LECTURE 2: ELLIPTIC GENERA AND NONCOMMUTATIVE MANIFOLDS

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Disclaimer: this is the writeup of a largely speculative lecture, given during an informal discussion session at the workshop “Novel approaches to the finite simple groups” in Banff. A more accurate paper based on some of the ideas described here is in preparation.

1. Elliptic genus, Dirac operators on loop spaces, and the Hirzebruch manifold

We recall briefly some of the main features of the Hirzebruch's theory of multiplicative genera on ordinary manifolds, before sketching some ideas for possible generalizations to the noncommutative world of spectral triples.

1.1. **Genera for ordinary manifolds.**

For ordinary manifolds, a multiplicative genus (see [13]) is a map ϕ from closed, oriented, smooth manifolds to a commutative, unital $\mathbb{Q}$-algebra Λ, with the following properties:

1. Additive on disjoint unions: $\phi(M \amalg N) = \phi(M) + \phi(N)$.
2. Multiplicative on products: $\phi(M \times N) = \phi(M)\phi(N)$.
3. Vanishing on boundaries: $\phi(\partial M) = 0$.

In particular, the value $\phi(M)$ depends only on the cobordism class $[M]$ of M.

1.2. **The genus series.**

Results of Thom showed that homomorphisms $\phi : \Omega_n^{SO} \to \Lambda$ can be dfined via combination of Pontrjagin numbers. The fact that the oriented cobordism ring $\Omega_*^{SO} \otimes \mathbb{Q} = \mathbb{Q}[\mathbb{CP}^n]_{n\geq 1}$ is a polynomial ring implies that a genus is determined by the series

$$\psi(t) = t + \frac{\phi(\mathbb{CP}^2)}{3} t^3 + \frac{\phi(\mathbb{CP}^4)}{5} t^5 + \cdots \in \Lambda[[t]]$$

1.3. **Elliptic genera.**

A multiplicative genus ϕ is *elliptic* if vanishes on the projectivizations $\mathbb{CP}(E)$ of complex vector bundles $E \to M$ over a closed oriented manifold M. In terms of the genus series this implies

$$\psi(t) = \int_0^t \frac{du}{\sqrt{1 - 2\delta u^2 + \epsilon u^4}}, \qquad \text{some } \epsilon, \delta \in \Lambda.$$

For $\Lambda = \mathbb{C}$ on obtains the signature for $\epsilon = \delta = 1$, or the $\hat{A}$-genus for $\delta = -1/8$, $\epsilon = 0$.

By expressing the Jacobi quartics $y^2 = x^4 - 2\delta x^2 + \epsilon$ as functions of τ, on obtains modular forms ϵ, δ of level $\Gamma_0(2)$. An elliptic genus $\phi(M)$ is polynomial in ϵ, δ, hence a modular form, with $\Lambda = M_*(\Gamma_0(2))$.

1.4. **Dirac operator on loop spaces.** The work of Witten connected the notion of elliptic genera to the formal geometry of Dirac operators on loop spaces, [24].

Given a spin manifold X with an action of a group G, one has a character valued Dirac index

$$F(g) = \mathrm{Tr}_{Ker(D)}(g) - \mathrm{Tr}_{Coker(D)}(g),$$

which can be expressed in terms of fixed points X_α, component $N = \oplus_\ell N_\ell$ of the normal bundle, with $g = e^{\theta P}$ and P acting on N_ℓ as $i\ell$. One gets

$$F_\alpha(\theta) = \epsilon_\alpha \langle \hat{A}(M_\alpha) \mathrm{ch}(\sqrt{\det(\otimes_{\ell>0} N_\ell)} \prod_\ell e^{i\theta\ell n_\ell/2} \bigotimes_{\ell>0} \frac{1}{1 - e^{i\ell\theta} N_\ell}), X_\alpha \rangle,$$

where $(1-tV)^{-1} = 1 \oplus tV \oplus t^2 S^2 V \oplus \cdots \oplus t^k S^k V \oplus \cdots$ and the sign ϵ_α depends on orientations.

