Focused Research Group 12frg163:
Geometrization of Smooth Characters

Pramod Achar (Louisiana State University)
Clifton Cunningham (University of Calgary)
Masoud Kamgarpour (Hausdorff Institute, Bonn)
David Roe (PIMS Calgary and Washington)
Hadi Salmasian (University of Ottawa)

7 – 12 May 2012

During the first two weeks in July 2011, at the Mathematisches Forschungsinstitut Oberwolfach (MFO), a group consisting of Pramod Achar, Clifton Cunningham, Masoud Kamgarpour and Hadi Salmasian worked on a geometric approach to the local Langlands correspondence as it pertains to algebraic groups over p-adic fields. The same research group, with one addition, David Roe, met for one week in May 2012 at the Banff International Research Station (BIRS) to continue this project. This report describes the work done by this group at both the MFO in July 2011 and at BIRS in May 2012.

1 Introduction

We seek to replace the basic ingredients of both sides of the local Langlands correspondence with geometric avatars (in this case, perverse sheaves) and then bring techniques from algebraic geometry to bear on the correspondence itself. We hope, in the process, to see how to make local Langlands correspondence more categorical. The main results we have established thus far are explained (but not proved) in this report, in sections corresponding to the four points below.

Throughout this report, F denotes a local non-Archimedean field with residue field $\mathbb{F}_q$ and $\bar{F}$ denotes a fixed separable closure of F. Let p be the characteristic of $\mathbb{F}_q$. Although we assume nothing regarding the characteristic of F, we are particularly interested in the case when F has characteristic 0. We write Γ for $\mathrm{Gal}(\bar{F}/F)$ and W_F (resp. W'_F) for the Weil (resp. Weil-Deligne) group of F. Let G be a connected reductive group over F, and T a torus over F.

(§ 2) We have found a category of perverse sheaves whose simple objects naturally correspond to complete Langlands parameters for $G(F)$; we refer to the simple objects in this category as geometric parameters for G. See Section 2.

(§ 3) We have sketched an argument that the category of geometric parameters is Koszul. See Section 3.

(§ 4) When T is an unramified induced torus, we have found a category of perverse sheaves whose simple objects naturally correspond to characters of admissible representations of $T(F)$; we refer to simple objects in this category as geometric characters for T. See Section 4.

(§ 5) We have found a function from geometric characters for $F^\times$ to geometric parameters for $F^\times$; this function is a bijection (on isomorphism classes) by class field theory. See Section 5

Since 12frg163 met, in related work with Aaron Christie and Anne-Marie Aubert, we have also found how to geometrize certain cusp forms appearing in the part of the local Langlands correspondence proved by Lusztig. However, no details of that progress will appear in this report.

Contents

1.1 The Local Langlands Correspondence

In order to give some context for our work, we give a brief description of the current status of the the local Langlands correspondence. A *complete Langlands parameter for* G is a pair (ϕ, ϵ), where $\phi : W'_F \to {}^LG$ is an admissible L-homomorphism and ϵ is an irreducible representation of the finite group $\mathcal{S}_\phi := Z_{\check{G}}(\phi)/Z_{\check{G}}(\phi)^\circ Z(\check{G})^{W_F}$. The local Langlands correspondence promises a bijection between complete parameters and characters Θ_π of admissible irreducible complex representations π of $G(F)$. The bijection $\Theta_\pi \leftrightarrow (\phi, \epsilon)$ must satisfy certain natural conditions, notably compatibility with the principle of functoriality and local class field theory.

The local Langlands correspondence has been proved for certain families of groups, including general linear (Harris-Taylor and Henniart), symplectic and odd-orthogonal groups (Arthur, building on recent work by Ngo and forthcoming work by Waldspurger). A slightly weaker statement is known for even-orthogonal groups (Arthur) and the proof for some other classes, including unitary groups, is currently under construction following Arthur's ideas. Besides these, the local Langlands correspondence has also been proved for a few low-rank groups, such as the rank-2 group of symplectic similitudes (Gan-Takeda). From a completely different perspective, the local Langlands correspondence is also fairly well understood for certain families of representations of quasi-split groups (recent work by Debacker, Reeder, Gross, and Yu), including some (but not all) depth-zero supercuspidal representations. From a different perspective again, the local Langlands correspondence was proved more than 15 years ago by Lusztig for cuspidal unipotent representations of connected algebraic groups over non-Archimedean local fields. The general case of the local Langlands correspondence remains open.

2 Geometric parameters

In this section we explain how to geometrize Langlands parameters of p-adic groups. There is considerable overlap between the ideas presented here and those appearing in [14] as they pertain to p-adic fields; a discussion of this overlap can be found at the end of Section 2.

Let G be a connected, reductive linear algebraic group over F; for simplicity, we assume here that G is also quasi-split (so all L-parameters are admissible L-parameters). Write F_G for the splitting field for G in $\bar{F}$ and Γ_G for the Galois group $\mathrm{Gal}(F_G/F)$. We use the finite model for the Langlands group: ${}^LG = \check{G} \rtimes \Gamma_G$ is a quasisplit reductive linear algebraic group over $\bar{\mathbb{Q}}_\ell$ (or $\mathbb{C}$, according to taste).

2.1 Cocycles

Let I_F be the inertia group for F; thus, $I_F = \mathrm{Gal}(\bar{F}/F^{\mathrm{nr}})$, where F^{nr} is the maximal unramified extension of F in $\bar{F}$. We being by explaining how to view $Z^1(I_F, \check{G})$ (cocycles continuous for the discrete topology on $\check{G}$) as an ind-variety. For every finite extension F' of F_G, set $I_{F'/F} = \mathrm{Gal}(F' \cdot F^{\mathrm{nr}}/F^{\mathrm{nr}})$ and let $I_{F'/F} \to \Gamma_G$ be the composition

$$I_{F'/F} = \mathrm{Gal}(F' \cdot F^{\mathrm{nr}}/F^{\mathrm{nr}}) \cong \mathrm{Gal}(F'/F' \cap F^{\mathrm{nr}}) \hookrightarrow \mathrm{Gal}(F'/F) \twoheadrightarrow \mathrm{Gal}(F_G/F) = \Gamma_G.$$

The finite group Γ_G acts algebraically on $\check{G}$ in the sense that, for every $\gamma \in \Gamma_G$, the function $\gamma : g \mapsto {}^{\gamma}g$ is a morphism of algebraic groups. It follows that we can interpret $Z^1(I_{F'/F}, \check{G})$ as an algebraic variety; indeed, it is a closed subvariety of the product of $|I_{F'/F}|$-copies of $\check{G}$:

$$Z_{F'} := \left\{ z = (z(\sigma))_{\sigma \in I_{F'/F}} \in \prod_{\sigma \in I_{F'/F}} \check{G} \,|\, z(\sigma\sigma') = z(\sigma)\, {}^{\sigma}z(\sigma'), \quad \forall \sigma, \sigma' \in I_{F'/F} \right\}.$$

