# Estimates for linear forms in logarithms

Yann Bugeaud

We give below some known estimates for linear forms in Archimedean and non-Archimedean logarithms. We give very few bibliographic references and direct the reader to the textbook of Waldschmidt [11] for further information.

## 1. Archimedean estimates

Let $n \geq 2$ be an integer. For $1 \leq i \leq n$, let $x_i/y_i$ be a non-zero rational number and $b_i$ a positive integer. Set

$$B := \max\{3, b_1, \ldots, b_n\}$$

and, for $1 \leq i \leq n$, set

$$A_i := \max\{3, |x_i|, |y_i|\}.$$

We assume that the rational number

$$\Lambda := \left(\frac{x_1}{y_1}\right)^{b_1} \cdots \left(\frac{x_n}{y_n}\right)^{b_n} - 1 \tag{1.1}$$

is non-zero. We wish to bound $|\Lambda|$ from below, thus we may assume that $|\Lambda| \leq 1/2$ and we get a *linear form in logarithms*:

$$|\Lambda| \geq \frac{|\log(1+\Lambda)|}{2} = \frac{1}{2}\left|b_1 \log \frac{x_1}{y_1} + \cdots + b_n \log \frac{x_n}{y_n}\right|.$$

A trivial estimate of the denominator of (1.1) gives

$$\log |\Lambda| \geq -\sum_{i=1}^{n} b_i \log |y_i| \geq -B \sum_{i=1}^{n} \log A_i.$$

The dependence on the $A_i$'s is very satisfactory, unlike the dependence on $B$. However, for applications to Diophantine problems, we need a better estimate in terms of $B$, even if it means to get a weaker one in terms of the $A_i$'s.

Alan Baker [1, 2] was the first to prove such a result, and we are now able to show that, under the above assumptions, there exists an effectively computable constant $c(n)$, depending only on the number $n$ of rational numbers involved, such that the lower estimate

$$\log |\Lambda| \geq -c(n) \log A_1 \ldots \log A_n \log B$$

holds.

More generally, one can get analogous lower bounds if the rational numbers $x_i/y_i$ are replaced by algebraic numbers $\alpha_i$, the quantity $\log A_i$ being then essentially the absolute logarithmic height of $\alpha_i$.

Let $\theta$ be an algebraic real number and

$$P(X) = a_d \prod_{i=1}^{d} (X - \theta_i)$$

be its minimal defining polynomial over $\mathbf{Z}$, with $a_d \geq 1$. The height of $\theta$, denoted by $\mathrm{h}(\theta)$, is by definition

$$\mathrm{h}(\theta) = \frac{1}{d}\Big(\log a_d + \sum_{i=1}^{d} \log \max\{1, |\theta_i|\}\Big).$$

Let $\alpha_1, \ldots, \alpha_n$ be algebraic numbers distinct from 0 and 1. Let $\log \alpha_1, \ldots, \log \alpha_n$ be any determination of their logarithms. Let $b_1, \ldots, b_n$ be non-zero integers such that

$$\Lambda_a := |b_1 \log \alpha_1 + \cdots + b_n \log \alpha_n|$$

is non-zero. Instead of making an historical survey, we rather quote a corollary of the, at present time, best estimate, due to Matveev [9]. Let $D$ be the degree of a number field $\mathbf{K}$ containing the $\alpha_i$, let $E \geq e$ and $A_1, \ldots, A_n$ be real numbers $> 1$ with

$$\log A_i \geq \max\Big\{\mathrm{h}(\alpha_i), \frac{|\log \alpha_i|}{D}, \frac{0.16}{D}\Big\}, \qquad 1 \leq i \leq n.$$

Set

$$B = \max\{|b_1|, \ldots, |b_n|\}.$$

The next result is a corollary of Theorem 2 of Matveev [9].

**Theorem A.** *Under the above assumption, we have*

$$\log |\Lambda_a| > -2 \times 30^{n+4} \, (n+1)^6 \, D^{n+2} \, \log(eD) \log A_1 \ldots \log A_n \log(eB). \qquad (1.2)$$

For $n = 2$, the numerical constant in (1.2) can be substantially reduced, see below. This is crucial for applications to the complete resolution of Diophantine equations.

Matveev's result is clean, easy to apply and has the currently best known dependence on the number $n$ of algebraic numbers involved in the linear form. However, it does not include other refinements which are crucial for many applications.