Formally, one takes $X = \mathcal{L}(M)$ to be the loop space over a manifold M, with M identified with the set of fixed points of the circle action (trivial loops). The normal bundle decomposes as $\oplus_\ell N_\ell$, with each $N_\ell = T = TM$, and with $n_\ell = d = \dim M$. Given a choice of a spin structure on M, one takes $\sqrt{\det(\otimes_{\ell>0} N_\ell)}$ and one has

$$F(q) = q^{-d/24} \langle \hat{A}(M) \mathrm{ch}(\otimes_{\ell=1}^\infty S_{q^\ell} T), M \rangle,$$

replacing $\prod_{\ell>0} e^{i\theta n_\ell/2}$ formally with

$$(\prod_{n=1}^\infty q^n)^{d/2} = (q^{\sum_n n})^{d/2} = q^{\zeta(-1)d/2} = q^{-d/24}.$$

Then one can write $F(q) = \Phi(q)/\eta(q)$, with $\eta(q) = q^{1/24} \prod_{\ell \geq 1}(1-q^\ell)$ the Dedekind eta function, and $\Phi(q)$ a modular form, which is the level one elliptic genus, under the assumption that $p_1(M) = 0$.

1.5. **A 24-dimensional manifolds for the Monster group.** Given a spin manifold M with $p_1(M) = 0$, the Witten genus Φ_M is in the ring of modular forms $\mathcal{M}_* = \mathbb{Z}[E-4, E-6, \Delta]/(E_4^3 - E_6^2 - 1728\Delta)$, with $\Delta = q\prod_n (1-q^n)^{24}$.

Hirzebruch conjectured the existence of a spin manifold M as above, of dimension 24, such that $\Phi_M = \hat{A}(M)\bar{\Delta} + \hat{A}(M, T_{\mathbb{C}})\Delta$, with $\bar{\Delta} = E_4^3 - 744\Delta$. This is equivalent to asking whether there exists a 24-dimensional compact spin manifold M, with $p_1(M) = 0$, and $\hat{A}(M) = 1$, and with $\hat{A}(M, T_{\mathbb{C}}) = 0$, or equivalently, such that $\Phi_M = \bar{\Delta}$, that is, with Witten genus the j function, after normalization by η^{24}.

This question was answered positively by Hopkins–Mahowald in [20].

Hirzebruch's conjecture also predicted that there would be a manifold as above with an action of the monster group $\mathbb{M}$ by isometries, such that one would obtains the Monster representations, whose dimensions are related to the coefficients of the modular function j, from the tensor powers of tangent bundle.

This second part of the Hirzebruch conjecture is still unsolved. Part of the purpose of this lecture is to suggest that an answer may be found, perhaps more naturally, not among ordinary manifolds, but among their noncommutative generalizations, spectral triples.

2. Multiplicative genera for noncommutative manifolds

The notion of spin manifold is extended in the world of noncommutative geometry by the notion of spectral triple.

2.1. **Spectral triples as noncommutative manifolds.** We introduce here an analogous notion of multiplicative genera for noncommutative manifolds.

The data of a spectral triple (see [5]) consist of:

- an involutive algebra $\mathcal{A}$
- a representation $\pi : \mathcal{A} \to \mathcal{B}(\mathcal{H})$ as bounded operators on a separable Hilbert space $\mathcal{H}$
- a self-adjoint operator D on $\mathcal{H}$, with dense domain and with compact resolvent, $(1 + D^2)^{-1/2} \in \mathcal{K}$.
- a compatibility condition between $\mathcal{A}$ and D requiring that the commutators $[a, D]$ are bounded operators on $\mathcal{H}$ for all $a \in \mathcal{A}$
- in the *even* case, a $\mathbb{Z}/2$- grading γ on $\mathcal{H}$ satisfying

$$[\gamma, a] = 0, \ \forall a \in \mathcal{A}, \quad \text{and} \quad D\gamma = -\gamma D.$$

A spectral triple is θ-summable if its heat kernel is trace class,

$$\mathrm{Tr}(e^{-tD^2}) < \infty, \quad \forall t > 0. \tag{2.1}$$

Smooth compact Riemannian spin manifolds M define spectral triples, where the data $(\mathcal{A}, \mathcal{H}, D)$ are given by $(C^\infty(M), L^2(M, S), \partial\!\!\!/_M)$. Under additional conditions, an abelian spectral triple determines a smooth compact Riemannian spin manifold via the reconstruction theorem of [6], [22]. In this sense spectral triples generalize ordinary manifolds and can be regarded as "noncommutative manifolds".

