\overline{M}_{0,n}$ has the natural structure of an operad, which comes from the possibility of gluing marked points together.

There is a close relationship between these moduli spaces and Grassmannians of planes, hence cluster algebras of finite type A. It seems possible to define some kind of operad structure on the collection of cluster algebras. This may lead either to interesting operads or to technical tools useful for cluster algebras.

Categories in which 2-clusters are objects

I have been interested in the study of the category of modules over the incidence algebra of the Tamari posets. Its derived category is worth studying, having many different descriptions and seeming to be a fractionally Calabi-Yau category. It may also have a geometric description related to some isolated hypersurface singularities. I have recently described the spectrum of the Coxeter transformation for this category, using operads (see http://fr.arxiv.org/abs/1103.3755). This is also related to the recently introduced Tamari posets of higher slope.

Another interesting point is that one can define objects in this category associated with 2-clusters. When looking at all 2-clusters that have a non-zero morphism from a fixed 2-cluster, one can see many polytopes appear, including all associahedra.

Current interests

Laurent Demonet

1 About semicanonical basis

Let G be a connected and simply connected simple Lie group, $\mathfrak{g}$ its Lie algebra and $\mathfrak{n}$ a maximal nilpotent subalgebra of $\mathfrak{g}$. Lusztig introduced the semicanonical basis of the enveloping algebra $U(\mathfrak{n})$ in type A, D and E. This basis is indexed by the irreducible components of the varieties of representations of the corresponding preprojective algebra. This construction leads to the categorifications of Geiß, Leclerc and Schröer of the cluster structure on the coordinate rings of unipotent subgroups of G. So, the cluster monomials correspond to the rigid representations of the preprojective algebra. In my thesis, I described a method which allows to see, for the non simply-laced types (B, C, F, G), the cluster monomials as coming from Γ-stable rigid irreducible components of the varieties of representations of a preprojective algebra Λ endowed with an action of a finite group Γ. So, one morally obtains a part of the semicanonical basis in this framework.

Problem 1. *Construct a semicanonical basis of $U(\mathfrak{n})$ in non simply-laced types.*

The original method for simply-laced case works indeed also for Kac-Moody case. I think now that it is difficult in such a full generality. So, I tried during last times to understand it at least for specific cases (B_n for example). Thus, the problem could be attacked by explicit computations on crystals. For example, it is known that, in Dynkin case, there is an easy to define bijection between isoclasses of representations of a quiver and the irreducible components of the varieties of representations of the corresponding preprojective algebra. Thus, this leeds to the structure of a crystal on the set of isoclasses of representations of the quiver. If we are able to compute explicitely this structure (it is far from being obvious), then, we can expect to get some much more explicit and computational description (as representations of a Dynkin quiver are easy to compute with). Thus, it could be a starting point for trying to give an analogue in non simply-laced case.

2 More general categorifications of skew-symmetrizable cluster algebras

I tried recently to adapt the work of Derksen, Weyman and Zelevinsky to cluster algebras whose exchange matrices are not skew-symmetric. Up to now, I got only a (quite small) subclass of these cluster algebras (containing for example acyclic ones), which seems to be reachable also by purely combinatorial methods. The natural idea which consists in taking a quiver with potential and an action of a group seems to be too restrictive. In other words, there should exist a generalization of such a situation.

To be more explicit, the problem in this case comes from the fact, if one takes the natural (or easy) candidate, is that the space of potentials becomes reducible (for Zariski topology). Thus, the natural argument for the existence of a non-degenerate potential vanishes (intersection of countably many open conditions). So, the problem could be to extend the notion of potentials in such a way that the space of potential becomes irreducible.

In the same spirit, we could probably construct in such a case generalized cluster categories if we had a good notion of potential (indeed, we can already, but, we often do not have generic potential, and therefore, these categories do not categorify the corresponding cluster algebras).

Philippe DI FRANCESCO, current research

My interest in cluster algebras arose from a physics point of view. In collaboration with R. Kedem, we found explicit solutions to the so-called Q- and T-systems that arise in various stages of the study of integrable quantum spin chains. These always take the form of systems of recursion relations with a discrete time variable. One of my main motivations was to understand combinatorially or by means of models of statistical physics the positive Laurent phenomenon conjecture of [Fomin,Zelevinsky].

We developed several approaches, using various statistical mechanical models to describe how general solutions are related to admissible sets of initial data (discrete Cauchy data). The models are essentially models of weighted paths on a target graph with both weights and graphs being determined by the initial data, or alternatively models of paths on networks (also referred to as "frieze" solutions sometimes). Both Q- and T-systems are particular instances of the so-called discrete Hirota equation, which plays a central role in discrete integrable systems [Krichever,Lipan,Wiegmann,Zabrodin]. Our method has consisted in constructing explicitly the (time-independent) conserved quantities of the systems, and to use them to compute generating functions of some infinite sets of cluster variables, obtained by iterated mutations, and put them in (finite) continued fraction forms. In particular, for Q- and T-systems, we were able to rephrase mutations of the graphs and weights of the path models as local rearrangements of the corresponding continued fractions, that are manifestly positive, thus proving the positivity conjecture for those cases. We also developed analogous models to solve the so-called quantum Q- and T-systems, defined by means of the quantum cluster algebra of [Berenstein,Zelevinsky], for indeterminates with quantum commutation relations.

After the recent work of [Kontsevich,Soibelman] on Donaldson-Thomas invariants, I became interested into non-commutative versions of discrete integrable systems, that is systems with a discrete time variable for non-commuting indeterminate, and with a number of (time-independent) algebraic conserved quantities. This led me to naturally study non-commutative versions of the Q-system, and to provide a proof of a conjecture by Kontsevich on the positive Laurent phenomenon in this case. The idea is that path models are already non-commutative objects, as walkers travel along edges of the target graph in a certain order. It turns out that a mild adaptation of our definitions is sufficient to solve the so-called A_1 Non-commutative Q-system of Kontsevich completely. This yields not only the non-commutative structure of this particular system, but also a nice simple definition of a non-commutative rank two cluster algebra, for which a more general positivity proof was derived by [Lee] in the skew-symmetric case.

My present interest is to try to push these ideas in higher rank cases. The structure of our general solution to Q-systems say for A type quantum spin chains goes over to the non-commutative setting via the use of continued fractions with non-commutative coefficients, corresponding to models of paths with non-commutative step weights, all of which are tapeable to non-commutative Cauchy initial data for the systems. So far we found that the discrete Hirota equation should be replaced with a non-commutative system involving discrete quasi-determinants/minors (in the sense of [Gelfand,Retakh]). This equation reduces to the classical Q- and T-systems for commuting variables, and quite interestingly to the quantum Q- and T-systems for indeterminate with quantum commutation relations.

I would also like to unify the solutions to mutation-finite cluster algebras and more specifically cluster algebras form surfaces [Musiker,Schiffler,Williams] in terms of path models, as a natural pathway to non-commutative generalizations.

Research Interests

M. Gekhtman

Our ongoing collaboration with M. Shapiro and A. Vainshtein is focused on applications of Poisson geometry to cluster algebras. The main goal is to undertake a systematic study of multiple cluster structures in coordinate rings of a certain varieties of importance in algebraic geometry, representation theory and mathematical physics and study an interaction between corresponding cluster algebras. Important examples include simple Lie groups, homogeneous spaces, configuration spaces of points, and are related to discrete and continuous integrable systems. On last subject, we also collaborate with S. Tabachnikov. Among the problems we are currently working on are

- cluster structures on simple Lie groups compatible with Poisson-Lie structures associated with the Belavin-Drinfeld classification;
- inverse problems for directed nets on surfaces of higher genus;
- generalizations of the pentagram map.