The next theorem is Theorem 10.22 of [11].

Here, we have $E \geq e$ and

$$\log A_i \geq \max\Big\{\mathrm{h}(\alpha_i), \frac{E}{D}|\log \alpha_i|, \frac{\log E}{D}\Big\}, \qquad 1 \leq i \leq n.$$

**Theorem B.** *Assume furthermore that $b_n$ is non-zero. Let $E^*$ and $B'$ be real numbers satisfying $B' \geq E^* \geq E^{1/D}$, $E^* \geq D/(\log E)$ and*

$$B' \geq \max_{1\leq j\leq n-1} \left\{ \frac{|b_n|}{\log A_j} + \frac{|b_j|}{\log A_n} \right\}.$$

*Then, there exists an effectively computable numerical constant $C$ such that*

$$\log|\Lambda_a| \geq -C^n\, n^{3n}\, D^{n+2}\, \log A_1 \dots \log A_n \log B' \log E^* (\log E)^{-n-1}.$$

The dependence on $n$ is not as good as in Theorem A. The refinement consisting in replacing the quantity $B$ by $B'$ is very interesting when $|b_n|$ is small. This has many applications, in particular to Thue equations. The parameter $E$ originates in papers by Shorey [10]. It is of interest when the $\alpha_i$ are real and very close to 1, in which case $E$ can be chosen to be very big. In the most favorables cases, this extra term allows us to replace the product of the $\log A_i$, as it occurs in the statement of Theorem A, by their sum. Further explanations are given in Section 10.4.3 of [11].

Lower bounds for linear forms in two or three logarithms occur in many Diophantine problems. While, in the case of three logarithms, we do not have very satisfactory estimates, Laurent, Mignotte & Nesterenko [8] obtained in 1995 a sharp lower bound for linear forms in two logarithms. The quality of their result is an illustration of the method of interpolation determinants, introduced in this context by Laurent [6]. The current best known estimate is established in [7].

**Theorem C.** *Assume $n = 2$ in (1.2) and that the algebraic numbers $\alpha_1$ and $\alpha_2$ are multiplicatively independent. Set $D' = [\mathbf{Q}(\alpha_1, \alpha_2) : \mathbf{Q}]/[\mathbf{R}(\alpha_1, \alpha_2) : \mathbf{R}]$. Let $A_1$ and $A_2$ be real numbers $> 1$ such that*

$$\log A_i \geq \max\left\{ \mathrm{h}(\alpha_i), \frac{1}{D'}, \frac{|\log \alpha_i|}{D'} \right\}, \qquad 1 \leq i \leq 2.$$

*Set*

$$B' = \frac{|b_1|}{D' \log A_2} + \frac{|b_2|}{D' \log A_1}.$$

*Then, we have the lower bound*

$$\log|\Lambda_a| \geq -30.9 D'^4 \log A_1 \log A_2 \left( \max\left\{ \log B' + 0.66, \frac{21}{D'}, \frac{1}{D'} \right\} \right)^2.$$

Observe that the dependence on $B'$ is worse than in Theorem B, since $\log B'$ occurs squarred. However, this does not restrict the applications. The fact that the numerical constant is small (namely equal to 30.9 in Theorem C) compensates largely the extra square.

We can be more precise when $\alpha_1$ and $\alpha_2$ are very close to 1. We quote Corollaire 3 of [8]. Note that $D' = D$ when $\alpha_1$ and $\alpha_2$ are real.

**Theorem D.** *Assume that* $\log\alpha_1$ *and* $\log\alpha_2$ *are real, positive, and linearly independent over* $\mathbf{Q}$. *Set*

$$E = 1 + \min\left\{\frac{D\log A_1}{\log\alpha_1}, \frac{D\log A_2}{\log\alpha_2}\right\}$$

*and*

$$\log B = \max\left\{\log B' + \log\log E + 0.47, \frac{10\log E}{D}, \frac{1}{2}\right\}.$$

*Assume furthermore that* $E \le \min\{A_1^{3D/2}, A_2^{3D/2}\}$. *Then,*

$$\log|\Lambda| \ge -35.1 D^4 (\log A_1)(\log A_2)(\log B)^2(\log E)^{-3}. \tag{1.3}$$

Let $x_1/y_1$ and $x_2/y_2$ be multiplicatively independent rational numbers, both $> 1$. Let $b_1$ and $b_2$ be positive rational integers and consider the linear form

$$\Lambda = b_2\log(x_2/y_2) - b_1\log(x_1/y_1).$$

Let $A_1$ and $A_2$ be real numbers such that

$$\log A_i \ge \max\{\log x_i, 1\}, \quad (i = 1, 2).$$

We highlight the special case of Theorem D where $\alpha_1$ and $\alpha_2$ are rational numbers.