2.2. **Real structure.** A real structure on a spectral triple consists of an antilinear isometry $J : \mathcal{H} \to \mathcal{H}$ with

$$J^2 = \varepsilon, \quad JD = \varepsilon' DJ, \quad \text{and} \quad J\gamma = \varepsilon'' \gamma J,$$

where the signs $\epsilon, \epsilon', \epsilon'' \in \{\pm 1\}$ determine the KO-dimension modulo eight, according to the table:

n	0	1	2	3	4	5	6	7
ε	1	1	-1	-1	-1	-1	1	1
ε'	1	-1	1	1	1	-1	1	1
ε''	1		-1		1		-1	

The real structure J satisfied the properties:

1. Commutation: $[a, b^0] = 0$ for all $a, b \in \mathcal{A}$, where $b^0 = Jb^*J^{-1}$, for all $b \in \mathcal{A}$.
2. Order one condition:

$$[[D, a], b^0] = 0 \qquad \forall\, a, b \in \mathcal{A}.$$

2.3. **Unions and products.** One can take disjoint unions and products of noncommutative manifolds according to the following prescription.

- Disjoint union (direct sum): the algebra is $\mathcal{A} = \mathcal{A}_1 \oplus \mathcal{A}_2$, the Hilbert space $\mathcal{H} = \mathcal{H}_1 \oplus \mathcal{H}_2$, and the Dirac operator

$$D = \begin{pmatrix} D_1 & 0 \\ 0 & D_2 \end{pmatrix}.$$

- Products (tensor products): the algebra is $\mathcal{A} = \mathcal{A}_1 \otimes \mathcal{A}_2$, the Hilbert space $\mathcal{H} = \mathcal{H}_1 \otimes \mathcal{H}_2$ and the Dirac operator (depending on the even/odd sign of the spectral triple)

$$D = D_1 \otimes 1 + \gamma_1 \otimes D_2$$

$$\gamma = \gamma_1 \otimes \gamma_2 \qquad J = J_1 \otimes J_2$$

2.4. **Dirac operator on manifolds with boundary.** The ordinary Dirac operator on an n-dimensional spin manifold M with boundary defines a class $[\not{D}]$ in $K_n(M, \partial M)$, which maps, under the long exact sequence

$$\to K_n(\partial M) \to K_n(M) \to K_n(M, \partial M) \stackrel{\partial}{\to} K_{n-1}(\partial M) \to$$

to the class $[\not{\partial}]$ of the Dirac operator on the boundary manifold ∂M, see [1]. The Chern character maps this to the long exact sequence in homology. In [11] it was shown, using Melrose b-calculus, that an identity $\partial Ch_*(\not{D}) = Ch_*(\not{\partial})$ holds at the level of cyclic cochains. More recent results based on b-calculus techniques were obtained in [19].

2.5. **Spectral triples with boundary.** Chamseddine and Connes recently introduced a notion of "spectral triples with boundary" [3], motivated by properties of the spectral action functional and boundary terms in quantum gravity [4]. We recall here the basic properties that describe the boundary of a spectral triple.

Suppose that $(\mathcal{A}, \mathcal{H}, D)$ is an even spectral triple with $\mathbb{Z}/2\mathbb{Z}$-grading γ on $\mathcal{H}$ with $[a, \gamma] = 0$ for all $a \in \mathcal{A}$ and assume that $Dom(D) \cap \gamma Dom(D)$ is dense in $\mathcal{H}$. The *boundary algebra* $\partial\mathcal{A}$ is then defined as the quotient $\mathcal{A}/(J \cap J^*)$, by the two-sided ideal $J = \{a \in \mathcal{A} | a Dom(D) \subset \gamma Dom(D)\}$. The *boundary Hilbert space* $\partial\mathcal{H}$ is the closure in $\mathcal{H}$ of $D^{-1} Ker D_0^*$, with D_0 a symmetric operator restricting D to $Dom(D) \cap \gamma Dom(D)$. The action of $\partial\mathcal{A}$ on the boundary Hilbert space is given by $a - D^{-2}[D^2, a]$. The *boundary Dirac operator* ∂D is defined on $D^{-1} Ker D_0^*$ with

$$\langle \xi, \partial D \eta \rangle = \langle \xi, D \eta \rangle,$$

for $\xi \in \partial\mathcal{H}$ and $\eta \in D^{-1} Ker D_0^*$. It has bounded commutators with $\partial\mathcal{A}$, [3], [7].

2.6. **The notion of dimension for noncommutative manifolds.** In noncommutative geometry there are at least three different notions of dimension for a spectral triple $(\mathcal{A}, \mathcal{H}, D)$.