CURRENT RESEARCH INTERESTS CONNECTED TO CLUSTER ALGEBRAS THEORY

DAVID HERNANDEZ

I am currently interested in the relations between cluster algebras and the representation theory of quantum affine algebras.

In the context of monoidal categorification of cluster algebras [HL], it is useful to establish the binarity property for tensor products of irreducible objects in certain monoidal categories : a tensor product of irreducible objects is irreducible is and only if, two by two, the tensor products are irreducible. By extending a result with Bernard Leclerc [HL], I proved this property [H] for the (full) tensor category of finite dimensional representations of a quantum loop algebras.

Recently, in a joint work in progress with Bernard Leclerc, we have shown that the t-deformed Grothendieck rings of certain tensor categories of representations of quantum loop algebras of type ADE, have a quantum cluster algebra structure (see the abstract of Bernard Leclerc).

References

[H] **D. Hernandez**, *Simple tensor products*, Invent. Math. **181** (2010), no. 3, 649–675

[HL] **D. Hernandez and B. Leclerc**, *Cluster algebras and quantum affine algebras*, Duke Math. J. **154** (2010), 265–341

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CURRENT RESEARCH

RINAT KEDEM

My recent research is in the following subjects:

(1) Cluster algebras coming from representation theory of affine algebras and quantum affine algebras: Q-systems, T-systems and their quantization. These are quite universal structures, and appear in various other contexts, although their origin is as a relation between characters of special representations of quantum affine algebras introduced in the context of the generalized Heisenberg spin chain. For example, the Q-system describes a certain (double Coxeter) subset of the double Bruhat cells of GL_n or, alternatively, the totally positive factorization. It also describes certain elements in the dual canonical basis (shown by the recent work by Nakajima).

(2) Application of results from cluster algebras to solve certain conjectures about fermionic formulas in representation theory, originating in integrable statistical mechanical models on the lattice and conformal field theory. I'm interested in the quantized version of these identities and its relation to the quantum cluster algebras related to the Q-systems.

(3) Discrete and quantum integrability manifested by such cluster algebras: Integrability survives quantization, and manifests itself in different aspects of the cluster algebra evolutions. For example, in the existence of conserved quantities and linear recursion relations which allow a solution of the system. The Toda flows of Gekhtman, Shapiro and Vainshtein are compatible with certain cluster algebra mutations which appear from the Q-system equations mentioned above. These are integrable in the Poisson sense, and Q-system mutations are coordinate transformations in the double Bruhat cells where the Toda flows take place. The quantized T-system is related to the quantum discrete Liouville equations of Faddeev and Volkov.

(4) Positivity proofs using techniques of statistical mechanics: The use of weighted statistical models gives explicit expressions for the cluster variables. Having a statistical model therefore gives more than positivity. Moreover, it may carry over to the non-commutative version with some work, as we have shown in the special case of Q-system. In cases where quasi-statistical models can be used, one may hope to lift the situation to the non-commutative setting. Our recent work on the non-commutative rank 2 models suggested by Kontsevich was recently generalized using such ideas to the symmetric non-integrable setting by Kyungyong Lee.

Most of this research is in collaboration with Philippe Di Francesco.

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Recent Interests Allen Knutson

Consider Poisson varieties carrying actions of a torus T, such that there are finitely many T-orbits of symplectic leaves, or "T·leaves". By Zwicknagl's work, these include cluster varieties. There are many other interesting examples coming from algebraic groups, not all of which seem to have cluster structures, and I would like to understand better when to hope for a cluster structure.

RESEARCH RELATED TO CLUSTER ALGEBRAS

DANIEL LABARDINI-FRAGOSO

My main research interest in relation with cluster algebras lies in the connection of these objects with the representation theory of algebras. More specifically, I am very interested in Derksen-Weyman-Zelevinsky's approach to cluster algebras using quivers with potentials (QPs).

My recent research has to do with finding potentials for the tagged triangulations of surfaces with non-empty boundary (by results of Fomin-Shapiro-Thurston, such tagged triangulations *are* clusters in the corresponding cluster algebras). In recent joint work, G. Cerulli Irelli and I have defined, for each tagged triangulation τ of a surface with marked points and non-empty boundary, a Jacobi-finite non-degenerate potential $S(\tau)$ on the signed-adjacency quiver $Q(\tau)$. We have shown that flips of tagged triangulations are compatible with QP-mutations, at least at the level of Jacobian algebras, and that every two tagged triangulations are related by a sequence of flips along which we have compatibility with QP-mutations (up to right-equivalence, not only at the level of Jacobian algebras). Furthermore, we have proved that the inclusion of the path algebra $R\langle Q(\tau)\rangle$ into the complete path algebra $R\langle\langle Q(\tau)\rangle\rangle$ induces an isomorphism between $R\langle Q(\tau)\rangle/J_0(S(\tau))$ and the Jacobian algebra $\mathcal{P}(Q(\tau), S(\tau))$, where $J_0(S(\tau))$ is the two-sided ideal of $R\langle Q(\tau)\rangle$ generated by the cyclic derivatives of $S(\tau)$. This has allowed us to apply Derksen-Weyman-Zelevinsky's homological interpretation of the *E-invariant* to obtain information about the cluster monomials of the cluster algebra associated to the surface.

Below you can find a list of papers directly related to my research.

References

[1] G. Cerulli Irelli and D. Labardini-Fragoso. *Quivers with potentials associated to triangulated surfaces, Part III: Tagged triangulations and cluster monomials.* arXiv:1108.1774

[2] H. Derksen, J. Weyman, A. Zelevinsky. *Quivers with potentials and their representations I: Mutations.* Selecta Mathematica, New Series. 14 (2008), 59-119. `http://arxiv.org/abs/0704.0649`

[3] H. Derksen, J. Weyman, A. Zelevinsky. *Quivers with potentials and their representations II: Applications to cluster algebras.* Journal of the American Mathematical Society 23 (2010), No. 3, 749-790. `http://arxiv.org/abs/0904.0676`

[4] S. Fomin, M. Shapiro, D. Thurston. *Cluster algebras and triangulated surfaces, part I: Cluster complexes.* Acta Mathematica 201 (2008), 83-146. `http://arxiv.org/abs/math/0608367`

[5] S. Fomin, A. Zelevinsky. *Cluster algebras I: Foundations.* Journal of the American Mathematical Society. **15** (2002), 497-529. `http://arxiv.org/abs/math/0104151`

[6] S. Fomin, A. Zelevinsky. *Cluster algebras II: Finite type classification.* Inventiones Mathematicae. **154** (2003), 63-121. `http://arxiv.org/abs/math/0208229`

[7] S. Fomin and A. Zelevinsky. *Cluster algebras IV: Coefficients.* Compositio Mathematica 143 (2007), 112-164. arXiv:math/0602259

[8] D. Labardini Fragoso. *Quivers with potentials associated to triangulated surfaces.* Proc. London Math. Soc. Volume 98 Part 3 (May 2009), 797-839. `http://arxiv.org/abs/0803.1328`

[9] D. Labardini Fragoso. *Quivers with potentials associated to triangulated surfaces, Part II: Arc representations.* `http://arxiv.org/abs/0909.4100`

[10] D. Labardini Fragoso. *Quivers with potentials associated with triangulations of Riemann surfaces.* Ph.D. Thesis. Northeastern University. 2010.