It is clear that there is a canonical bijection between the $\bar{\mathbb{Q}}_\ell$-rational points on $Z_{F'}$ and the set $Z^1(I_{F'/F}, \check{G})$. If F'' is a finite extension of F', itself a finite extension of F_G, then restriction defines a morphism of algebraic varieties $Z_{F'} \to Z_{F''}$. With this in mind, it is easy to see how to view $Z^1(I_F, \check{G})$ as an ind-variety:

$$Z := \varinjlim_{F'/F_G} Z_{F'}.$$

It is now clear that there is a canonical bijection between the $\bar{\mathbb{Q}}_\ell$-rational points on Z and the set $Z^1(I_F, \check{G})$.

$$Z^1(I_F, \check{G}) \cong Z(\bar{\mathbb{Q}}_\ell) \tag{1}$$

During our programme we proved:

Lemma 2.1 *For every finite extension F' of F_G, the group $\check{G}$ (resp. $\check{G}_{\Gamma\text{-}ad} := \check{G}/Z(\check{G})^{\Gamma}$) is reductive and acts on $Z_{F'}$ in the category of algebraic varieties; moreover, $\check{G}$ (resp. $\check{G}_{\Gamma\text{-}ad}$) acts on Z in the category of ind-varieties.*

2.2 Langlands parameters with trivial monodromy

In order to recognize $\mathrm{Hom}_{\Gamma_G}(W_F, {}^LG)$ as an ind-variety, it seems necessary to make a slightly disagreeable choice: we fix a lift $\mathrm{Fr} \in W_F$ of arithmetic Frobenius for $\mathbb{F}_q$; equivalently, we fix a splitting of the short exact sequence

$$1 \longrightarrow I_F \longrightarrow W_F \underset{\curvearrowleft\ \mathrm{Fr}_F}{\longrightarrow} W_{\mathbb{F}_q} \longrightarrow 1.$$

Using this choice we may identify elements $\phi \in \mathrm{Hom}_{\Gamma_G}(W_F, {}^LG)$ with pairs (z, s) tied together by the condition

$$z({}^{\mathrm{Fr}}\sigma) = s\ {}^{\mathrm{Fr}}z(\sigma)\ {}^{{}^{\mathrm{Fr}}\sigma}(s^{-1}), \quad \forall \sigma \in I_F;$$

to recover such a pair from ϕ let s be the image of Fr in $\check{G}$ and let $z \in Z^1(I_F, \check{G})$ be the restriction of ϕ to I_F.

To pass from $\mathrm{Hom}_{\Gamma_G}(W_F, {}^LG)$ to L-parameters with trivial monodromy we need one more condition. Let $\check{G}^{\mathrm{Fr\text{-}ss}}$ be the subvariety (neither open nor closed, in general) of $s \in \check{G}$ such that $s \rtimes \mathrm{Fr}$ lies in the variety ${}^LG^{\mathrm{ss}}$ of semisimple elements in ${}^LG$. For every finite extension F' of F_G, define

$$Y_{F'} := \{(z, s) \in Z_{F'} \times \check{G}^{\mathrm{Fr\text{-}ss}} \,|\, z({}^{\mathrm{Fr}}\sigma) = s\ {}^{\mathrm{Fr}}z(\sigma)\ {}^{{}^{\mathrm{Fr}}\sigma}(s^{-1}), \quad \forall \sigma \in I_{F'/F}\}$$

and set $Y := \varinjlim_{F'/F_G} Y_{F'}$. We may now write

$$Y = \{(z,s) \in Z \times \check{G}^{\text{Fr-ss}} \mid z({}^{\text{Fr}}\sigma) = s\ {}^{\text{Fr}}z(\sigma)\ {}^{\text{Fr}\sigma}(s^{-1}), \quad \forall \sigma \in I_F\}.$$

Lemma 2.2 *For each lift* Fr *of arithmetic Frobenius for* $\mathbb{F}_q$*, making use of Equation 1 and Lemma 2.1, there is a canonical,* $\check{G}$*-equivariant bijection between the* $\bar{\mathbb{Q}}_\ell$*-rational points on* $Y = Y_{\text{Fr}}({}^LG)$ *and the set of Langlands parameters for* G *with trivial monodromy.*

$$\operatorname{Hom}_{\Gamma_G}(W_F,\, {}^LG^{ss}) \longrightarrow Y(\bar{\mathbb{Q}}_\ell)$$

defined by $\lambda \mapsto (\lambda|_{I_F}, \lambda(\text{Fr}))$.

2.3 The geometric parameter ind-variety

Let $X = X_{\text{Fr}}({}^LG)$ be the ind-variety defined by $X = \varinjlim_{F'/F_G} X_{F'}$, where $X_{F'}$ is the variety of triples $(z,s,N) \in Z_{F'} \times \check{G}^{\text{Fr-ss}} \times \check{\mathfrak{g}}^{\text{nilp}}$ such that, for each $\sigma \in I_{F'/F}$,

$$s\ {}^{\text{Fr}}z(\sigma)\ {}^{\text{Fr}\sigma}(s^{-1}) = z({}^{\text{Fr}}\sigma) \tag{2}$$

$$s\ {}^{\text{Fr}}N\ s^{-1} = qN, \tag{3}$$

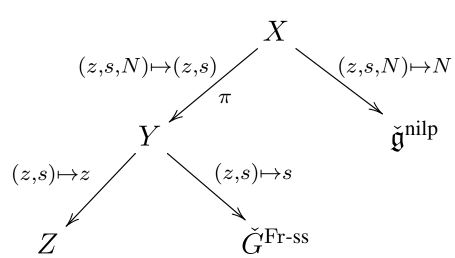
$$z(\sigma)\ {}^{\sigma}N\ z(\sigma)^{-1} = N. \tag{4}$$

Lemma 2.3 *Although the ind-variety* $X = X_{\text{Fr}}({}^LG)$ *does depend on the choice* Fr *made above, the isomorphism class of* X *does not. For each lift* Fr *of arithmetic Frobenius for* $\mathbb{F}_q$*, there is a canonical,* $\check{G}$*-equivariant bijection between the* $\bar{\mathbb{Q}}_\ell$*-rational points on* X *and the set of Langlands parameters for* G*.*

For reasons that will be apparent later (looking ahead to Theorem 2.1), we refer to $X_{\text{Fr}}({}^LG)$ as the *geometric parameter ind-variety* for G.