- The metric dimension is measured by the rate of growth of the eigenvalues of the Dirac operator;
- The KO-dimension (which is an integer mod 8) is determined by the signs of the commutation relations of the operators J, γ, D, mimicking the analogous relations for real spin manifolds and the mod 8 periodicity of real K-theory;
- The dimension spectrum is a subset of the complex plane, consisting of the poles of the zeta functions $\zeta_{a,D}(s) = \mathrm{Tr}(a|D|^{-s})$. At each point s of the dimension spectrum there is a corresponding integration in dimension s, given by the residues of the zeta functions at that point.

For ordinary spin manifolds the first two notions of dimension agree (mod 8), while the dimension spectrum contains the usual dimension, along with other integer points less than or equal to the dimension. For truly noncommutative spaces the KO-dimension is not always equal to the metric dimension mod 8, and the dimension spectrum can contain non-integer and also non-real points.

2.7. **The local index formula.** The Connes–Moscovici local index formula [10] can be thought of as a way to provide an analog of Pontrjagin classes in noncommutative geometry.

Given an even, finitely summable spectral triple $(\mathcal{A}, \mathcal{H}, D)$ with simple dimension spectrum, the Chern character local formula looks like

$$\phi_n(a_0, \ldots, a_n) = \sum c_{n,k} \mathrm{ResTr}(a^0 [D, a_1]^{(k_1)} \cdots [D, a_n]^{(k_n)} |D|^{-n-2|k|})$$

where

$$c_{n,k} = \frac{(-1)^{|k|}\Gamma(|k| + n/2)}{k!((k_1 + 1) \cdots (k_1 + k_2 + \cdots + k_n + n))}$$

and $\nabla(a) = [D^2, a]$ and $a^{(k)} = \nabla^k(a)$.

It defines a pairing of cyclic cohomology $HC^*(\mathcal{A})$ and K-theory $K_*(\mathcal{A})$.

It is natural to ask whether, in the noncommutative context, there is a good notion of cobordism ring, with a possible distinguished set of generators, and whether there is a possible description of cobordism in terms of the (ϕ_n), seen as noncommutative Pontrjagin classes. This would not be a straightforward generalization of the Thom argument, which relies on notions of embeddings and normal bundles for manifolds, which do not have a good noncommutative analog. However, in noncommutative geometry one has good analogs of vector bundles (projective modules, Hilbert modules), and morphisms of spectral triples (bimodules with connections) among which some qualify as the right notion of "embeddings", so it is possible that parts of the Thom argument may turn out to have a suitable noncommutative analog. This question is closely related to the question of whether there may exist anything like a power series description of genera in noncommutative geometry.

2.8. **Multiplicative genera for spectral triples.** One can then define multiplicative genera for noncommutative manifolds by mimicking the definition for the commutative case.

Let Λ be a unital commutative algebra. Then a multiplicative genus is a function ϕ from θ-summable spectral triples to Λ with the following properties:

(1) Additive on disjoint unions (direct sums):

$$\phi((\mathcal{A}_1, \mathcal{H}_1, D_1) \oplus (\mathcal{A}_2, \mathcal{H}_2, D_2)) = \phi(\mathcal{A}_1, \mathcal{H}_1, D_1) + \phi(\mathcal{A}_2, \mathcal{H}_2, D_2). \tag{2.2}$$

(2) Multiplicative on products (tensor products):

$$\phi((\mathcal{A}_1, \mathcal{H}_1, D_1) \otimes (\mathcal{A}_2, \mathcal{H}_2, D_2)) = \phi(\mathcal{A}_1, \mathcal{H}_1, D_1)\phi(\mathcal{A}_2, \mathcal{H}_2, D_2). \tag{2.3}$$

(3) Vanishing on boundaries:

$$\phi(\mathcal{A}, \mathcal{H}, D) = 0 \quad \text{if} \quad (\mathcal{A}, \mathcal{H}, D) = \partial(\mathcal{A}', \mathcal{H}', D'). \tag{2.4}$$

2.9. **Elliptic genera?** A natural question regarding the definition of multiplicative genera for noncommutative manifold is what should be the right notion that generalizes ellipticity.

If one defines ellipticity, in the case of ordinary manifolds, as the vanishing on the projective bundles $\mathbb{CP}(E)$ of complex vector bundles $E \to M$, then it would be natural to seek an analog of $\mathbb{CP}(E)$ in the noncommutative world.