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RESEARCH DESCRIPTION

PHILIPP LAMPE
UNIVERSITY OF BIELEFELD

My focus is on the connection between *cluster algebras* and Lusztig's *canonical basis.*

Given an acyclic quiver Q, one associates two algebras: the path algebra $\mathbb{C}Q$ and the preprojective algebra Λ of Q. Representation theory is concerned with the study of the module categories $\mathrm{mod}(\mathbb{C}Q)$ and $\mathrm{mod}(\Lambda)$. Out of these one can construct two triangulated categories which categorify cluster algebras: the cluster category $\mathcal{C}_Q$ which is an orbit category of the bounded derived category of $\mathrm{mod}(\mathbb{C}Q)$, see [1], and the stable module category $\underline{\mathrm{mod}}(\Lambda)$ of the selfinjective algebra Λ. Both $\mathcal{C}_Q$ and $\underline{\mathrm{mod}}(\Lambda)$ are Calabi-Yau of dimension two. The cluster category $\mathcal{C}_Q$ is triangle equivalent to the stable category $\underline{\mathcal{C}}_w$ of a suitable subcategory $\mathcal{C}_w$ of $\mathrm{mod}(\mathbb{C}Q)$ which is parametrized by a particular Weyl group element of length $2n$, where $n = |Q_0|$, see [2].

We are interested in the associated cluster algebra $\mathcal{A}(\mathcal{C}_w)$. Let $\mathfrak{g} = \mathfrak{n} \oplus \mathfrak{h} \oplus \mathfrak{n}_-$ be the corresponding Kac-Moody Lie algebra. The quantization $\mathcal{A}(\mathcal{C}_w)_q$ is then a quantum cluster algebra; it can be realized as an integral form of a subalgebra $U_q(w) \subset U_q(\mathfrak{n})$ of the quantized universal enveloping algebra of $\mathfrak{n}$. The algebra $U_q(w)$ was introduced by Lusztig [6]. It is constructed via braid automorphisms and it admits several bases. Among these are the various (non-canonical) Poincaré-Birkhoff-Witt bases, and moreover, there is a geometrically constructed canonical basis.

Instead of Lusztig's geometric approach, Leclerc [5] uses the quantum shuffles introduced by Rosso [7] to obtain a combinatorial description of Lusztig's canonical basis.

My own work in this circle of ideas is to investigate whether quantum cluster variables belong to the dual of Lusztig's canonical basis (with respect to Kashiwara's bilinear form) which we confirm [3, 4] in the case when Q is either the Kronecker quiver or an alternating quiver of type A_n. In both cases we give explicit recursions for the quantum cluster variables. Our proof features Leclerc's combinatorial description of the dual canonical basis and the explicit form of the straightening relations among the generators of $U_q(w)$. For $Q = A_n$ we also explore the combinatorics of the recursions in terms of quantum shuffles and alternating permutations.

REFERENCES

[1] A.B. Buan, R. Marsh, M. Reineke, I. Reiten, G. Todorov, *Tilting theory and cluster combinatorics*, Adv. Math. **204** (2006), no. 2, 572–618.
[2] C. Geiß, B. Leclerc, J. Schröer, *Cluster algebra structures and semicanonical bases for unipotent groups*, Preprint arXiv:math/0703039 (2007).
[3] P. Lampe, *A Quantum Cluster Algebra of Kronecker Type and the Dual Canonical Basis*, Int. Math. Res. Notices **2011** (2011), no. 13, 2970–3005.
[4] P. Lampe, *Quantum cluster algebras of type A and the dual canonical basis*, Preprint arXiv:1101.0580 (2011).
[5] B. Leclerc, *Dual canonical bases, quantum shuffles and q-characters*, Math. Z. **246** (2004), 691–732.
[6] G. Lusztig *Introduction to Quantum Groups*, Progr. Math. **110**, Birkhäuser, Boston (1993).
[7] M. Rosso, *Quantum groups and quantum shuffles*, Invent. Math. **133** (1998), 399–416.

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Description of current research interests connected to cluster algebras

B. Leclerc

21 August 2011

I am currently interested in quantum cluster algebras and their connections with quantum groups.

In a joint work with Christof Geiss and Jan Schröer [**GLS**], we have shown that many quantum coordinate rings occuring in Lie theory have a quantum cluster algebra structure.

Recently, in a joint work in progress with David Hernandez, we have shown that the t-deformed Grothendieck rings of certain tensor categories of representations of quantum loop algebras of type A, D, E, have a quantum cluster algebra structure.

References

[**GLS**] C. Geiss, B. Leclerc, J. Schröer, *Cluster structures on quantum coordinate rings*, arXiv:1104.0531.

Research interests - Greg Muller

The geometry of cluster algebras. For a (complex) cluster algebra $\mathcal{A}$, the affine scheme $X := Spec(\mathcal{A})$ has many interesting properties. Each cluster determines an embedding of an algebraic torus, and the union of these tori determines an open subscheme X' (called the *cluster manifold* in [GSV03]) whose ring of global functions is the upper cluster algebra $\mathcal{U}$. I am interested in the complement $X_d := X - X'$, about which very little seems to be known in general. For fixed $\mathcal{A}$, several basic questions are already non-trivial, even when $\mathcal{A}$ is finite-type:

- Is X_d empty?
- Is X_d smooth in X?
- What is the codimension of X_d in X?
- Does the Poisson structure on X' (as defined in [GSV03]) extend to X_d? For acyclic $\mathcal{A}$, this was shown to be true in [Mul].

Cluster algebras of surfaces and skein algebras. Given an oriented surface Σ (possibly with boundary $\partial\Sigma$) with a finite collection of marked points M, there are two related algebras associated to Σ.[1] The first is the cluster algebra $\mathcal{A}(\Sigma)$ of Σ, as defined in [GSV05] and [FST08] (and also the upper cluster algebra $\mathcal{U}(\Sigma)$). The second is the *Kauffman skein algebra* $\mathsf{Sk}(\Sigma)$, an algebra of formal products of arcs and loops together with a local relation (the definition involves a parameter q).[2] When $q = 1$, there are natural inclusions

$$\mathcal{A}(\Sigma) \subseteq \mathsf{Sk}^o(\Sigma) \subseteq \mathcal{U}(\Sigma)$$

where $\mathsf{Sk}^o(\Sigma)$ is a certain localization of $\mathsf{Sk}(\Sigma)$. When $M \subset \partial\Sigma$, I can show that $\mathsf{Sk}^o(\Sigma) = \mathcal{U}(\Sigma)$, and there is evidence that this is true for general M.

Cluster algebras of surfaces and character algebras. Following [FG06], one may define *decorated local systems* on Σ, by taking an $SL_2(\mathbb{C})$ local system on Σ and adding extra data at the M. To this moduli problem, one can associate an affine *character scheme* $Char(\Sigma)$ (which is the schemification of the moduli stack). For $M \subset \partial\Sigma$, it is possible to (non-canonically)[3] identify the ring of functions $\mathcal{O}Char(\Sigma)$ with the skein algebra $\mathsf{Sk}(\Sigma)$; and so a localization of $\mathcal{O}Char(\Sigma)$ may be identified with $\mathcal{U}(\Sigma)$. This provides a regular version of a birational result obtained in [FG06]. This work is joint with Peter Samuelson.