2.4 Vogan varieties and a stratification of the geometric parameter ind-variety

Notice that the geometric parameter ind-variety comes equipped with various $\check{G}$-equivariant projections, given below.



Lemma 2.4 *The* $\check{G}$*-equivariant morphism* $\pi : X \to Y$ *given by* $(z,s,N) \mapsto (z,s)$ *determines a stratification of* X *into disjoint, connected* $\check{G}$*-stable (resp.* $\check{G}_{\Gamma\text{-}ad}$*-stable) subvarieties of* X*:*

$$X = \coprod_{\mathcal{O} \subseteq Y} \pi^{-1}(\mathcal{O}), \tag{5}$$

where the disjoint union is taken over all $\check{G}$*-orbits in* Y*. Moreover, the group* $\check{G}$ *(resp.* $\check{G}_{\Gamma\text{-}ad}$*) acts on* $\pi^{-1}(\mathcal{O})$ *with only finitely many orbits, which are locally closed subvarieties of* X*. Thus,* X *admits a stratification into locally closed,* $\check{G}$*-stable (resp.* $\check{G}_{\Gamma\text{-}ad}$*-stable) subvarieties.*

As with all the results in this report, the proof of this lemma will appear elsewhere. However, it is worth taking a moment to discuss the construction of this stratification, since it is will play an important role in the main result of this section, Theorem 2.1.

Fix $y = (z, s) \in Y$. By construction, there is some finite extension F' of F_G such that $y \in Y_{F'}$. Observe that $\pi^{-1}(Y_{F'}) = X_{F'}$ and that the projection $\pi_{F'} : X_{F'} \to Y_{F'}$, given by $(z, s, N) \mapsto (z, s)$, is a morphism of algebraic varieties. Thus, $\pi^{-1}(y)$ is a closed subvariety in X. The proof of Lemma 2.4 shows that

$$\pi^{-1}(z, s) = \{(z, s, N) \in X_{F'} \mid N \in Z_{\check{\mathfrak{g}}}(z)_{s,q}\}$$

where $Z_{\check{\mathfrak{g}}}(z)_{s,q}$ is the q-eigenspace of the semisimple automorphism of the Lie algebra of

$$Z_{\check{G}}(z) = \{g \in \check{G} \mid \ z(\sigma)\ ({}^{\sigma}g)\ z(\sigma)^{-1} = g,\ \forall \sigma \in I_{F'/F}\}$$

given by $s \times \mathrm{Fr} : N \mapsto s\ ({}^{\mathrm{Fr}}N)\ s^{-1}$. Observe that $Y_{F'}$ is a subvariety of $|I_{F'/F}|$-copies of $\check{G}$, on which $\check{G}$ acts, component-wise, by conjugation. The proof of Lemma 2.4 also shows that $\check{G}$-orbit $\mathcal{O}_{\check{G}}(y)$ of $y = (z, s) \in Y$ is a subvariety in Y, and provides an isomorphism

$$\pi^{-1}(\mathcal{O}_{\check{G}}(y)) \cong \check{G} \times^{Z_{\check{G}}(y)} \pi^{-1}(y)$$

where

$$Z_{\check{G}}(y) = Z_{\check{G}}(z, s) = \{g \in Z_{\check{G}}(z) \mid \ s\ ({}^{\mathrm{Fr}}g)\ s^{-1} = g\}.$$

Since the varieties $\pi^{-1}(\mathcal{O}_{\check{G}}(y))$ appear in [14] (though without situating them in the geometric parameter ind-variety X) we refer to them as *Vogan varieties*. Since some of the arguments used to prove Lemma 2.4 also appear in Vogan's work. As shown in [14], each Vogan variety is stratified into finitely-many locally closed subvarieties formed by $\check{G}$-orbits. As the proof of Lemma 2.4 shows, each Vogan variety is also $\check{G}_{\Gamma\text{-ad}}$-stable and each Vogan variety is stratified into finitely-many locally closed subvarieties formed by $\check{G}_{\Gamma\text{-ad}}$-orbits. In summary, the geometric parameter ind-variety X is stratified by the $\check{G}$-orbits (resp. $\check{G}_{\Gamma\text{-ad}}$-orbits) in the Vogan varieties appearing in X. That is the content of Lemma 2.4.

2.5 Geometric parameters

The geometrization of Langlands parameters is achieved by introducing the categories

$$\mathsf{Perv}_{\check{G}}(X) := \bigoplus_{\mathcal{O}} \mathsf{Perv}_{\check{G}}(\pi^{-1}(\mathcal{O})) \tag{6}$$

and

$$\mathsf{Perv}_{\check{G}_{\Gamma\text{-ad}}}(X) := \bigoplus_{\mathcal{O}} \mathsf{Perv}_{\check{G}_{\Gamma\text{-ad}}}(\pi^{-1}(\mathcal{O})), \tag{7}$$

where the categorical sum of abelian categories is taken over $\check{G}$-orbits in Y. Objects in this category are *finite* direct sums of perverse sheaves on Vogan varieties. Note that any finite union of Vogan varieties in X form a variety in the geometric parameter ind-variety $X = X_{\mathrm{Fr}}({}^{L}G)$.

Theorem 2.1 *There is a canonical bijection between isomorphism classes of simple objects in the abelian category*

$$\mathsf{Perv}_{\check{G}_{\Gamma\text{-}ad}}(X)$$

and equivalence classes of pairs (ϕ, ϵ) *where* ϕ *is a Langlands parameter and* ϵ *is an irreducible representation of the finite group*

$$\mathcal{S}_\phi := Z_{\check{G}}(\phi)/Z_{\check{G}}(\phi)^0\ Z(\check{G})^\Gamma.$$

Likewise, there is a canonical bijection between isomorphism classes of simple objects in the abelian category

$$\mathsf{Perv}_{\check{G}}(X),$$

and equivalence classes of pairs (ϕ, τ) *where* ϕ *is a Langlands parameter and* τ *is an irreducible representation of the finite group*

$$Z_{\check{G}}(\phi)/Z_{\check{G}}(\phi)^0.$$

Theorem 2.1 (and Lemma 2.4, upon which the theorem depends) is a variation on results due to Vogan; see [14, Cor. 4.6]. Because of this theorem, we refer to $\mathsf{Perv}_{\check{G}_{\Gamma\text{-ad}}}(X)$ as the *category of geometric parameters* and refer to simple objects in $\mathsf{Perv}_{\check{G}_{\Gamma\text{-ad}}}(X)$ as *geometric parameters*. We also refer to $\mathsf{Perv}_{\check{G}}(X)$ as the category of *geometric pure parameters* and refer to simple objects in $\mathsf{Perv}_{\check{G}}(X)$ as *geometric pure parameters*; the use of the term 'pure' in this context will be justified elsewhere. For a simple example of category $\mathsf{Perv}_{\check{G}_{\Gamma\text{-ad}}}(X)$, see Section 5.1.