**Theorem E.** *Keep the above notation. Let* $E \ge 3$ *be a real number such that*

$$E \le 1 + \min\left\{\frac{\log A_1}{\log(x_1/y_1)}, \frac{\log A_2}{\log(x_2/y_2)}\right\},$$

*and set*

$$\log B = \max\left\{\log\left(\frac{b_1}{\log A_1} + \frac{b_2}{\log A_2}\right) + \log\log E + 0.47, 10\log E\right\}.$$

*Assuming that* $E \le \min\{A_1^{3/2}, A_2^{3/2}\}$, *we have*

$$\log\Lambda \ge -35.1\,(\log A_1)(\log A_2)(\log B)^2(\log E)^{-3}. \tag{1.4}$$

When $x_1/y_1$ and $x_2/y_2$ are *very* close to 1, the factor $(\log E)^{-3}$ allows us, roughly speaking, to replace the product $(\log A_1)(\log A_2)$ occurring in the 'classical estimate' (1.3) by the sum $(\log A_1) + (\log A_2)$. Indeed, assume for instance that we have $|(x_i/y_i) - 1| \le x_i^{-1/2}$ for $i = 1, 2$. Then, we get $\log(x_i/y_i) \le x_i^{-1/2}$ for $i = 1, 2$ and, if $x_1 \ge x_2 \ge 5$, we see that we can choose $\log E = (\log x_2)/2 = (\log A_2)/2$ in Theorem E. By (1.4) this gives

$$\log\Lambda \ge -71\,(\log A_1 + \log A_2)\,(10 + \log(b_1 + b_2))^2.$$

To be even more precise, we display a useful consequence of Theorem E. It deals with a particular situation that occurs in many applications. We assume that $b_2 = 1$, $x_1 \ge 3$,

$3 \le x_2 < 2y_2$ and that $|\Lambda|$ is very small. It follows that $x_2/y_2 < x_1/y_1$, thus the parameter $E$ satisfies

$$\log E \le -\log\log(x_2/y_2).$$

Define $\varepsilon$ by

$$\frac{x_2}{y_2} = 1 + x_2^{\varepsilon-1}.$$

Then we have $0 < \varepsilon < 1$ and $-\log\log(x_2/y_2) \ge (1-\varepsilon)\log x_2$. Taking into account the assumption $E \le \min\{A_1^{3/2}, A_2^{3/2}\}$ in Theorem E, we set

$$\log E = \min\{\log x_1, (1-\varepsilon)\log x_2\}.$$

**Corollary 1.** *Under the above assumption, we have*

$$\begin{aligned}\log\Lambda &\ge -35.1\,\frac{(\log x_1)(\log x_2)}{\min\{\log x_1, (1-\varepsilon)\log x_2\}}\,(10+\log b_1)^2\\ &\ge -\frac{35.1}{1-\varepsilon}\,\max\{\log x_1, \log x_2\}\,(10+\log b_1)^2.\end{aligned}$$

This crucial improvement upon the 'classical' estimate (1.3) turns out to have many spectacular applications.

Before finishing this section, we highlight an indirect consequence of Theorem C, which was established in [3].

We use the following notation. Let $\alpha_1, \ldots, \alpha_n, \alpha_{n+1}$ be complex algebraic numbers and $\log\alpha_1, \ldots, \log\alpha_n, \log\alpha_{n+1}$ any determinations of their logarithms. Let $b_1, \ldots, b_n$ be non-zero integers and set

$$D = [\mathbf{Q}(\alpha_1, \ldots, \alpha_{n+1}) : \mathbf{Q}], \qquad \Lambda = b_1\log\alpha_1 + \cdots + b_n\log\alpha_n + \log\alpha_{n+1}.$$

Let $A_1, \ldots, A_{n+1}$ be real numbers such that

$$\log A_i \ge \max\Big\{\mathrm{h}(\alpha_i), \frac{|\log\alpha_i|}{D}, \frac{1}{D}\Big\} \quad (i = 1, \ldots, n+1).$$

**Theorem F.** *Assume that*

$$0 < |\Lambda| < e^{-\varepsilon B},$$

*where $\varepsilon$ is a positive real number and $B = \max\{|b_1|, \ldots, |b_n|\}$. We then have*

$$B \le (\log A_{n+1})B_0,$$

*where $B_0$ is an effectively computable constant depending only on $A_1, \ldots, A_n$, $D$ and $\varepsilon$.*

Clearly, this statement is much weaker than Theorems A and B. However, its proof is simpler. And it is sufficient for many interesting applications (including that to Thue equations).