References

[FG06] Vladimir Fock and Alexander Goncharov, *Moduli spaces of local systems and higher Teichmüller theory*, Publ. Math. Inst. Hautes Études Sci. (2006), no. 103, 1–211. MR 2233852 (2009k:32011)

[FST08] Sergey Fomin, Michael Shapiro, and Dylan Thurston, *Cluster algebras and triangulated surfaces. I. Cluster complexes*, Acta Math. **201** (2008), no. 1, 83–146. MR 2448067 (2010b:57032)

[GSV03] Michael Gekhtman, Michael Shapiro, and Alek Vainshtein, *Cluster algebras and Poisson geometry*, Mosc. Math. J. **3** (2003), no. 3, 899–934, 1199, {Dedicated to Vladimir Igorevich Arnold on the occasion of his 65th birthday}. MR 2078567 (2005i:53104)

[GSV05] ———, *Cluster algebras and Weil-Petersson forms*, Duke Math. J. **127** (2005), no. 2, 291–311. MR 2130414 (2006d:53103)

[Mul] Greg Muller, *The Weil Petersson form on an acyclic cluster variety.*, preprint, arxiv: 1103.2341.

[1] We assume there are 'enough' points in M.

[2] When there are marked points not on the boundary, it is also necessary to add tagged arcs, as in [FST08].

[3] This identification may be made canonical by replacing decorated local systems with *twisted decorated local systems.*

Research Statement on Cluster Algebras
Gregg Musiker

My current research interests in cluster algebras focus on cluster algebras from surfaces and connections to Teichmüller theory. I have also recently worked with REU students exploring cluster algebras related to integrable systems, such as those corresponding to Gale-Robinson sequences $x_n x_{n-k} = x_{n-r}x_{n-k+r} + x_{n-s}x_{n-k+s}$, and worked with Christian Stump developing software in SAGE for working with cluster algebras and quivers.

Regarding my research on cluster algebras from surfaces, I have been working with Ralf Schiffler and Lauren Williams to extend our previous work, "Positivity for cluster algebras from surfaces" [MSW11], which provided combinatorial formulas for cluster variables in terms of perfect matchings of graphs. The long term goal of this work is to provide positive canonical, also known as atomic bases (for example in the recent works of Cerulli and Dupont-Thomas) for such cluster algebras. Towards this end, Williams and I posted a recent paper to the arxiv, 1108.3382, connecting our graph theoretic formula to matrix product formulas in $PSL_2(\mathbb{R})$ related to previous work of Fock-Goncharov, and others. We work in the generality of principal coefficients, and write down explicit skein relations that correspond to multiplying together cluster variables in this context.

With Schiffler and Williams, we are now wrapping up a paper exhibiting a vector space basis for cluster algebras associated to unpunctured surfaces with principal coefficients, based on earlier work of Fock-Goncharov and Fomin-Shapiro-Thurston. In particular, we exhibit a bijection between $\mathbb{Z}^n$ and g-vectors associated to the basis elements to illustrate linear independence.

Work in this area has also led me to further study the associated geometric background. This past year, I met regularly with geometers and topologists in Minnesota, Ren Guo and Helen Wong, to compare constructions of quantum Teichmüller space to constructions of quantum cluster algebras.

I also recently gave a five lecture course on cluster algebras and Teichmüller theory at MSRI, `http://www.msri.org/web/msri/scientific/workshops/show/-/event/Wm550`. Stump and I have been writing software for SAGE, currently available through the SAGE Combinat queue at `http://combinat.sagemath.org/patches/`, for computations related to cluster algebras. Some of the code's features thus far include cluster mutations, recognizing the Dykin or Extended Dynkin type of a cluster algebra from its matrix or quiver, and computing a list of the exchange matrices or cluster variables mutation-equivalent to a given seed. A compendium describing these features is available on the arxiv, 1102.4844.

Research Interests of Kentaro Nagao

Recently, I'm interested in the product of quantum dilogarithms associated to a mapping class.

Let (Q, W) be a QP and $\mathbf{k} = (k_1, \ldots, k_l)$ be a sequence of vertices. The QP $\mu_{\mathbf{k}}(Q, W)$ obtained by the successive mutations is derived equivalent to the original one. The equivalence is given by a torsion pair $(\mathcal{T}_{\mathbf{k}}, \mathcal{F}_{\mathbf{k}})$ of the module category of the Jacobi algebra of (Q, W). The generating series of the motivic DT type invariants of objects in $\mathcal{T}_{\mathbf{k}}$ is described as a product of quantum dilogarithms.

Assume that $\mu_{\mathbf{k}}(Q, W) = (Q, W)$ and the derived equivalence is identity functor. Then we have $\mathcal{T}_{\mathbf{k}} = 0$ and so the product of the quantum dilogarithm is 1. This is the quantum dilogarithm identity proved by Keller.

On the other hand, assume that $\mu_{\mathbf{k}}(Q, W) = (Q, W)$ but the derived equivalence is not identity functor. Then the product of the quantum dilogarithm is not 1 and difficult to compute explicitly, but commutes with the generating series of DT invatriants.

A typical example is obtained from a triangulation of a surface. For a triangulation, a quiver with a potential is associated and the mapping class group acts on the derived category. Then, for any mappling class the assciated product of dilogarithms commutes with the generating series of motivic DT invariants. In other word, the motivic DT series satisfies the constraints indueced from the mapping class group symmetry.

From this viewpoint, it will be impertant to study the product of quantum dilogarithms associated to a mapping class.

Given a mapping class group, we take the mapping torus to get a 3-manifold. It is natural to expect that there is a "TQFT-like" construction of an invariant of such a 3-manifold.

Recently, Tarashima and Yamazaki propose to study the trace of a mapping class in quantum Teichmuller theory. They checked that the asymptotic behaviors of the trace of a pseudo-Anosov mapping classes coincides with the volume of the mapping torus in some examples.

Kashaev's volume conjecture claims that the asymptotic behaviors of Jones polynomial of a knot coincides with the volume of the compliment of the knot. Tarashima-Yamazaki's proposal implies the product of quantum dilogarithms would play an important role in the volume conjecture.

Let $U_q^-(w)$ be the quantum unipotent subgroup associated with a Weyl group element w of a symmetric Kac-Moody Lie algebra $\mathfrak{g}$. Kimura showed that it is compatible with Kashiwara-Lusztig's dual canonical base. By the work of Geiss-Leclerc-Schröer, it has a structure of a quantum cluster algebra. It is conjectured that the dual canonical base contains quantum cluster monomials.

Recall that Geiss-Leclerc-Schröer showed that Lusztig's dual semicanonical base contains cluster monomials. And a dual semicanonical base element corresponds to an irreducible component of Lusztig's lagrangian subvariety. Let Λ_b be an irreducible component, which corresponds to a cluster monomial m. By Kashiwara-Saito, Λ_b corresponds to a (dual) canonical base element b. I conjecture a following statement: Let b' be another dual canonical base element in $U_q^-(w)$ and $P_{b'}$ be the corresponding perverse sheaf. If its singular support $SS(P_{b'})$ contains Λ_b, then $b' = b$.

Assuming this conjecture, I can prove that b, specialized at $q = 1$, is the given cluster monomial m, i.e., the dual semicanonical base element is the specialization of the canonical base element in this case. Probably with a little more effort, I can also prove that b is a quantum cluster monomial.

Research Interests
Tomoki Nakanishi
August, 2011

Since 2008, I have been working on the project of studying the interrelation between *cluster algebras* and the *T- and Y-systems*. The latter arose from the integrable systems based on the conformal field theory (CFT) and the Yang-Baxter equation (or quantum groups) in the 90's. I have done this in collaboration with specialists in various fields, Rei Inoue, Osamu Iyama, Rinat Kashaev, Bernhard Keller, Atsuo Kuniba, Roberto Tateo, Suzuki, and Andrei Zelevinsky. We clarified the integrity of T- and Y-systems, and dilogarithm as well, with cluster algebras; and, as a fruitful outcome we obtained the following results:

— proof of the conjecture on the periodicity of Y-systems in full generality

— proof of the conjecture of the dilogarithm identities in CFT in full generality

— generalization of dilogarithm identities by cluster algebras and their quantum counterpart

— generalization of T- and Y-systems by cluster algebras and their connections to some integrable differential equations

— some basic properties of cluster algebras (duality of C- and G-matrices, extension theorem of periods, etc.)