As mentioned above, there is considerable overlap between the ideas presented in this section and [14]. While the geometric parameter ind-scheme X does not appear in [14], the Vogan varieties do, and the idea of interpreting complete Langlands parameters as equivariant perverse sheaves on Vogan varieties is one of the key ideas in [14], although our group arrived at this idea independently. In particular, if, in the second part of Theorem 2.1, one replaces the ind-variety X by a single Vogan variety and if one also replaces equivalence classes of complete pure Langlands parameters by equivalence classes of complete pure Langlands parameters with given infinitesimal character, then one recovers a result that can also be found in [14].

3 Geometric parameters and Koszul duality

In Section 2 we saw how to geometrize (and categorify) complete Langlands parameters (resp. complete pure Langlands parameters): by Theorem 2.1, isomorphism classes of simple objects in the category $\mathsf{Perv}_{\check{G}_{\Gamma\text{-ad}}}(X_{\mathrm{Fr}}({}^LG))$ (resp. in the category $\mathsf{Perv}_{\check{G}}(X_{\mathrm{Fr}}({}^LG))$ correspond to equivalence classes of complete Langlands parameters (resp. complete pure Langlands parameters).

In this section we sketch an argument, developed during our programme, showing that these categories are Koszul, in a sense made precise below. Here we consider only the category of geometric pure parameters, leaving a treatment of the category of geometric parameters for another time. As mentioned in Section 2.4, the Vogan varieties $\pi^{-1}(\mathcal{O}_{\check{G}}(y))$ appearing in the parameter ind-variety $X_{\mathrm{Fr}}({}^LG)$ lie in distinct components of $X_{\mathrm{Fr}}({}^LG)$. In this section we use Lemma 2.2 to match y with an L-homomorphism $\lambda : W_F \to {}^LG$ and use the notation

$$X^\lambda = \pi^{-1}(\mathcal{O}_{\check{G}}(y))$$

for the Vogan variety determined by the orbit $\mathcal{O}_{\check{G}}(y)$. Consequently, Equation (5) yields a categorical direct sum decomposition

$$\mathsf{Perv}_{\check{G}}(X) = \bigoplus_\lambda \mathsf{Perv}_{\check{G}}(X^\lambda)$$

where the sum is taken over all equivalence classes (for the action of $\check{G}$) of L-homomorphisms $\lambda : W_F \to {}^LG$. In our study of Koszulness, it is therefore enough to treat each summand category, $\mathsf{Perv}_{\check{G}}(X^\lambda)$, separately.

In this section we also wish to emphasise the fact the category under consideration, $\mathsf{Perv}_{\check{G}}(X^\lambda)$, is completely determined by the quasi-split reductive algebraic group $\check{G}$, equipped with an action of $\Gamma = \mathrm{Gal}(\bar{F}/F)$. For this reason we break from the notation of Section 2 and write G for any connected complex reductive group equipped with an action of Γ.

3.1 Overview of Koszul duality

Consider a nonnegatively graded ring $A = \bigoplus_{i \geq 0} A^i$. Given a graded module $M = \bigoplus_{i \in \mathbb{Z}} M^i$, let $M(j)$ be the graded module whose i-th component is given by $M(j)^i = M^{i-j}$. The ring A is said to be *Koszul* if the following conditions hold:

- A^0 is a semisimple ring.
- Regarding A^0 as a graded A-module, we have $\mathrm{Ext}^i(A^0, A^0(j))$ vanishes unless $i = j$.

Under certain finiteness conditions, there is a duality phenomenon that occurs: the graded ring $A^\dagger = \bigoplus_{i \geq 0} \mathrm{Ext}^i(A_0, A_0(i))$ is again Koszul, and there is a natural isomorphism $(A^\dagger)^\dagger \cong A$.

The importance of this notion in representation theory was established by the breakthrough discovery by Beilinson–Ginzburg–Soergel [3] that certain rings related to Lie algebra representations in category $\mathcal{O}$ are Koszul, and, moreover, that the Koszul duals of these rings also admit descriptions in terms of category $\mathcal{O}$. Since then, a number of additional examples of Koszul duality have been established: see, for instance, [1, 6, 13].

3.2 Geometric Koszul duality

Geometric examples of Koszul duality have been particularly important. In the seminal work [3], the authors considered the flag variety X for a reductive group G, stratified by orbits of a Borel subgroup B. They show that the category $\mathsf{Perv}_{B,c}(X)$ of perverse sheaves that are constructible with respect to this stratification is Koszul.

More precisely, they show that $\mathsf{Perv}_{B,c}(X)$ is equivalent to a category of *ungraded* modules over a Koszul ring (whose grading has been forgotten). In order to bring graded phenomena into the geometric setting, one must make use of the richer structure of "mixed geometry": either mixed ℓ-adic perverse sheaves on a variety over a finite field, or mixed Hodge modules on a complex variety. This step is quite delicate: the category of *all* mixed perverse sheaves or mixed Hodge modules is too large, and has unwanted Ext-groups that contradict Koszulity. But suitable modified categories can sometimes play the role of graded modules. In this report, for simplicity, we denote these modified categories (in either the mixed ℓ-adic or Hodge setting) with notation such as "$\mathsf{Perv}^{\mathrm{mix}}_{B,c}(X)$," suppressing technical issues in their definition.

In the case of the flag variety, the category $\mathsf{Perv}^{\mathrm{mix}}_{B,c}(X)$ turns out to be equivalent to its Koszul dual. A far-reaching generalization of this result in the setting of Kač–Moody groups has recently been established by Bezrukavnikov–Yun [5].

A key property of $\mathsf{Perv}^{\mathrm{mix}}_{B,c}(X)$ is that it is equipped with a *de-grading functor* $\varkappa : \mathsf{Perv}^{\mathrm{mix}}_{B,c}(X) \to \mathsf{Perv}_{B,c}(X)$ (see [3, §4.3]) that allows one to make a comparison of Ext-groups between the two categories. Further ingredients in the proof of Koszulity are discussed in Section 3.4 below.

