The $\overline{\partial}$-Method: Inverse Scattering, Nonlinear Waves, and Random Matrices

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1 Overview of the Field

The $\overline{\partial}$-method in inverse scattering enables, in principle at least, an explicit solution to certain completely integrable, dispersive nonlinear equations in two space dimensions. The $\overline{\partial}$-method also has potential application to problems arising in the study of random matrix models and orthogonal polynomials in the plane. The following two examples indicate the directions open to exploration and the nature of the common, underlying mathematical problem.

The Davey-Stewartson (DS) II equation. The DS II equation is a completely integrable model that describes monochromatic, weakly nonlinear waves in shallow water. The solution u gives the (complex) amplitude $u(x,y,t)$ of such a wave. The defocussing DS II equation is the system ($\epsilon > 0$ is a parameter)

$$i\epsilon q_t + 2\epsilon^2 \left(\partial^2 + \overline{\partial}^2\right) q + (g+\overline{g})q = 0 \quad \text{and} \quad \overline{\partial} g = -\partial\left(|q|^2\right), \quad \partial := \frac{\partial}{\partial z}, \quad \overline{\partial} := \frac{\partial}{\partial \overline{z}}, \quad z := x+iy. \tag{1}$$

For the initial-value problem we fix an initial condition: $q(x,y,0) = q_0(x,y)$. The elliptic equation for g is to be solved subject to the condition that $g \to 0$ as $x, y \to \infty$.

To solve by inverse scattering, suppose that $q_0 \in \mathcal{S}(\mathbb{R}^2)$. There is a nonlinear map $\mathcal{R}$ taking q_0 to a function $r_0 \in \mathcal{S}(\mathbb{R}^2)$, the scattering transform. The solution is constructed by solving the $\overline{\partial}$-problem

$$\overline{\partial}_k \nu(\kappa,\sigma) = \frac{1}{2}\overline{r_0(\kappa,\sigma)} e^{-2iS(\kappa,\sigma)/\epsilon} \sigma_1 \overline{\nu(\kappa,\sigma)}, \quad \sigma_1 := \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \overline{\partial}_k := \frac{\partial}{\partial \overline{k}}, \quad k = \kappa + i\sigma,$$

where ν is a 2-component vector that tends to $(1,0)^{\mathsf{T}}$ as $k \to \infty$, and where $S(\kappa,\sigma) := \Im\{kz\} + 2t\Re\{k^2\}$ is a real-valued phase function. The solution $u(x,y,t)$ is then computed from the reconstruction formula

$$q(x,y,t) = 2 \lim_{k\to\infty} \overline{k}\, \overline{\nu_2(\kappa,\sigma)} = \frac{1}{\pi\epsilon} \iint_{\mathbb{R}^2} e^{2iS(\kappa,\sigma)/\epsilon} r_0(\kappa,\sigma) \nu_1(\kappa,\sigma)\, d\kappa\, d\sigma.$$

Moreover, the problem of computing the scattering transform map $\mathcal{R}$ can be formulated also as a quite similar $\overline{\partial}$-problem, but this time set in the complex z-plane. Thus, the technical core of this problem is the analysis of a $\overline{\partial}$-problem involving parameters z, t, and ϵ that enter in a singular fashion.

Normal Matrix Models. The joint probability measure of complex eigenvalues $z_1, \ldots, z_N$ for a unitary-invariant ensemble of normal random $N \times N$ matrices can be taken in the form

$$P(z_1,\ldots,z_N) dm(z_1)\cdots dm(z_N) := \frac{1}{Z_N} \prod_{j\neq k} |z_j - z_k|^2 \prod_{n=1}^{N} e^{-NV(z_n)}\, dm(z_n) \tag{2}$$

where $V : \mathbb{C} \to \mathbb{R}$ is a (confining) potential, Z_N is the normalization constant (partition function), and $dm(z)$ is Lebesgue measure in the z-plane. As in Hermitian matrix models, the statistics of eigenvalues may be studied through the associated monic orthogonal polynomials $\{P_n\}_{n=0}^{\infty}$ defined by:

$$\int_{\mathbb{C}} P_n(z)\overline{P_m(z)}e^{-NV(z)}\,dm(z) = h_n\delta_{mn}, \quad h_n > 0, \quad P_n(z) = z^n + \cdots . \tag{3}$$

These orthogonal polynomials can be equivalently obtained via the solution of a matrix $\overline{\partial}$-problem in which the matrix size N and degree n enter as parameters. Indeed, if $Y_n(z,\overline{z})$ denotes the 2×2 matrix that satisfies

$$\overline{\partial} Y_n(z,\overline{z}) = \overline{Y_n(z,\overline{z})}\begin{pmatrix} 0 & -e^{-NV(z)} \\ 0 & 0 \end{pmatrix}$$

then $P_n(z) = Y_{n,11}(z,\overline{z})$. To analyze the asymptotic distribution of eigenvalues as $N \to \infty$ along with the fine structure of local correlations, one needs information about $P_n(z)$ for n as large as N, and in this setting the $\overline{\partial}$-problem is *in principle* well-suited to asymptotic analysis because the large parameters N and n appear *explicitly* in the conditions on Y_n. Thus, the details of eigenvalue statistics can be worked out if the $\overline{\partial}$-problem can be analyzed accurately in the limit $N, n \to \infty$. Again, the technical core of this problem is the accurate asymptotic analysis of a $\overline{\partial}$-problem involving large parameters.