## 2. Non-Archimedean estimates

The $p$-adic analogue of Baker's theory has been studied by Coates, van der Poorten, Dong and Yu. We quote a result of Yu [12] (see also his previous papers for bibliographic references).

Keep the above notation.

Let $p$ be a prime number and denote by $\overline{\mathbf{Q}}_p$ an algebraic closure of the $p$-adic field $\mathbf{Q}_p$. We equip the field $\overline{\mathbf{Q}}_p$ with the ultrametric absolute value $|x|_p = p^{-v_p(x)}$, where $v_p$ denotes the unique extension to $\overline{\mathbf{Q}}_p$ of the standard $p$-adic valuation over $\mathbf{Q}_p$ normalized by $v_p(p) = 1$ (we set $v_p(0) = +\infty$). Let $\alpha_1, \ldots, \alpha_n$ be algebraic over $\mathbf{Q}$ and we regard them as elements of the field $\overline{\mathbf{Q}}_p$. We look for a lower bound for the ultrametric absolute value of

$$\Lambda_u := v_p(\alpha_1^{b_1} \ldots \alpha_n^{b_n} - 1), \tag{2.1}$$

where $b_1, \ldots, b_n$ denote rational integers.

We assume that $\alpha_1^{b_1} \ldots \alpha_n^{b_n} - 1$ is non-zero, thus $\Lambda_u$ is finite. Let $A_1, \ldots, A_n$ be real numbers with

$$\log A_i \geq \max\{\mathrm{h}(\alpha_i), 1/(16e^2D^2)\}, \qquad 1 \leq i \leq n.$$

The next result is a crude simplification of the estimate given on page 190 of [12].

**Theorem G.** *With the above notation, we have*

$$\Lambda_u < (16eD)^{2(n+1)} n^{5/2} (\log(2nD))^2\, D^n \frac{p^D}{(\log p)^2} \log A_1 \ldots \log A_n \log B.$$

Theorem G should be compared with Theorem A. We have exactly the same dependence on the parameters $n$, $\log A_i$ and $B$.

The next theorem reproduces one of the corollaries of the main result of [5].

**Theorem H.** *Assume $n = 2$ in (2.1) and that $\alpha_1$ and $\alpha_2$ are multiplicatively independent in $\mathbf{Q}_p(\alpha_1, \alpha_2)$ and satisfy $v_p(\alpha_1) = v_p(\alpha_2) = 0$. Set $D = [\mathbf{Q}(\alpha_1, \alpha_2) : \mathbf{Q}]$. Let $A_1$ and $A_2$ be real numbers $> 1$ such that*

$$\log A_i \geq \max\left\{\mathrm{h}(\alpha_i), \frac{\log p}{D}\right\}, \qquad 1 \leq i \leq 2.$$

*Set*

$$B' = \frac{b_1}{D \log A_2} + \frac{b_2}{D \log A_1}.$$

*Then, we have the upper bound*

$$\Lambda_u \leq \frac{24p(p^D - 1)}{(p-1)(\log p)^4} D^4 \log A_1 \log A_2 \left(\max\left\{\log B' + \log\log p + 0.4, \frac{10 \log p}{D}, 10\right\}\right)^2.$$

Theorem H should be compared with Theorem C. In particular, the numerical constant is very small and the quantity $\log B'$ also occurs squarred.

We end this section with an improvement of Theorem H when $\alpha_1$ and $\alpha_2$ are rational numbers.