3.3 Aim of the project

We hope to show that the category $\mathsf{Perv}_G(X^\lambda)$ of G-equivariant perverse sheaves on X^λ is "Koszul." As above, this means that a certain "mixed" (ℓ-adic or Hodge) category $\mathsf{Perv}^{\mathrm{mix}}_G(X^\lambda)$ is Koszul, and that there is a de-grading functor $\varkappa : \mathsf{Perv}^{\mathrm{mix}}_G(X^\lambda) \to \mathsf{Perv}_G(X^\lambda)$.

(Note that this is, in general, a smaller category that the category $\mathsf{Perv}_{G,c}(X^\lambda)$ of all perverse sheaves that are constructible with respect to the stratification by G-orbits. In contrast with the flag variety case considered above, the category $\mathsf{Perv}_{G,c}(X^\lambda)$ contains unwanted objects of no representation-theoretic significance.)

3.4 Outline for the proposed research project

Some of the themes that have arisen in previous work on Koszul duality in geometric settings include: pointwise purity and parity vanishing; quasi-hereditary categories; and derived equivalences for the perverse t-structure. Below, we consider these themes in the context of Vogan varieties.

3.4.1 Pointwise purity and parity vanishing

A simple object $L \in \mathsf{MHM}^\diamond(X)$ is said to be *pointwise pure* if, for every orbit $S \subset X$, the restriction $L|_C$ is a pure object of $D^{\mathrm{b}}\mathsf{MHM}(C)$. The close relationship between pointwise purity and Koszul duality has been observed by a number of authors; see, for instance, [4, Remark 4]. It plays a prominent role in [3, 5]. Another key feature is *parity vanishing*: this is the requirement that the cohomology sheaves $H^i(L|_C)$ vanish for all odd i (or perhaps all even i, depending on the dimensions of C and of the support of L). This type of condition holds on the flag variety [8] and on the nilpotent cone [12].

For Vogan varieties, it seems that both properties can be deduced from the work of Lusztig on perverse sheaves on graded Lie algebras [9]. Indeed, Lusztig's motivation seems to have been the study of Vogan varieties, and the precise link between his work and these varieties is likely well understood by experts, but we have been unable to find a thorough account of this link in the literature. Thus, this aspect of the project will be mainly expository; we nevertheless believe it will be useful contribution.

3.4.2 Quasi-hereditary property

Vogan varieties shares the property with the flag variety that the push-forward functors attached to orbits are t-exact. In other words, for an orbit $C \subset X^\lambda$ and a local system E on C, the objects

$$j_!E[\dim C] \qquad \text{and} \qquad j_*E[\dim C]$$

(where $j : C \to X^\lambda$ is the inclusion map) are perverse. These objects, called *standard* and *costandard* perverse sheaves, respectively, satisfy at least the first five of the six axioms in [3, §3.2]. By an argument of Ringel explained in *loc. cit.*, one can then deduce that the categories $\mathsf{Perv}_G(X)$ and $\mathsf{Perv}_G^{\mathrm{mix}}(X)$ have enough projectives and injectives.

For the flag variety, the next step is to establish a derived equivalence $D^{\mathrm{b}}\mathsf{Perv}(X) \to D^{\mathrm{b}}(X)$, using a key Ext^2-vanishing property for standard and costandard objects. (This is the sixth axiom in [3, §3.2].) Unfortunately, the relevant Ext^2-group can be nonzero on the Vogan variety, and indeed, the derived category $D^{\mathrm{b}}\mathsf{Perv}_G(X^\lambda)$ is not, in general, equivalent either to $D^{\mathrm{b}}(X^\lambda)$ or to the G-equivariant derived category $D^{\mathrm{b}}_G(X)$.

3.4.3 Realization functor

To rephrase the last observation: the Ext-groups in $\mathsf{Perv}_G^{\mathrm{mix}}(X)$ cannot directly be identified with Hom-groups in any "geometric" derived category. Thus, a study of these Ext-groups is the most difficult aspect of the project.

A rather general construction [2] gives us a t-exact functor $\rho : D^{\mathrm{b}}\mathsf{Perv}_G^{\mathrm{mix}}(X) \to D^{\mathrm{b}}_{G,\mathrm{m}}(X)$, called a *realization functor*. This functor induces an isomorphism on Ext^1-groups and an injective map on Ext^2-groups, but beyond that, little can be said in general. In our setting, we hope to use parity-vanishing phenomena in $\mathsf{Perv}_G^{\mathrm{mix}}(X)$ to establish a tighter relationship between the two triangulated categories, and ultimately to deduce the Koszulity of $\mathsf{Perv}_G^{\mathrm{mix}}(X)$ from known Ext-vanishing facts in $D^{\mathrm{b}}_{G,\mathrm{m}}(X)$.

3.4.4 Identifying the Koszul dual

As noted above, many of the most celebrated results on the theme of Koszul duality have two parts: they establish the Koszulity of some ring arising in representation theory, and they identify the Koszul dual ring as an object having representation-theoretic significance on its own. Unfortunately, for the moment, we do not know of a suitable candidate category that might be the Koszul dual of the (putatively) Koszul category $\mathsf{Perv}_G(X^\lambda)$. We hope to study this question through examples in the future.

4 Geometric characters for p-adic tori

During our programme we understood how to geometrize admissible characters of unramified, induced p-adic tori, generalising earlier work on geometrization of admissible characters of $F^\times$.

4.1 Classical geometrization

Let us begin by recalling classical geometrization. For the moment, let G be a connected, commutative algebraic group over $\mathbb{F}_q$. In this context, geometrization is well-understood: use the Lang morphism for G to define $\pi_1(G,\bar{e}) \to G(\mathbb{F}_q)$ and thus convert each character $\chi : G(\mathbb{F}_q) \to \bar{\mathbb{Q}}_\ell$ into a character of the fundamental group $\pi_1(G,\bar{e}) \to \bar{\mathbb{Q}}_\ell^\times$. In this way we define an (isomorphism class of an) ℓ-adic local system $\mathcal{L}_\chi$ on the etale site of G, from the character χ. By base change, the local system $\mathcal{L}_\chi$ defines a local system $\bar{\mathcal{L}}_\chi$ on $\bar{G} := G \otimes_{\mathbb{F}_q} \bar{\mathbb{F}}_q$ equipped with an isomorphism $\phi_\chi : \mathrm{Fr}^*\bar{\mathcal{L}}_\chi \to \bar{\mathcal{L}}_\chi$ such that the trace of Frobenius, $t^{\mathcal{L}_\chi}_{\mathrm{Fr}} : G(\mathbb{F}_q) \to \bar{\mathbb{Q}}_\ell^\times$, defined by the diagramme