The $\overline{\partial}$ method is, potentially, as powerful a tool in these sets of related problems as the Riemann-Hilbert method has proven to be in the study of completely integrable systems "in one space dimension" such as the KdV, mKdV and NLS equations, random matrix distributions for symmetric, orthogonal, and unitary matrices, and orthogonal polynomials on the circle or the line. The purpose of this Focussed Research Group was to bring together researchers in completely integrable systems and dispersive equations, together with experts in harmonic analysis and PDE, to better develop the $\overline{\partial}$-methods.

2 Recent Developments and Open Problems

Dispersive nonlinear partial differential equations in two dimensions have been extensively studied in recent years, both by PDE methods and inverse scattering methods. The former methods yield much stronger local existence and well-posedness results than can be expected from inverse scattering methods, but the latter promise to yield much more detailed behavior on semi-classical asymptotics and long-time behavior if parameter dependence of solutions to the underlying $\overline{\partial}$ problems can be controlled. Model equations include the Davey-Stewartson (DS), Kadomtsev-Petviashvilli (KP), and Novikov-Veselov (NV) equations. The $\overline{\partial}$-method was developed by Fokas-Ablowitz [1, 2, 3] and Beals-Coifman [5, 6, 7] Its application to inverse scattering has been studied by many authors including Ablowitz, Fokas and their collaborators (see the monograph [4] for references up to 1990), and Grinevich, Grinevich-Manakov, and Grinevich-Novikov [15, 16, 17, 18, 19, 20, 21, 22]. Mathematically rigorous treatments of the scattering maps for the DS and NV equation include those of Brown [9], Sung [31], Lassas-Mueller-Siltanen [24], Nachman [28], Perry [29, 30]. A major challenge involves the classification and analysis of singularities of scattering maps that lead to soliton solutions, blow-up in finite time, and other dynamical phenomena. Another major challenge involves understanding semiclassical limits of two-dimensional dispersive equations, as described in greater detail below. The semiclassical method has yielded insights into the dynamics of dispersive equations in one dimension: see, for example, the recent work of Buckingham-Miller [11, 12]. We expect that similar insights will be gained from the study of semiclassical limits, for example, in the DS II equation.

Two-dimensional random matrix models and orthogonal polynomials in the plane have been the subject of intensive investigation in recent years. Its and Takhtajan [23] outlined a program for studying large-N asymptotics of orthogonal polynomials and random matrix models by $\overline{\partial}$ methods. Elbau and Felder [14] studied certain perturbations of the Gaussian case $V(z) = z^2$ (cf. (2)) and showed that the density of eigenvalues converges, in the limit $N \to \infty$, to the characteristic function of an explicit bounded region in the complex plane. Bagh, Bertola, Lee, and McLaughlin [8] carried out a complete analysis of certain random matrix models by reducing the underlying $\overline{\partial}$ problem for the orthogonal polynomials to a Riemann-Hilbert problem. An understanding of the full $\overline{\partial}$ problem remains elusive.

3 Presentation Highlights

The FRG began with presentations by Peter Perry on inverse scattering for the Davey-Stewartson equation, based on [29] and by Ken McLaughlin introducing random matrix models. The purpose of these lectures was to establish a knowledge base among all participants in the two key areas of research considered. Samuli Siltanen lectured on electrical impedance imaging, the inverse scattering transform, and $\overline{\partial}$ methods. Michael Christ gave an illuminating lecture on the Brascamp-Lieb-type inequalities which underlie much of the progress in analysis of scattering maps in [9] and [29] (see the Appendix to [29], written by Michael Christ, for details and references to the literature).

McLaughlin and Miller led an ongoing discussion on semi-classical analysis for the defocussing DS II equation (1) with $q = q(x, y, t, \varepsilon)$ a complex-valued function having initial data of the form

$$q(x, y, 0) = A(x, y) \exp\left(iS(x, y)/\varepsilon\right)$$

The problem is to study solutions in the limit $\varepsilon \to 0$. Passing to the inverse scattering method, one sees that the solution to the semiclassical DS II problem is obtained in two steps.