Let $x_1/y_1$ and $x_2/y_2$ be non-zero rational numbers. Assume that there exist a positive integer $g$ and a real number $E$ such that

$$v_p\big((x_1/y_1)^g - 1)\big) \geq E > 1/(p-1) \quad \text{and} \quad v_p\big((x_2/y_2)^g - 1)\big) > 0.$$

Theorem I below, established in [4], provides an explicit upper bound for the $p$-adic valuation of

$$\Lambda = \left(\frac{x_1}{y_1}\right)^{b_1} - \left(\frac{x_2}{y_2}\right)^{b_2},$$

where $b_1$ and $b_2$ are positive integers. We let $A_1 > 1, A_2 > 1$ be real numbers such that

$$\log A_i \geq \max\{\log|x_i|, \log|y_i|, E\,\log p\}, \;\; (i = 1, 2).$$

and we put

$$b' = \frac{b_1}{\log A_2} + \frac{b_2}{\log A_1}.$$

**Theorem I.** *With the above notation, if $x_1/y_1$ and $x_2/y_2$ are multiplicatively independent, then we have the upper estimates*

$$v_p(\Lambda) \leq \frac{36.1\,g}{E^3\,(\log p)^4}\big(\max\{\log b' + \log(E\,\log p) + 0.4, 6\,E\,\log p, 5\}\big)^2 \log A_1 \log A_2$$

*and*

$$v_p(\Lambda) \leq \frac{53.8\,g}{E^3\,(\log p)^4}\big(\max\{\log b' + \log(E\,\log p) + 0.4, 4\,E\,\log p, 5\}\big)^2 \log A_1 \log A_2,$$

*if $p$ is odd or if $p = 2$ and $v_2(x_2/y_2 - 1) \geq 2$. Otherwise, we have*

$$v_2(\Lambda) \leq 208\,\big(\max\{\log b' + 0.04, 10\}\big)^2 \log A_1 \log A_2.$$

The parameter $E$ in Theorem I should be compared with the parameter $E$ in Theorems D and E. Roughly speaking, provided that the algebraic numbers involved are $p$-adically close to 1, this means that one can replace the product $(\log A_1)(\log A_2)$ in the estimate by the sum $(\log A_1) + (\log A_2)$.

## References

[1] A. Baker, *Linear forms in the logarithms of algebraic numbers*, Mathematika 12 (1966), 204–216.

[2] A. Baker, *A sharpening of the bounds for linear forms in logarithms I–III*, Acta Arith. 21 (1972), 117–129; 24 (1973), 33–36; 27 (1975), 247–252.

[3] Yu. Bilu et Y. Bugeaud, *Démonstration du théorème de Baker-Feldman via les formes linéaires en deux logarithmes*, J. Théor. Nombres Bordeaux 12 (2000), 13–23.

[4] Y. Bugeaud, *Linear forms in p-adic logarithms and the Diophantine equation* $\frac{x^n-1}{x-1} = y^q$, Math. Proc. Cambridge Phil. Soc. 127 (1999), 373–381.

[5] Y. Bugeaud et M. Laurent, *Minoration effective de la distance p-adique entre puissances de nombres algébriques*, J. Number Theory 61 (1996), 311–342.

[6] M. Laurent, *Linear forms in two logarithms and interpolation determinants*, Acta Arith. 66 (1994), 181–199.

[7] M. Laurent, *Linear forms in two logarithms and interpolation determinants. II*, Acta Arith. 133 (2008), 325–348.

[8] M. Laurent, M. Mignotte et Y. Nesterenko, *Formes linéaires en deux logarithmes et déterminants d'interpolation*, J. Number Theory 55 (1995), 285–321.

[9] E. M. Matveev, *An explicit lower bound for a homogeneous rational linear form in logarithms of algebraic numbers. II*, Izv. Ross. Acad. Nauk Ser. Mat. 64 (2000), 125–180; English translation in Izv. Math. 64 (2000), 1217–1269.

[10] T. N. Shorey, *Linear forms in the logarithms of algebraic numbers with small coefficients I, II*, J. Indian Math. Soc. (N. S.) 38 (1974), 271–284; ibid. 38 (1974), 285–292.

[11] M. Waldschmidt, Diophantine Approximation on Linear Algebraic Groups. Transcendence Properties of the Exponential Function in Several Variables, Grundlehren Math. Wiss. 326, Springer, Berlin, 2000.

[12] K. Yu, *p–adic logarithmic forms and group varieties, III*, Forum Math. 19 (2007), 187–280.

Yann Bugeaud
Université de Strasbourg
Mathématiques
7, rue René Descartes
67084 STRASBOURG (FRANCE)

bugeaud@math.unistra.fr