$$
\begin{array}{ccc}
(\mathrm{Fr}^*\bar{\mathcal{L}}_\chi)_{\bar{g}} & \xrightarrow{(\phi_\chi)_{\bar{g}}} & (\bar{\mathcal{L}}_\chi)_{\bar{g}} \\
\text{canonical isomorphism} \Big| & \nearrow_{t^{\mathcal{L}_\chi}_{\mathrm{Fr}}(g)} & \\
(\bar{\mathcal{L}}_\chi)_{\bar{g}}, & &
\end{array}
$$

recovers the character $\chi : G(\mathbb{F}_q) \to \bar{\mathbb{Q}}_\ell$. It is easy to characterise the local systems on G that arise in this manner: if $\mathcal{L}$ is a local system on G and if there is an isomorphism

$$\mathcal{L} \boxtimes \mathcal{L} \cong m^*\mathcal{L}, \tag{8}$$

where $m : G \times G \to G$ is the multiplication map for G, then $t^{\mathcal{L}}_{\mathrm{Fr}} : G(\mathbb{F}_q) \to \bar{\mathbb{Q}}_\ell^\times$ is a character, and all ℓ-adic characters of $G(\mathbb{F}_q)$ are produced in this way. The final miracle is this: if $\mathcal{L}$ and $\mathcal{L}'$ both admit isomorphisms as in (8) and if $t^{\mathcal{L}}_{\mathrm{Fr}} = t^{\mathcal{L}'}_{\mathrm{Fr}}$, then $\mathcal{L} \cong \mathcal{L}$. Consequently, the trace of Frobenius $\mathcal{L} \mapsto t^{\mathcal{L}}_{\mathrm{Fr}}$ defines an isomorphism of groups

$$\left\{ \begin{array}{c} \text{local systems } \mathcal{L} \text{ on } G \\ \exists\, \mathcal{L} \boxtimes \mathcal{L} \cong m^*\mathcal{L} \end{array} \right\}_{/\mathrm{iso}} \longrightarrow \mathrm{Hom}_{\mathrm{grp}}(G(\mathbb{F}_q), \bar{\mathbb{Q}}_\ell)$$

These facts are well-known. Since isomorphism classes of local systems appearing on the left-hand side above correspond to characters of $G(\mathbb{F}_q)$, it is common to refer to such local systems as *character sheaves* on G. We will revisit this definition in the next two sections.

4.2 Geometric characters for commutative groups

In order to justify the claims made above, one must make crucial use of the fact that G is connected and finitely generated over $\mathbb{F}_q$, in that section. But we wish to loosen these conditions on G to admit non-connected, commutative group schemes over $\mathbb{F}_q$. As we understood during our programme, for that we require a new definition, given here.

Let G be a commutative group scheme over $\mathbb{F}_q$. A *geometric character* on G is an ℓ-adic local system $\bar{\mathcal{L}}$ on $\bar{G} := G \times_{\mathrm{Spec}(\mathbb{F}_q)} \mathrm{Spec}\left(\bar{\mathbb{F}}_q\right)$, with three supplementary structures:

1. an isomorphism $\mu : m^*\bar{\mathcal{L}} \longrightarrow \bar{\mathcal{L}} \boxtimes \bar{\mathcal{L}}$;
2. a rigidification $r : \bar{\mathcal{L}}_{\bar{e}} \xrightarrow{\cong} \bar{\mathbb{Q}}_\ell$ at the geometric point $\bar{e}$ of $\bar{G}$ lying above the origin e of G;
3. an isomorphism $\phi : \mathrm{Fr}^*\bar{\mathcal{L}} \to \bar{\mathcal{L}}$.

The quartuple $\mathcal{L} = (\bar{\mathcal{L}}, \mu, r, \phi)$ must also satisfy some natural compatability conditions which we omit from this report. It is a consequence of this definition that if $\mathcal{L} = (\bar{\mathcal{L}}, \mu, r, \phi)$ is a geometric character then $\bar{\mathcal{L}}$ is an irreducible local system on $\bar{G}$. We write $\mathrm{GC}(G)$ for the additive category generated by geometric characters on G, with obvious definition for morphisms. (This category will be treated carefully in one of the papers based on our programme.) Simple objects in $\mathrm{GC}(G)$ are geometric characters on G.

4.3 Comparison with character sheaves

If we return to the case when G is a connected, commutative algebraic group over $\mathbb{F}_q$, then the forgetful functor $(\bar{\mathcal{L}}, \mu, r, \phi) \mapsto (\bar{\mathcal{L}}, \phi)$ takes geometric characters on G to character sheaves on G, as defined in Section 4.1. While this functor is full and essentially surjective, it is not faithful. If $G = T$ is also an algebraic torus, and $(\bar{\mathcal{L}}, \mu, r, \phi)$ is a geometric character such that $\bar{\mathcal{L}}^n = \bar{\mathbb{Q}}_\ell$ for some positive integer n then $\bar{\mathcal{L}}$ is a character sheaf on $\bar{T}$, as defined by Lusztig, and all Frobenius-stable character sheaves on $\bar{T}$ arise in this manner.

4.4 Greenberg of Neron

In this section we introduce a geometric space needed to geometrize admissible characters of p-adic tori. Let us set some notation and briefly recall the filtration of admissible characters by depth. Let F be a non-Archimedean local field with residual field $\mathbb{F}_q$ and let T be an algbraic torus over F. Let $\chi : T(F) \to \bar{\mathbb{Q}}_\ell^\times$ be an admissible character. Then the depth of χ is given by

$$\inf\{r \geq 0 \,|\, \forall s > r,\ T(F)_s \subset \ker(\chi)\},$$

where the filtration

$$T(F) \supseteq T(F)_0 \supseteq \cdots \supseteq T(F)_s \supseteq \cdots,$$

is defined in [10] or equally in [11]. Let $\mathrm{Hom}_d(T(F), \bar{\mathbb{Q}}_\ell^\times)$ be the group of ℓ-adic characters of $T(F)$ with depth less than or equal to d. In the next few sections we will explain how to geometrize elements of the group $\mathrm{Hom}_d(T(F), \bar{\mathbb{Q}}_\ell^\times)$, along the lines of Section 4.1

4.4.1 Neron models

The Neron model for T is a smooth group scheme T_R locally of finite type over R with generic fibre T, such that for every smooth group scheme Y over R, the canonical function

$$\mathrm{Hom}_R(Y, T_R) \xrightarrow[\substack{\text{base change along} \\ \mathrm{Spec}(F)\to\mathrm{Spec}(R)}]{\substack{\text{restriction to} \\ \text{generic fibres}}} \mathrm{Hom}_F(Y \times_S \mathrm{Spec}(F), T)$$

is bijective; in particular, $T_R(R) \cong T(F)$. Neron models exist for all p-adic tori, and are unique up to isomorphism.