I continue to study a further development of the basic theory of cluster algebras and its application to various fields, especially, to integrable differential equations.

References

[1] R. Inoue, O. Iyama, A. Kuniba, T. Nakanishi, and J. Suzuki. Periodicities of T and Y-systems. *Nagoya Math. J.*, 197:59–174; arXiv:0812.0667 [math.QA], 2010.

[2] A. Kuniba, T. Nakanishi, and J. Suzuki. T-systems and y-systems in integrable systems. *J. Phys. A: Math. Theor.*, 44:103001 (146pp); arXiv:1010.1344 [hep–th], 2010.

[3] T. Nakanishi. Dilogarithm identities for conformal field theories and cluster algebras: Simply laced case. *Nagoya Math. J.*, 202:23–43; arXiv:math.0909.5480 [math.QA], 2011.

[4] R. Inoue, O. Iyama, B. Keller, A. Kuniba, and T. Nakanishi. Periodicities of T and Y-systems, dilogarithm identities, and cluster algebras I: Type B_r, 2010. arXiv:1001.1880 [math.QA], to appear in Publ. Res. Inst. Math. Sci.

[5] R. Inoue, O. Iyama, B. Keller, A. Kuniba, and T. Nakanishi. Periodicities of T and Y-systems, dilogarithm identities, and cluster algebras II: Types C_r, F_4, and G_2, 2010. arXiv:1001.1881 [math.QA], to appear in Publ. Res. Inst. Math. Sci.

[6] T. Nakanishi. T-systems and Y-systems, and cluster algebras: Tamely laced case. In et al. Feigin, B. L., editor, *Proceedings of the Infinite Analysis 09, New Trends in Quantum Integrable Systems*, pages 325–355; arXiv:1003.1180 [math.QA], Singapore, 2010. World Scientific.

[7] T. Nakanishi and R. Tateo. Dilogarithm identities for sine-Gordon and reduced sine-Gordon Y-systems. *SIGMA*, 6:085, 34 pages; arXiv:1005.4199 [math.QA], 2010.

[8] T. Nakanishi. Periodicities in cluster algebras and dilogarithm identities, 2010. arXiv:1006.0632 [math.QA], to appear in "Representation of Algebras and Related Topics", Proceedings of ICRA XIV, Tokyo, 2010.

[9] R Inoue and T. Nakanishi. Difference equations and lcuster algebras i: Poisson bracket for integrable difference equations, 2010. arXiv:1012.5574 [math.QA], to appear in RIMS Kokyuroku Bessatsu.

[10] T. Nakanishi and A. Zelevinsky. On tropical dualities in cluster algebras, 2011. arXiv:1101.3736 [math.RA].

[11] R. M. Kashaev and T. Nakanishi. Classical and quantum dilogarithm identities, 2011. arXiv:1104.4630 [math.QA].

Yann Palu - **current interests**

Cluster algebras can be categorified by means of certain triangulated categories. In that context, clusters correspond to some specific objects: the maximal rigid objects. Reflecting the mutation of clusters, a mutation theory for maximal rigid objects has been studied extensively. I am currently interested, in collaboration with Robert Marsh, in the mutation theory of (not necessarily maximal) rigid objects, as it resembles the mutation theory of maximal rigid objects in d-Calabi–Yau categories. Inspired by [BT09] we studied, in [MP], the combinatorics of these mutations in terms of coloured quivers. This included, in particular, the case of the Amiot cluster categories associated with Riemann surfaces. In that setup, it was proved in [BZ10] that rigid objects correspond to partial triangulations. We proved that Iyama–Yoshino reduction at the categorical level corresponds to cutting along arcs in the associated Riemann surface.

We are now tackling the following problem:

As proved in [BMR07], mutations of maximal rigid objects in cluster categories induce nearly-Morita equivalences (in the sense of Ringel) between their endomorphism algebras. Does this phenomenon hold in the non-maximal case? Even though it is easily seen to fail in most cases, some weaker kinds of Morita equivalence do hold.

References

[BMR07] Aslak Bakke Buan, Robert J. Marsh, and Idun Reiten. Cluster-tilted algebras. *Trans. Amer. Math. Soc.*, 359(1):323–332, 2007.

[BT09] Aslak Bakke Buan and Hugh Thomas. Coloured quiver mutation for higher cluster categories. *Adv. Math.*, 222(3):971–995, 2009.

[BZ10] Thomas Brüstle and Jie Zhang. On the cluster category of a marked surface. *preprint arXiv:1005.2422v2 [math.RT]*, 2010.

[MP] Robert Marsh and Yann Palu. Coloured quivers for rigid objects and partial triangulations: The unpunctured case. *Preprint arXiv:1012.5790v2 [math.RT].*

RESEARCH INTERESTS CONNECTED WITH CLUSTER ALGEBRAS – PIERRE-GUY PLAMONDON

Cluster algebras. Cluster algebras, introduced by S. Fomin and A. Zelevinsky [3], are commutative algebras with a distinguished set of generators called *cluster variables*. These generators are constructed using an iterated process called *mutation*. The combinatorial information needed to initiate this process is encoded in a finite quiver.

Additive categorification of cluster algebras. The combinatorial phenomena appearing in the theory of cluster algebras are known to be analogous to others appearing inside some triangulated categories. Namely, given a finite quiver without loops or 2-cycles and a non-degenerate potential on that quiver, C. Amiot [1] defined the cluster category associated to that quiver with potential. Using a formula similar to that of P. Caldero and F. Chapoton, one can then prove a series of nice correspondences between the cluster algebra and the cluster category associated to the same quiver. For instance, cluster variables in one correspond to some indecomposable objects in the other; mutation, to some distinguished triangles; products, to direct sums; **g**-vectors, to indices; and so on. The rich structure of cluster categories provides us with a good means to prove properties of cluster algebras (see, for instance, [5]).

Generic bases of cluster algebras. One of the main problems in the theory of cluster algebras is that of finding a basis which should, among other properties, contain a set of distinguished elements called *cluster monomials*. Despite several results for some classes of examples, the best ones being those of C. Geiss, B. Leclerc and J. Schröer [4], this problem is still largely open. One possible way of attacking it is by using the additive categorification described above. As in the work of G. Dupont [2], to each index in the cluster category (equivalently, to each n-tuple of integers) is associated an element of the *upper* cluster algebra. This element is defined to be the "generic" value taken by the Caldero–Chapoton formula under certain conditions given by the index. The elements thus constructed can be shown (see [6]) to be linearly independent, to be well-behaved with respect to mutation and to form the basis of [4] in the cases considered there. Hopefully, they form a basis of the upper cluster algebra in general.

References

[1] Claire Amiot. *Cluster categories for algebras of global dimension 2 and quivers with potential*, Annales de l'Institut Fourier, Vol. 59 no. 6 (2009), p. 2525-2590.

[2] Grégoire Dupont. *Generic Cluster Characters*, International Mathematics Research Notices IMRN, rnr024, 2011.

[3] Sergey Fomin and Andrei Zelevinsky. *Cluster algebras I: Foundations*, J. Amer. Math. Soc. 15 (2002), no. 2, 497-529.