A possible way to proceed may be to reformulate the usual notion of projectivization of a vector bundle. A projective bundle on M can be thought of, by

Banach–Steinhaus, as a principal $PU(\mathcal{H})$-bundle. Isomorphism classes correspond to sheaf cohomology $H^1(M, PU(\mathcal{H})_M)$. Projective bundles that are projectivizations $P = \mathbb{CP}(E)$ of a complex vector bundle E are then characterized as being those P for which the Dixmier–Douady class $\delta(P) \in H^3(M,\mathbb{Z})$ is trivial, $\delta(P) = 0$.

This formulation appears suitable to generalizations to the noncommutative case, whenever one can make sense of a Dixmier–Douady class. This is the case, for instance, within the class of continuous trace C^*-algebras. Ellipticity of a multiplicative genus for noncommutative manifolds would then be characterized by the vanishing on all P with $\delta(P) = 0$, in this more general setting.

It is unclear whether such an approach to ellipticity would maintain any connection to modularity, as in the ordinary manifold case.

3. The JLO cocycle

3.1. **A noncommutative geometry view of the elliptic genus.** Part of the reason for expecting that a good generalization of multiplicative genera and elliptic genera in noncommutative geometry may exist is coming from the fact that one already has a formulation of the elliptic genera for ordinary manifolds in the language of noncommutative geometry, due to Jaffe [15], [16].

Given a θ-summable spectral triple $(\mathcal{A},\mathcal{H},D)$, for which $\mathrm{Tr}(|D|^{-s})$ need not be finite but $\mathrm{Tr}(e^{-tD^2}) < \infty$, for all $t > 0$, the JLO-cocycle gives a Chern character pairing with K-theory $K_0(\mathcal{A})$ by

$$\tau_n^{JLO}(a_0,\dots,a_n;g) = \int_{\Sigma_n} \mathrm{Tr}(\gamma U(g) a_0 e^{-s_0 D^2} da_1 e^{-s_1 D^2} \cdots da_n e^{-s_n D^2}) dv$$

where $da = [D,a]$, $a \in \mathcal{A}$ and the integration chain is a simplex $\Sigma_n = \{\sum_j s_j = 1\}$, $dv = ds_0 \cdots ds_n$. The JLO cocycle is a super-KMS-functional generalizing the notion of a Gibbs state.

Jaffe realized the elliptic genus as a partition function

$$\mathrm{Tr}_{\mathcal{H}}(\gamma e^{-i\theta J - i\sigma P - \beta H})$$

with Hamiltonian $H = Q^2 - P$, supercharge Q, twisting angle J, and translations P. The supercharge operator Q can be viewed as a Dirac operator of a θ-summable spectral triple, and the equivariant index of the Dirac operator Q on the loop space can be computed by evaluation of a JLO cocycle, [15], [16].

3.2. **The JLO cocycle as multiplicative genus.** The results of [15] and [16] suggest that, more generally, the JLO cocycle of [14] should provide the main example of a multiplicative genus for noncommutative manifolds.

As above, given operators T_i, for $i = 0,\dots,k$ on the even Hilbert space $\mathcal{H}$ with $\mathbb{Z}/2\mathbb{Z}$ grading γ, one defines

$$\langle T_0,\dots,T_k\rangle = \int_{\Delta_k} \mathrm{Tr}(\gamma(T_0 e^{-t_0 D^2} \cdots T_k e^{-t_k D^2}) dt_0 \cdots dt_k$$

with Δ_k the k-simplex in $\mathbb{R}_+^{k+1}$.

The JLO cocycle of a θ-summable triple $(\mathcal{A},\mathcal{H},D)$ is the entire cyclic cochain defined by [14]

$$\varphi_k(a_0,\dots,a_k) = \langle a_0, [D,a_1],\dots,[D,a_k]\rangle.$$

This satisfies $(b+B)\varphi_* = 0$ and it represents the Chern character from K-theory to entire cyclic homology.

The multiplicative behavior of the JLO-cocycle over products of spectral triples was in fact already shown in a recent paper of Uuye, [23].

3.3. **The loop space of a noncommutative manifold?** A related question is whether there is a good geometric construction of an analog of the (formal) loop space for an arbitrary noncommutative manifold (spectral triple).

In the case of an ordinary manifold, one defines the loop space as the space of maps $Maps(S^1, X)$. At the level of their (commutative) algebras of functions these would translate into homomorphisms $\chi : C(X) \to C(S^1)$ of commutative C^*-algebras. A version of (infinitesimal) loops for schemes was similarly formulated by Kapranov–Vasserot, essentially as ring homomorphisms $\mathrm{Hom}(A, R[[t]])$.