4.4.2 Greenberg transform

Let A be an Artin local ring; let k be its residual field. Marvin Greenberg [7] has defined a functor

$$\big(\mathrm{Sch}_{/A}\big)_{\text{lft}} \xrightarrow{\substack{\text{Greenberg} \\ \text{transform}}} \big(\mathrm{Sch}_{/k}\big)_{\text{lft}} \qquad\qquad X \longmapsto \mathrm{Gr}(X)$$

with a number of agreeable properties, including, for every X and Y, locally of finite type over A: a canonical bijection $X(A) \cong \mathrm{Gr}(X)(k)$; if X if affine (resp. smooth, finite etale) then so is $\mathrm{Gr}(X)$; if $X \to Y$ is an open subscheme (resp. a closed subscheme) then so is $\mathrm{Gr}(X) \to \mathrm{Gr}(Y)$.

4.5 Geometrization of characters of bounded depth

During our programme we put together a proof of the following result.

Theorem 4.1 *Let T be an induced, unramified torus over F. For each $d \in \mathbb{N}$, let T_d be the Greenberg transform of $T_R \times_{\mathrm{Spec}(R)} \mathrm{Spec}\big(R/\mathfrak{p}^{d+1}\big)$, where T_R is a Neron model for T. The trace of Frobenius defines an isomorphism of groups from isomorphism classes of simple objects in* $\mathrm{GC}(T_d)$ *to* $\mathrm{Hom}_d(T(F), \bar{\mathbb{Q}}_\ell^\times)$.

4.6 Admissible geometric characters

Consider the commutative pro-algebraic group $T_{\mathbb{F}_q} := \varprojlim_{d\in\mathbb{N}} T_d$. Amazingly, this limit exists in the category of groups schemes over $\mathbb{F}_q$. It comes equipped with a canonical isomorphism

$$T_{\mathbb{F}_q}(\mathbb{F}_q) \cong T(F).$$

A geometric character on $T_{\mathbb{F}_q}$ is *admissible* if there is an integer $d \in \mathbb{N}$ and a geometric character on T_d such that $\mathcal{L} = f^*\mathcal{L}_d$ where $f : T_{\mathbb{F}_q} \to T_d$ is the obvious map. Let $\mathrm{GC}_{\text{ad}}(T_{\mathbb{F}_q})$ be the category of admissible geometric characters on $T_{\mathbb{F}_q}$. For a simple example of category $\mathrm{GC}_{\text{ad}}(T_{\mathbb{F}_q})$, see Section 5.2.

The main result of Section 4 is the following theorem, which follows from Theorem 4.1, the definition above, and a small amount of extra work.

Theorem 4.2 *Let T be an induced, unramified torus over F. The trace of Frobenius defines an isomorphism of groups, compatible with the fitration by depth on both sides, from isomorphism classes of simple objects in* $\mathrm{GC}_{ad}(T_{\mathbb{F}_q})$ *to* $\mathrm{Hom}_{ad}(T(F), \bar{\mathbb{Q}}_\ell^\times)$.

5 Geometric reciprocity

In Section 2 we saw how to geometrize Langlands parameters for quasisplit groups G over F by introducing the category $\mathsf{Perv}_{\check{G}_{\Gamma\text{-ad}}}(X_{\mathrm{Fr}}({}^LG))$ and studying isomorphism classes of simple objects in this category. In Section 4 we saw how to geometrize admissible characters of unramified, induced T tori over F by introducing the category $\mathrm{GC}_{\mathrm{ad}}(T_{\mathbb{F}_q})$ and studying its simple objects. This raises the question: supposing $G = T$, is there a functor from $\mathrm{GC}_{\mathrm{ad}}(T_{\mathbb{F}_q})$ to $\mathsf{Perv}_{\check{G}_{\Gamma\text{-ad}}}(X_{\mathrm{Fr}}({}^LT))$ that defines the reciprocity map for T by restriction to (isomorphism classes of) simple objects? In this section we answer that question when $G = T = \mathbb{G}_{\mathbf{m},F}$.

5.1 Geometric parameters for $\mathbb{G}_{\mathbf{m},F}$

Set $G = \mathbb{G}_{\mathbf{m},F}$. Then $\check{G} = \mathbb{G}_{\mathbf{m},\bar{\mathbb{Q}}_\ell}$ and $\Gamma_G = 1$ so ${}^LG = \mathbb{G}_{\mathbf{m},\bar{\mathbb{Q}}_\ell}$. Recall that the definition of Y (see Section 2.2) and X (see Section 2.3) require the choice of a lift Fr of Fr_q. We will revisit this choice in Section 5.3. Observe that $\check{G}$ acts trivially on Z (see Section 2.1) and Y and X. The ind-variety Y is a totally disconnected space. For each admissible $\lambda : W_F \to \bar{\mathbb{Q}}_\ell^\times$, the corresponding Vogan variety $X^\lambda = \pi^{-1}(\mathcal{O}_{\check{G}}(y))$ is $\{y\}$, where $y \in Y$ is the point corresponding to λ under Lemma 2.2. So $X = Y$ is a totally disconnected space with trivial $\mathbb{G}_{\mathbf{m},\bar{\mathbb{Q}}_\ell}$-action. With reference to Section 2.5, $\mathsf{Perv}_{\check{G}}(X) = \mathsf{Perv}(Y)$; thus,

$$\mathsf{Perv}_{\check{G}}(X_{\mathrm{Fr}}({}^LG)) = \bigoplus_{y\in Y} \mathsf{Perv}(\{y\}) \cong \bigoplus_{\lambda\in \mathrm{Hom}_{\mathrm{ad}}(W_F,\bar{\mathbb{Q}}_\ell^\times)} \mathsf{Perv}(\mathrm{Spec}\,(\bar{\mathbb{Q}}_\ell)).$$

It follows that $\mathsf{Perv}_{\check{G}}(X_{\mathrm{Fr}}({}^LG))$ is equivalent to the category of finite-dimensional admissible ℓ-adic representations of W_F^{ab}:

$$\mathsf{Perv}_{\check{G}}(X_{\mathrm{Fr}}({}^LG)) \cong \mathrm{Rep}_{\bar{\mathbb{Q}}_\ell,\mathrm{ad}}(W_F^{\mathrm{ab}}). \tag{9}$$