First, one solves the following $\overline{\partial}$-problem for $\mu = (\mu_1, \mu_2)^T$ to compute the scattering transform $r$of the initial data:

$$\begin{aligned} \varepsilon\overline{\partial}\mu &= \frac{q}{2} \exp\frac{1}{\varepsilon}\left(\overline{kz} - kz\right)\sigma_1\overline{\mu} \\ \lim_{|z|\to\infty} \mu(z, k) &= (1, 0)^T. \end{aligned} \tag{4}$$

and recovers the scattering transform r from the formula

$$r(k) = 2 \lim_{|z|\to\infty} z\mu_2(z, k)$$

The scattering transform of the full solution then evolves according to

$$r(k, t) = r(k, 0) \exp\frac{2it}{\varepsilon}\left(k^2 + \overline{k}^2\right).$$

Second, to recover $q(x, y, t, \varepsilon)$, one solves the $\overline{\partial}$ problem for $\nu = (\nu_1, \nu_2)^T$:

$$\begin{aligned} \varepsilon\overline{\partial}\nu &= \frac{\overline{r}}{2} \exp\frac{1}{\varepsilon}\left(\overline{kz} - kz\right)\sigma_1\overline{\nu} \\ \lim_{|k|\to\infty} \nu(z, k) &= (1, 0)^T. \end{aligned} \tag{5}$$

One recovers the potential from the formula

$$q(x, y, t) = 2 \lim_{|k|\to\infty} \overline{k}\overline{\nu_2}.$$

Thus, analytically, one needs to understand the small-ε limit of the $\overline{\partial}$-problems (4) and (5).

4 Scientific Progress Made

- Peter Miller and Ken McLaughlin initiated a study of semiclassical limits for the defocussing Davey-Stewartson II equation. Subsequently, Sarah Hamilton, a postdoctoral research fellow working with Samuli Siltanen, carried out numerical computations which show some interesting features of the semiclassical limits. One of the themes of the upcoming conference and workshop at the University of Kentucky will be the analytical study of semiclassical limits, as a direct outgrowth of these discussions.

- Ken McLaughlin led discussions on the analysis of the $\overline{\partial}$ probem for 2D orthogonal polynomials. By summarizing known results obtained via reductions to Riemann-Hilbert methods for special examples, team members developed a colection of approximations which should be valid for a large family of orthogonal polynomials, and attempted to arrive at a small-norm $\overline{\partial}$ probem amendable to known analytical methods.

- Peter Perry, in discussions with Peter Miller, Ken McLaughlin, and Samuli Siltanen, studied soliton solutions to the focussing DS II equation and the more general problem of so called *exceptional sets* where the scattering transforms have singularities. He initiated a study of determinants and soliton solutions for the Davey-Stewartson II equation. This led to a collaborative paper with colleague Russell Brown and graduate student Michael Music on determinants in inverse scattering [10]. One of the key insights that led to this paper–that the determinant itself satisfies a $\overline{\partial}$-equation that allows the determinant to be computed in terms of scattering data–originated in discussions at the 2012 FRG.

5 Outcome of the Meeting

The following publications have already resulted in part from discussions at this meeting: [10, 13].

The following subsequent meetings have been organized in part by participants in the FRG (participants in the FRG in boldface type):

- Exceptional Circle Helsinki Workshop, University of Helsinkii,[1] August 12-16, 2013 . Organized by **Samuli Siltanen** and Sarah Hamilton. This meeting involved eleven researchers from the United States and Europe.
- Conference and Workshop on Scattering and Inverse Scattering in Multi-Dimensions, University of Kentucky, [2] May 15-23, 2014, co-sponsored by the National Science Foundation, the Institute for Mathematics and its Applications, and the University of Kentucky. Organized by **Ken McLaughlin**, **Peter Miller**, and **Peter Perry**. This conference and workshop will include lectures by **Kari Astala** (Helsinki), James Colliander (Toronto), **Ken McLaughlin** (Arizona), **Peter Miller** (Michigan), **Peter Perry** (Kentucky), Andreas Stahel (Bern University), Paolo Santini (Rome), and Jean-Claude Saut (Paris-Sud). Approximately 40 participants,including 20 graduate and postdoctoral students, are expected.

The May 2014 meeting in the University of Kentucky will have the following research foci: (1) Semiclassical limits of $\overline{\partial}$-problems and dispersive equations, (2) Eigenvalue distributions of random normal matrices, (3) Direct scattering and exceptional sets, (4) Inverse scattering and exceptional sets, (5) One-dimensional limits of two-dimensional inverse problems, All of these foci grow out of discussions at the FRG and its successor meeting in Helsinki.

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[1] See conference webpage, https://wiki.helsinki.fi/display/mathstatHenkilokunta/Exceptional+Circle+Workshop+2013

[2] See conference webpage, http://math.as.uky.edu/scattering-and-inverse-scattering-multidimensions

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