Consider the genus 2 curve
\[ X : y^2 = -x^6 - x^2 + x + 2 \]

We have the following bits of information:
(i) We have the points \( P_1 = (1, 1) \in X(\mathbb{Q}) \) and \( P_2 = (1, -1) \in X(\mathbb{Q}) \)
(ii) \( \text{Pic}^0(X/\mathbb{Q}) = \langle G = [P_2 - P_1] \rangle \cong \mathbb{Z} \).
(iii)
\[ \Lambda_3 = \langle 5G \rangle = \langle [Q_1 + Q_2 - 2P_1] \rangle, \]
where
\[ Q_1 = (10\sqrt{3} - 44 + O(3^4), \sqrt{3} + 7 + O(3^4)) \]
\[ Q_2 = (-10\sqrt{3} - 44 + O(3^4), -\sqrt{3} + 7 + O(3^4)) \]
(iv) An annihilating 3-adic differential is \( \omega = \frac{1}{y}dx + O(3) \).

Together you can use this to determine all rational points on \( X \).
(a) Determine \( X(\mathbb{F}_3) \).
(b) Determine the points where \( \omega \) vanishes modulo 3.
(c) Determine \( X(\mathbb{Q}) \).
(d) Verify (iii) assuming (ii)
(e) Verify (iv)

A classic curve for explicit Chabauty equations is a curve considered by Poonen, Schaefer and Stoll, arising from considering periodic points under quadratic polynomial maps.
\[ X : y^2 = x^6 + 8x^5 + 22x^4 + 22x^3 + 5x^2 + 6x + 1 \]

(a) Verify that \( X \) has good reduction at \( p = 3 \).
(b) Assume that \( [(-3, 1) - (0, 1)] \) generates \( \Lambda_3 \). Determine an annihilating 3-adic differential. \( [x] \) is a good uniformizer for \( (0, 1) \).
(c) Determine \( X(\mathbb{F}_3) \).
(d) In order to analyze the points with \( x = \infty \), change coordinates to \( (z, w) = (\frac{1}{z}, \frac{y}{x}) \). Verify that modulo 3, the annihilating differential does not have a zero with \( z = 0 \).
(e) By expanding \( \omega \) to a little higher precision around \( (x, y) = (0, 1) \), you can read off that there are at most two rational points that reduce to \( (0, 1) \) modulo 3.
(f) You can assemble this to a full determination of all rational points on \( X \).

Prove the baby version of Strassman’s Lemma: Let \( f(z) \sum_{i=0}^{\infty} a_i z^i \in \mathbb{Z}_p[[z]] \) be a power series that converges on \( \mathbb{Z}_p \) (i.e., \( \lim_{i \to \infty} v_p(a_i) = \infty \)). Suppose that \( v_p(a_i) > v_p(a_1) \) for all \( i = 2, \ldots \). Then \( f(z) \) has only one root in \( \mathbb{Z}_p \).

Worthwhile questions from previous exercise batches: \( \text{N7, N3} \), except you probably want to do \( x^3 - 2y^3 = 5 \) instead, in view of \( 27 - 16 = 11 \).