Under this equivalence, simple objects in $\mathsf{Perv}_{\check{G}}(X_{\mathrm{Fr}}({}^LG))$ correspond to one-dimensional representations of W_F. The equivalence above induces a bijection between isomorphism classes of simple objects in $\mathsf{Perv}_{\check{G}}(X_{\mathrm{Fr}}({}^LG))$ and admissible characters of W_F:

$$\mathrm{Hom}_{\mathrm{ad}}(W_F, \bar{\mathbb{Q}}_\ell^\times) \longrightarrow \mathrm{simp.\,obj}\,\mathsf{Perv}_{\check{G}}(X_{\mathrm{Fr}}({}^LG))_{/\mathrm{iso}}$$

defined by $\lambda \mapsto (\bar{\mathbb{Q}}_\ell)_{\{y\}}$, where y corresponds to λ under Lemma 2.2 and $(\bar{\mathbb{Q}}_\ell)_{\{y\}}$ is the sheaf on $X = Y$ supported at $\{y\}$ where it is the constant sheaf $\bar{\mathbb{Q}}_\ell$. This is a special case of Theorem 2.1.

5.2 Geometric characters for $\mathbb{G}_{\mathbf{m},F}$

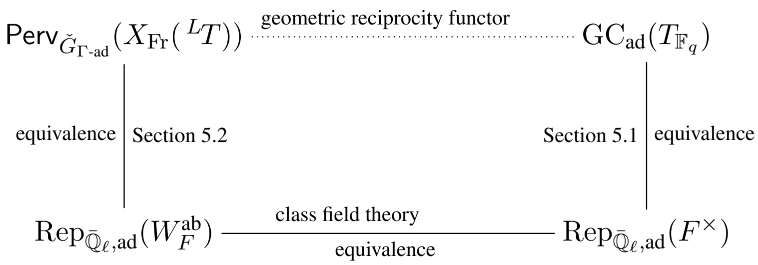
Set $T = \mathbb{G}_{\mathbf{m},F}$, so $T_{\mathbb{F}_q}$ is the Greenberg transform of the Neron model of $\mathbb{G}_{\mathbf{m},F}$. Theorem 4.2 can be strengthened to an equivalence of categories

$$\mathrm{GC}_{\mathrm{ad}}(T_{\mathbb{F}_q}) \cong \mathrm{Rep}_{\bar{\mathbb{Q}}_\ell,\mathrm{ad}}(T(F)) \tag{10}$$

between the category of admissible geometric parameters on $T_{\mathbb{F}_q}$ and the category of finite-dimensional, admissible ℓ-adic representations of $T(F)$. This equivalence (and the proof of Theorem 4.2) is too complicated to describe in this report, but is currently being prepared for publication.

5.3 Geometric reciprocity for non-Archimedean local fields

Class field theory provides an isomorphism $W_F^{\mathrm{ab}} \cong F^\times$ and thus an equivalence of categories between $\mathrm{Rep}_{\bar{\mathbb{Q}}_\ell,\mathrm{ad}}(W_F^{\mathrm{ab}})$ and $\mathrm{Rep}_{\bar{\mathbb{Q}}_\ell,\mathrm{ad}}(F^\times)$. In light of Sections 5.1 and 5.2, this determines an equivalence between $\mathsf{Perv}_{\check{G}_{\Gamma\text{-ad}}}(X_{\mathrm{Fr}}({}^LT))$ and $\mathrm{GC}_{\mathrm{ad}}(T_{\mathbb{F}_q})$, when $T = \mathbb{G}_{m,F}$.



During the last day of our programme we discussed a geometric construction which leads to a functor directly from $\mathsf{Perv}_{\check{G}_{\Gamma\text{-ad}}}(X_{\mathrm{Fr}}({}^LT))$ to $\mathrm{GC}_{\mathrm{ad}}(T_{\mathbb{F}_q})$ without recourse to class field theory. We (presumptuously) call this a geometric reciprocity functor. If it agrees with the equivalence given by class field theory, such a functor would actually recover the isomorphism $W_F^{\mathrm{ab}} \cong F^\times$ from our geometric reciprocity functor. This is now a topic of research in progress.

References

[1] E. Backelin, *Koszul duality for parabolic and singular category $\mathcal{O}$*, Represent. Theory **3** (1999), 139–152.

[2] A. Beĭlinson, *On the derived category of perverse sheaves*, F-theory, arithmetic and geometry (Moscow, 1984–1986), Lecture Notes in Mathematics, vol. 1289, Springer-Verlag, Berlin, 1987, pp. 27–41.

[3] A. Beĭlinson, V. Ginzburg, and W. Soergel, *Koszul duality patterns in representation theory*, J. Amer. Math. Soc. **9** (1996), 473–527.

[4] R. Bezrukavnikov, *Cohomology of tilting modules over quantum groups and t-structures on derived categories of coherent sheaves*, Invent. Math. **166** (2006), 327–357.

[5] R. Bezrukavnikov and Z. Yun, *On Koszul duality for Kac–Moody groups*, arXiv:1101.1253.

[6] T. Braden, *Koszul duality for toric varieties*, Trans. Amer. Math. Soc. **359** (2007), 385–415.

[7] M. Greenberg, *Schemata over local rings. II*, Ann. of Math. (2) **78** (1963), 256-266.

[8] D. Kazhdan and G. Lusztig, *Schubert varieties and Poincaré duality*, Geometry of the Laplace operator (Proc. Sympos. Pure Math., Univ. Hawaii, Honolulu, Hawaii, 1979), Proc. Sympos. Pure Math., XXXVI, Amer. Math. Soc., Providence, RI, 1980, pp. 185–203.

[9] G. Lusztig, *Study of perverse sheaves arising from graded Lie algebras*, Adv. Math. **112** (1995), 147–217.

[10] A. Moy and G. Prasad, *Jacquet functors and unrefined minimal F-types*, Comment. Math. Helv. **71** No. 1 (1996), 98–121.

[11] P. Schneider and U. Stuhler, *Representation theory and sheaves on the Bruhat-Tits building*, Inst. Hautes Études Sci. Publ. Math. **85** (1997), 97–191.

[12] T. A. Springer, *A purity result for fixed point varieties in flag manifolds*, J. Fac. Sci. Univ. Tokyo Sect. IA Math. **31** (1984), 271–282.

[13] S. Riche, *Koszul duality and modular representations of semisimple Lie algebras*, Duke Math. J. **154** (2010), 31–134.

[14] D. A. Vogan, *The local Langlands conjecture*, Representation theory of groups and algebras, Contemp. Math., vol. 145, Amer. Math. Soc., Providence, RI, 1993, pp. 305–379.