However, this kind of definition runs into the same problem one encounters in trying to define "points" on noncommutative spaces. For noncommutative algebras there are typically not enough characters (or equivalently, two sided ideals).

With points, a good replacement notion that corrects for the lack of characters, is to think of points as extremal measures. The notion of probability measures on an ordinary space generalizes to the notion of *states* on a C^*-algebra, that is, continuous linear functionals $\varphi : \mathcal{A} \to \mathbb{C}$ that satisfy a positivity condition $\varphi(a^*a) \geq 0$, for all $a \in \mathcal{A}$, and are normalized (for unital algebras) by $\varphi(1) = 1$.

This suggests that a possible way of generalizing the notion of loops and loop spaces may be via a version of "states", given in terms of linear functionals $\ell : C(X) \to C(S^1)$, that are no necessarily algebra-homomorphisms, but that satisfy a suitable analog of the "positivity" condition of states. Following this point of view would lead to a definition of a loop space $\mathcal{L}(X)$ of a noncommutative space X that is not itself an algebra, but at best a module or bimodule over the algebra of coordinates of the noncommutative space. However, this may still suffice to the purpose of building a noncommutative analog of the Dirac operator on loop spaces, as a spectral triple.

3.4. **Formal Dirac operators and noncommutative loop spaces?** Notice that, in the case of ordinary manifolds, in order to define a formal Dirac operator on loop spaces, one does not need a full rigorous construction of the loop space as an infinite dimensional spin manifold, but only an "infinitesimal neighborhood" of the constant loops. The main ingredient is the normal bundle $N = \oplus_{\ell \neq 0} T_\ell$ of a manifold M in the loop space $\mathcal{L}(M)$, with $T_\ell \simeq TM$. The bundle $\mathcal{T} = T\mathcal{L}(M)$ is the pullback of TM to loops $\gamma : S^1 \to M$. Given $\mathcal{E}$, a notrivial real line bundle on S^1, and $\hat{\mathcal{T}} = \mathcal{E} \otimes T\mathcal{L}(M)$, one has

$$\hat{\mathcal{T}}|_M = \oplus_{m \in \mathbb{Z}+1/2} q^m T_m$$

with $T_m \simeq TM$, where q^m corresponds to the S^1-action.

Thus, if one wants to phrase a similar construction in the setting of spectral triples, one can, for instance, aim at constructing a spectral triple for $\mathcal{L}(M)$ with $\mathcal{H} = \oplus_m q^m \mathcal{H}_m$, where the graded pieces are identified with copies of (sections of) the spinor bundle $\mathcal{H}_m = L^2(M, S)$.

A twisted Dirac operator on M is obtained by

$$\not{D}_M \otimes \otimes_{n \geq 1} S^{q^n}(TM_{\mathbb{C}}) \otimes S \otimes \otimes_{n>0} \Lambda^{q^n}(TM_{\mathbb{C}}).$$

In trying to build the right setup for a spectral triple formulation of the Dirac operator on loop spaces, one should keep in mind that the construction should be compatible with known results (Landweber) about Dirac operators for loop groups

LG and possibly with recent approaches relating *string structures* on manifolds to spin connections on the loop space.

4. Looking for a noncommutative Hirzebruch manifold

The considerations elaborated in the previous sections were aimed at providing a framework of supporting evidence and intuition for the idea that the manifold conjectured by Hirzebruch, along with the desired isometric action of the monster group may in fact not be a manifold in the ordinary sense but a noncommutative manifold, in the form of a spectral triple.

More direct evidence supporting this possibility lies in two different sources:

- Recently discovered relations between the noncommutative spaces underlying the quantum statistical mechanics of the GL_2-system of [8] (see also Chapter 3 of [9]) and the moonshine of the monster group. This is the content of ongoing work of Jorge Plazas, [21].
- An operator algebra approach to conformal field theory, developed by Longo, Kawahigashi, [17], [18]

The first work provides a natural connection between a noncommutative space of lattices with additional structure (degenerate level structures) and modular functions, which can be adapted to the replicable functions of moonshine theory.

The second, on the other hand, provides a framework where the vertex operator algebra of Borcherds' formulation of moonshine [2] can be seen in terms of von Neumann algebras and data closely related to JLO cocycles and spectral triples.

Making these connections precise is the purpose of some ongoing work of the author and the results will be presented elsewhere.

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