[4] Christof Geiss, Bernard Leclerc and Jan Schröer. *Generic bases for cluster algebras and the Chamber Ansatz*, to appear in J. Amer. Math. Soc., arXiv:1004.2781v2 [math.RT].

[5] Pierre-Guy Plamondon. *Cluster algebras via cluster categories with infinite-dimensional morphism spaces*, to appear in Comp. Math., arXiv:1004.0830v2 [math.RT].

[6] Pierre-Guy Plamondon. *Catégories amassées aux espaces de morphismes de dimension infinie, applications*, PhD thesis, available at the author's webpage.

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QUIVER VARIETIES AND QUANTUM CLUSTER ALGEBRAS

FAN QIN

ABSTRACT. Inspired by Nakajima's previous work [Nak11] on bipartite (quantum) cluster algebras, we construct all acyclic quantum cluster algebras via perverse sheaves over graded quiver varieties. The construction follows the spirit of the conjectural monoidal categorification proposed by Hernandez and Leclerc [HL10]. As a consequence, we obtain that all the quantum cluster variables have positive cluster expansions, whenever there exists an acyclic seed.

This is joint work with Yoshiyuki Kimura.

REFERENCES

[HL10] David Hernandez and Bernard Leclerc, *Cluster algebras and quantum affine algebras*, Duke Math. J. **154** (2010), no. 2, 265–341, arXiv:0903.1452.

[Nak11] Hiraku Nakajima, *Quiver varieties and cluster algebras*, Kyoto J. Math. **51** (2011), no. 1, 71–126, arXiv:0905.0002v5.

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1. **Cluster automorphisms:** Ibrahim Assem, Vasilisa Shramchenko and I have introduced the notion of cluster automorphism of a cluster algebra $\mathcal{A}$ as an automorphism of the $\mathbf{Z}$-algebra $\mathcal{A}$ which sends a cluster to a cluster and commutes with the mutations at that cluster. We study the cluster automorphism group for acyclic cluster algebras and for cluster algebras from surfaces. Using the combinatorial structure of the Auslander-Reiten quiver of the cluster category, we compute the cluster automorphism group in the finite types and in the euclidean (i.e. affine) types. Using the surface approach, we show that (a variation of) the mapping class group of the surface is isomorphic to a subgroup of the cluster automorphism group.

2. **Cluster algebras and representation theory:** Relation between tilting theory and cluster-tilting theory. Cluster-tilted algebras are endomorphism algebras (over the cluster category) of cluster-tilting objects, so to every cluster in a cluster algebra corresponds a cluster-tilted algebra. Tilted algebras are endomorphism algebras (over a hereditary algebra) of a tilting module. There is a surjective map ϕ from tilted algebras to cluster-tilted algebras given by taking a trivial extension of the tilted algebra. The map ϕ is defined on all algebras of global dimension at most two, and there are many examples of algebras that are not tilted but whose image under ϕ is cluster-tilted. In collaboration with Lucas David-Roesler, we are studying the algebras whose image under ϕ are cluster-tilted algebras of type corresponding to surfaces without punctures.

3. **Expansion formulas for cluster variables in rank 2:** Kyungyong Lee and I have found a combinatorial formula for the cluster variables of skew-symmetric cluster algebras of rank two in terms of subpaths of a specific lattice path in the plane. The formula is manifestly positive, providing a new proof of a result by Nakajima and Qin.

4. **Cluster algebras from surfaces:** Gregg Musiker, Lauren Williams and I have found combinatorial formulas for the cluster variables in cluster algebras from surfaces. Cluster variables correspond to arcs in the surface. The formula is in terms of perfect matchings of a so-called snake graph associated to the arc of the cluster variable. Adapting the combinatorial formulas in order to associate cluster algebra elements also to *other* curves in the surface, we are now working on constructing bases for the corresponding cluster algebra.

The order in this list is alphabetical on the names of the authors.

Cluster-related interests of Hugh Thomas

Atomic bases (=canonically positive bases). An element of a cluster algebra is called *positive* if its expansion in terms of any cluster has non-negative coefficients. The positive elements of a cluster algebra form a cone in the cluster algebra. An atomic basis for a cluster algebra is a basis whose non-negative linear combinations generate the positive cone. (Such a basis need not exist, but if it exists, it is unique.)

Until recently, only few examples of cluster algebras with atomic bases were known. Giovanni Cerulli recently showed that for finite type cluster algebras, the cluster monomials form an atomic basis. Subsequently, building on his approach, Grégoire Dupont and I found the atomic basis for coefficient-free cluster algebras of type $\widetilde{A}_n$ (confirming a conjecture of Dupont). We hope that it may be possible to extend this approach to all cluster algebras arising from surfaces (using bases provided by Musiker-Schiffler-Williams).

Higher dimensional analogues of cluster algebra theory. Two of the best-understood classes of cluster algebras have a "two" associated to them: for cluster algebras with an additive categorification, the "two" is the 2-Calabi-Yau property, while for cluster algebras arising from surfaces, the "two" is the two-dimensionality of the surface. It is natural to ask if these twos can be (simultaneously) increased.

In recent work with Steffen Oppermann, we showed that the link between the combinatorics of the A_n cluster algebra (triangulations of a polygon) and the type A_n cluster category, can be generalized by replacing the polygon by an even-dimensional cyclic polytope, and replacing the cluster category by a similar construction starting from the higher Auslander algebra of an A_n quiver. Various elements of cluster algebra theory persist; in particular, there is an interpretation of a tropical version of exchange relations.

However, many elements of cluster algebra theory are still missing (including, most obviously, an analogue of cluster algebras themselves). We are also curious if there is any connection to (higher-dimensional) hyperbolic geometry, Poisson geometry, or any of the other flavours of representation theory which are associated to cluster algebras.

Preprojective algebras and total positivity. Fix Q a quiver (Dynkin or not). Let Λ_Q be the associated preprojective algebra, and W the corresponding Weyl group. For $w \in W$, there is an ideal I_w in Λ_Q. Write $(I_w)_{kQ}$ for I_w viewed as a kQ-module. Define $\mathcal{C}_w$ to consist of the finitely-generated kQ-modules contained in $\mathrm{add}(I_w)_{kQ}$.

In ongoing work with Steffen Oppermann and Idun Reiten, we show that the indecomposable modules in $\mathcal{C}_w$ encode the *positive distinguished* subexpression for w inside the word $(s_1 s_2 \dots s_n)^\infty$ if Q is non-Dynkin, or inside the $s_1 \dots s_n$-sorting word for w_0 if Q is Dynkin. The positive distinguished subexpression for w in closely tied to total positivity. Given the link between total positivity and preprojective algebras, one is led to expect a conceptual explanation for our result, which should also be related to the Geiss-Leclerc-Schröer approach to categorification of cluster algebras associated to Kac-Moody groups.

BIRS, Meeting on Cluster Algebras
September 5-10, 2011, Banff, Canada

Continuous Cluster Categories

Gordana Todorov, Northeastern University, Boston, MA
Joint work with Kiyoshi Igusa, Brandeis Iniversity

Motivation: Continuous cluster categories of type **A** are uncountably infinite categories with cluster structures, where the clusters correspond to ideal geodesic triangulations of the hyperbolic plane in one case, or clusters of the cluster categories of type $\mathbf{A_n}$ in the other case. (The hyperbolic plane is the universal cover of once punctured surfaces.)

