Philippe DI FRANCESCO, current research

My interest in cluster algebras arose from a physics point of view. In collaboration with R. Kedem, we found explicit solutions to the so-called Q- and T-systems that arise in various stages of the study of integrable quantum spin chains. These always take the form of systems of recursion relations with a discrete time variable. One of my main motivations was to understand combinatorially or by means of models of statistical physics the positive Laurent phenomenon conjecture of [Fomin,Zelevinsky].

We developed several approaches, using various statistical mechanical models to describe how general solutions are related to admissible sets of initial data (discrete Cauchy data). The models are essentially models of weighted paths on a target graph with both weights and graphs being determined by the initial data, or alternatively models of paths on networks (also referred to as “frieze” solutions sometimes). Both Q- and T-systems are particular instances of the so-called discrete Hirota equation, which plays a central role in discrete integrable systems [Krichever,Lipan,Wiegmann,Zabrodin]. Our method has consisted in constructing explicitly the (time-independent) conserved quantities of the systems, and to use them to compute generating functions of some infinite sets of cluster variables, obtained by iterated mutations, and put them in (finite) continued fraction forms. In particular, for Q- and T-systems, we were able to rephrase mutations of the graphs and weights of the path models as local rearrangements of the corresponding continued fractions, that are manifestly positive, thus proving the positivity conjecture for those cases. We also developed analogous models to solve the so-called quantum Q- and T-systems, defined by means of the quantum cluster algebra of [Berenstein,Zelevinsky], for indeterminates with quantum commutation relations.

After the recent work of [Kontsevich,Soibelman] on Donaldson-Thomas invariants, I became interested into non-commutative versions of discrete integrable systems, that is systems with a discrete time variable for non-commuting indeterminate, and with a number of (time-independent) algebraic conserved quantities. This led me to naturally study non-commutative versions of the Q-system, and to provide a proof of a conjecture by Kontsevich on the positive Laurent phenomenon in this case. The idea is that path models are already non-commutative objects, as walkers travel along edges of the target graph in a certain order. It turns out that a mild adaptation of our definitions is sufficient to solve the so-called A_1 Non-commutative Q-system of Kontsevich completely. This yields not only the non-commutative structure of this particular system, but also a nice simple definition of a non-commutative rank two cluster algebra, for which a more general positivity proof was derived by [Lee] in the skew-symmetric case.

My present interest is to try to push these ideas in higher rank cases. The structure of our general solution to Q-systems say for A type quantum spin chains goes over to the non-commutative setting via the use of continued fractions with non-commutative coefficients, corresponding to models of paths with non-commutative step weights, all of which are tapeable to non-commutative Cauchy initial data for the systems. So far we found that the discrete Hirota equation should be replaced with a non-commutative system involving discrete quasi-determinants/minors (in the sense of [Gelfand,Retakh]). This equation reduces to the classical Q- and T-systems for commuting variables, and quite interestingly to the quantum Q- and T-systems for indeterminate with quantum commutation relations.

I would also like to unify the solutions to mutation-finite cluster algebras and more specifically cluster algebras form surfaces [Musiker,Schiffler,Williams] in terms of path models, as a natural pathway to non-commutative generalizations.