We define $\mathcal{A}_\mathbb{R}$ to be the category of k-representations of the real line $\mathbb{R}$, where k is a field. For each $a < b \in \mathbb{R}$ we denote by $V_{(a,b]}$ the special representation: $V_{(a,b]}(x) = k, \forall x \in (a,b]$ and $V_{(a,b]}(y) \to V_{(a,b]}(x)$ is 1_k for all $a < x < y \leq b$. We also define the full subcategory $\mathcal{B} \subset \mathcal{A}_\mathbb{R}$ to be additively generated by the indecomposable objects $\{V_{(a,b]} \mid a < 0 < b\}$.

A particularly nice correspondence between indecomposable objects of $\mathcal{B}$ and points in $\mathbb{R}^2$ is obtained by:

$$V_{(a,b]} \leftrightarrow M(x,y) \quad \text{where} \quad (x,y) = (-ln(-a), ln(b)),$$

and we will use this correspondence to identify the k-representations of $\mathbb{R}$ with points in the plane $\mathbb{R}^2$.

For a positive real number $c \in \mathbb{R}$, we define the full subcategory $\mathcal{B}_{\geq c} \subset \mathcal{B}$ by defining the indecomposable objects of $(\mathcal{B}_{\geq c})$ as $\{M(x,y) \in \mathcal{B} \mid |x-y| \geq c\}$. *The continuous derived category* $\mathcal{D}_c$ is defined using "two way $\mathcal{B}_{\geq c}$-approximations" in $\mathcal{B}$ and defining triangulated structure on $\mathcal{D}_c := \mathcal{B}/\mathcal{B}_{\geq c}$.

For each positive real number $d \in \mathbb{R}$ we define functor $F_d : \mathcal{D}_c \to \mathcal{D}_c$ by $F_d(M(x,y)) = M(y+d, x+d)$ which can be used to define a triangulated automorphism of the doubled derived category $\mathcal{D}_c^{(2)}$. Using the functor F_d the orbit category of $\mathcal{D}_c^{(2)}$ is defined and denoted by $\mathcal{C}_{(c,d)} := \mathcal{D}_c^{(2)}/F_d$. With these definitions we have the following results.

Theorem: The orbit category $\mathcal{C}_{(c,d)}$ is triangulated if $c \leq d$.

Theorem: The orbit category $\mathcal{C}_{(c,d)}$ has a cluster structure if and only if either $c = d$ or $c = \frac{n+1}{n+3}d$ for some positive integer n.

Relation between continuous cluster category and ideal triangulation of hyperbolic plane by geodesics is obtained in the case $c = d = \pi$. To each representation $M(x,y) \in \mathcal{C}_{(\pi,\pi)}$ we associate geodesic starting at angle x and ending at $y+\pi$. With this correspondence, each cluster in the continuous cluster category $\mathcal{C}_{(\pi,\pi)}$ corresponds to an ideal triangulation of the hyperbolic plane.

Objects of $\mathcal{C}_{(c,\pi)}$ can also be viewed as representations of the circle S^1.

Pavel Tumarkin
Jacobs University Bremen

I am mainly interested in mutation-finite cluster algebras. All the projects (and results) below are joint with A. Felikson and M. Shapiro.

One of the aims is to construct a geometric object realizing mutations of skew-symmetrizable mutation-finite matrices. It occurs that this can be done by considering triangulations of marked orbifolds. As in the case of cluster algebras from surfaces introduced by Fomin, Shapiro and Thurston (extending earlier works of Fock and Goncharov), complex of triangulations of orbifolds (considered earlier by Chekhov) coincides with exchange graph of some mutation-finite cluster algebra. Further refinements of the construction allow to produce all but several finite mutation classes of skew-symmetrizable matrices. All the exclusions originate from extended affine root systems.

Moreover, slight modifications of the arguments of Fomin and Thurston allow to prove that cluster variables for cluster algebras from orbifolds can be realized by lambda lengths. In particular, this implies that exchange graph does not depend on coefficients.

One of the applications is a construction of an unfolding (introduced by Zelevinsky) for almost all mutation-finite skew-symmetrizable matrices. In case of orbifolds unfolding can be understood as a combination of ramified covering and a "cut-and-paste" procedure. At the same time, for one series of mutation-finite matrices no unfolding is known.

The unfolding, in its turn, immediately leads to a proof of the positivity conjecture for corresponding cluster algebras (by using the result of Musiker, Schiffler and Williams for algebras from surfaces). Another application of unfolding (together with geometric interpretation of mutations) is a quasi-isometry of corresponding exchange graphs. In particular, this allows to find a growth rate of all cluster algebras from orbifolds. Growth rate of the exclusions can be computed separately (this is a joint work with H. Thomas).

1 Lauren Williams

My current research projects related to cluster algebras are as follows.

In joint work with Yuji Kodama, we demonstrate a link between cluster algebras and soliton solutions to the KP equation. More specifically, for each point $A \in (Gr_{k,n})_{\geq 0}$, there is a soliton solution $u_A(x, y, t)$ to the KP equation. If one fixes the time t, one can draw the *contour plot* of the solution, the locus in the plane where the solution is maximized. Such solutions to the KP equation model shallow water wave, and one may think of this contour plot as showing the locations of the peaks of the waves. What we've shown is a precise relation between these contour plots and the *reduced plabic graph* or *Postnikov diagrams* that correspond to clusters in the cluster algebra of the coordinate ring of the Grassmannian. More generally one can get a solution to the KP equation from any point A in the real Grassmannian (not necessary the non-negative part), and we are working to extend our results to this case. We are in the process of proving that any *regular* soliton solution that arises in this way must actually come from $(Gr_{k,n})_{\geq 0}$.

I also have two works in progress on cluster algebras from surface. In joint work with Musiker and Schiffler, we have constructed (conjectural) vector space bases for cluster algebras from surfaces, using principal coefficients. We have verified this conjecture for unpunctured surfaces. The main idea is to parameterize elements of the basis by collections of (tagged) arcs and also *closed loops* in the surface, and extend the notion of g-vector to closed loops. We define the elements associated to closed loops combinatorially, associating a certain *band graph* (on an annulus or Mobius strip) to each closed loop, and taking a weighted sum of the *good* matchings of that band graph. One can show that the set of good matchings has the structure of a distributive lattice, which in turn implies that the corresponding cluster algebra element has a well-defined g-vector. One then needs to show that our basis elements span (by proving a number of skein relations), and show that their g-vectors are all distinct. (Conceptually there is no major problem with extending our proofs to surfaces with punctures, but checking the 20 or so different kinds of skein relations involving notched arcs seems very painful.)

Musiker and I also have a paper in progress which provides some necessary tools for the paper on bases. This paper with Musiker provides expressions for our basis elements in terms of products of two by two matrices, and then proves skein relations, using principal coefficients. These results generalize results of Fock and Goncharov for the coefficient-free case. Note that we need such results for principal coefficients for our paper on vector space bases of these cluster algebras, in order to make sense of g-vectors.

My interests are in the area of ring theory of quantum function algebras, their underlying Poisson geometry and relations to cluster algebras.

The first family of quantum function algebras which I investigate are the quantum nilpotent algebras $\mathcal{U}_q^-(w)$ defined by De Concini, Kac and Procesi (associated to an arbitrary simple Lie algebra $\mathfrak{g}$ and a Weyl group element $w \in W$). In [6] I classified their torus invariant prime spectra$\mathcal{U}_q^-(w)$, described the inclusions between those ideals, and gave an explicit description of all such prime ideals in term of Demazure modules. A set theoretic classification of the torus invariant primes was also obtained by Mériaux and Cauchon [3]. Using [6] in a later paper I classified the torus invariant prime spectra of all partial flag varieties. Consequently, Geiß, Leclerc and Schröer [2] proved that in the case of simply laced $\mathfrak{g}$ the algebras $\mathcal{U}_q^-(w)$ are quantum cluster algebras. A celebrated (unpublished) theorem of Gabber establishes that the universal enveloping algebra of an arbitrary solvable Lie algebras is catenary. In [8] I proved that all algebras $\mathcal{U}_q^-(w)$ are catenary. My interests are how questions in cluster algebras and ring theory for $\mathcal{U}_q^-(w)$ interplay with each other. Zwicknagl's preprint [9] addresses this, but I believe that it has mistakes.

In a related direction I work on the spectra of quantum function algebras $R_q[G]$. In [7] I proved separation of variables type results for all Joseph's localizations [5] and the algebras $\mathcal{U}_q^-(w)$, and classified the maximal spectra of $R_q[G]$. The Joseph localizations are precisely the quantized coordinate rings of (all) double Bruhat cells in G. They play a key role in the study [5, 4] of the spectra of $R_q[G]$. Here ring theory relates once again to cluster algebras in connection with the Berenstein–Zelevinsky work on quantum double Bruhat cells and upper cluster algebras.

From ring theoretic perspective for both families of algebras $\mathcal{U}_q^-(w)$ and $R_q[G]$ there are many difficult open problems. The most important one is to describe the topology of their spectra.

References

[1] A. Berenstein and A. Zelevinsky, *Quantum cluster algebras*, Adv. Math **155** (2005), 405–455.

[2] C. Geiß, B. Leclerc and J. Schröer, *Cluster structures on quantized coordinate rings*, preprint arXiv:1104.0531.

[3] A. Mériaux and G. Cauchon, *Admissible diagrams in quantum nilpotent algebras and combinatoric properties of Weyl groups,* Represent. Theory **14** (2010), 645–687.

[4] T. J. Hodges, T. Levasseur and M. Toro, *Algebraic structure of multiparameter quantum groups*, Adv. Math. **126** (1997), 52–92.

[5] A. Joseph, *On the prime and primitive spectra of the algebra of functions on a quantum group*, J. Algebra **169** (1994), 441–511.

[6] M. Yakimov, *Invariant prime ideals in quantizations of nilpotent Lie algebras*, Proc. London Math. Soc. (3) **101** (2010), no. 2, 454–476,

[7] M. Yakimov, *On the spectra of quantum groups*, preprint arXiv:1106.3821.

[8] M. Yakimov, *A proof of the Goodearl–Lenagan polynormality conjecture*, in preparation.

[9] S. Zwicknagl, *Poisson ideals in cluster algebras and the spectra of quantized coordinate rings*, preprint arXiv:1107.2760.

Current research interests connected to cluster algebras

Jie Zhang

September 2, 2011

My current research interest is cluster algebras arising from a marked surface without punctures. In a recent joint work with Thomas Brüstle ([BZ11]), we give a module-theoretic interpretation of Schiffler's expansion formula ([S10]) which is defined combinatorially in terms of complete $(\Gamma, \gamma)-$paths in order to get the expansion of the cluster variables in the cluster algebra of a marked surface. Based on the geometric description of the indecomposable objects of the cluster category of the marked surface in [BZ10], we show the coincidence of Schiffler-Thomas' expansion formula ([ST09]) and the cluster character defined by Palu([P08]).

References

[BZ10] T. Brüstle and J. Zhang, *On the cluster category of a marked surface*, to appear in J. Algebra and Number Theory, arXiv:1005.2422v2.

[BZ11] T. Brüstle and J. Zhang, *A module-theoretic interpretation of Schiffler's expansion formula* arXiv:1105.5673

[P08] Y. Palu, *Cluster characters for 2-Calabi-Yau triangulated categories*, Annales de l'institut Fourier **58** (2008), no. 6, 2221-2248.

[S10] R. Schiffler, *On cluster algebras arising from unpunctured surfaces II*, Adv. Math. **223** (2010), no. 6, 1885-1923.

[ST09] R. Schiffler and H. Thomas, *On cluster algebras arising from unpunctured surfaces*, Int. Math. Res. Not. **17** (2009), 3160-3189.

Current research interests

Bin Zhu

Tsinghua University

An important integrant in defining cluster algebras is the notion of mutation, which can be modeled by the mutation of cluster tilting objects in cluster categories. One can define mutation in arbitrary triangulated categories, which was done by Iyama-Yoshino. In a recent joint work with Yu Zhou, a notion of mutation of torsion pairs in triangulated categories is given, and the mutation of torsion pairs is proved a torsion pair again. If the triangulated categories are the cluster categories of type A_n (or of type A_∞), a geometric interpretation of the mutation is given via the mutation of Ptolemy diagrams of a regular $(n+3)-$gon P_{n+3} (resp. a $\infty-$gon P_∞). The mutation of Ptolemy diagrams is defined by generalizing the flip of triangulations of P_{n+3} (resp. P_∞).

RESEARCH INTERESTS SEBASTIAN ZWICKNAGL

0.1. **Overview.** I am currently interested in the connections between cluster algebras, representation theory on the one hand side and Poisson geometry and quantum groups on the other. I recently proved that if a cluster algebra $\mathfrak{A}$ is Noetherian (and its complexified version the ring of functions on an affine complex variety X) and $\{\cdot,\cdot\}$ a compatible Poisson structure on X, then X can be stratified into finitely many torus orbits of symplectic leaves. This stratification is indeed independent of the choice of compatible Poisson structure. Many known examples e.g. Grassmannians and nilpotent Lie algebras U_w fall into this pattern.

0.2. **Questions.** However, I would now like to understand whether there are related structures in the realm of monoidal or additive categorifications of cluster algebras. For example what sub-or quotient categories of representations of preprojective algebras correspond to these ideals. Or are there related structures in the twisted/motivic Hall algebras. I care about these structures for the following reasons.

0.3. **Some Background, results and conjectures.** The stratification introduced above can be described entirely using data from the exchange matrix of the initial cluster. I can show that we can use this data to classify all the symplectic leaves on such a variety X. Here, of course the stratification will depend on the Poisson structure. This description applies methods from the theory of quantum groups and non-commutative ring theory (Goodearl-Letzter stratifications) in the classical limit of Poisson structures. I would like to note that the Poisson version of Goodearl-Letzter stratifications was studied in a much more general context by Goodearl and coauthors. However, this approach does not directly yield an explicit description of these ideals in terms of generators and relations.

But these results provide a new approach the following rather old conjecture in the theory of quantum groups, resp. quantized function algebras: There exists a homeomorphism between primitive ideals in quantized function algebras and the symplectic leaves in the respective semiclassical limits. One should check Milen Yakimov's recent work "On the spectra of Quantum Groups" for more results and references. Among these algebras are quantum Grassmannians, quantum groups $\mathcal{O}_q(G)$ for G semisimple, and DeConcini-Kac-Procesi algebras U_w^q. These algebras have a natural quantum cluster algebra structure due to Geiß, Leclerc and Schröer. We have the following natural conjecture which I am able to prove for all acyclic quantum cluster algebras.

Conjecture 0.1. *Let $\mathfrak{A}_q$ be a quantum cluster algebra defined by a quantum seed $(\mathbf{x}, B, \Lambda)$, $\mathfrak{A}$ a cluster algebra given by a seed $(\mathbf{x}, B)$ and let Λ define a Poisson bracket on $\mathfrak{A}$. Then the primitive spectrum is homeomorphic to the topological space of symplectic leaves.*

I believe that to prove the conjecture in general one might need to understand more about the ideals themselves and this is where I hope the categorical approach might be useful.