Ideals of operators on $(\bigoplus E_n)_{c_0}$ and $(\bigoplus E_n')_{\ell^1}$

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Statement of Problem

Let E be a Banach space. The space $L(E)$ of all bounded linear operators on E is a Banach algebra. Determine the structure of the closed ideals in $L(E)$.

Particularly, look at E of the form $(\oplus E_n)_{c_0}$ and $(\oplus E_n')_{\ell^1}$, where E_n's are certain finite dimensional spaces.

- $(\oplus \ell^2(n))_{c_0}$ [Laustsen, Loy & Read, JFA, 2004].
- $(\oplus \ell^2(n))_{\ell^1}$ [Laustsen, Schlumprecht & Zsák, JOT 2006].
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Notation

X, Y separable Banach spaces.

$$\mathcal{G}_{c_0}(X,Y) = \{T \in L(X,Y) : \exists R \in L(X,c_0), S \in L(c_0,Y), T = SR\}.$$

$$\mathcal{G}^s(X,Y) = \{T \in L(X,Y) : \exists \text{ quotient } Q : \ell^1 \to X, TQ \in \overline{\mathcal{G}_{c_0}(\ell^1,Y)}\}.$$

T_1, T_2 operators. $T_1 \prec T_2$ if T_1 factors through T_2, i.e., $T_1 = AT_2B$ for some operators A and B.

Theorems

LLR Let $E = (\oplus \ell^2(n))_{c_0}$. If $T \notin \overline{\mathcal{G}_{c_0}(E)}$, then $1_E \prec T$.

LOSZ Let $E = (\oplus \ell^1(n))_{c_0}$. If $T \notin \mathcal{G}^s(E)$, then $1_E \prec T$.

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Steps in the proofs

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If $I \subseteq \mathbb{N}$, let $E_I = (\oplus_{n\in I} E_n)_{c_0}$ and P_I be the projection onto E_I.
$P_{\{n\}} = P_n$.
An operator $T \in L(E)$ has a matrix representation (T_{mn}), where $T_{mn} = P_n T P_m$.
$T \in L(X, Y)$, $\varepsilon > 0$. Let

$$\mathrm{Fac}_0^{\varepsilon}(T) = \inf\{\|R\|\,\|S\| : R \in L(X, c_0), S \in L(c_0, Y), \|T - SR\| \leq \varepsilon\}.$$

1. Let $T \in L(E)$ and $T_n = P_n T$. If $\sup_n \mathrm{Fac}_0^{\varepsilon}(T_n) < \infty$, then $d(T, \mathcal{G}_{c_0}(E)) \leq \varepsilon$.
2. Let $T_n : (\oplus_{i=1}^n \ell^2(m_i))_{\ell^\infty(n)} \to \ell^2(j_n)$ be uniformly bounded. Then either
   1. $\sup_n \mathrm{Fac}_0^{\varepsilon}(T_n) < \infty$ for all $\varepsilon > 0$, or
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1. Let $T \in L(E)$ and $T_n = P_n T$. If there exists a quotient $Q : \ell^1 \to E$ such that $\sup_n \mathrm{Fac}_0^\varepsilon(T_n Q) < \infty$, then for any quotient $\pi : \ell^1 \to E$, $d(T\pi, \mathcal{G}_{c_0}(\ell^1, E)) \le \varepsilon$.
2. Let X be a separable Banach spaces and let $T_n : X \to L^1$ be uniformly bounded. Then either
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Dual cases

Let E_n be finite dimensional spaces and let $F = (\oplus E_n')_{c_0}$, $E = F' = (\oplus E_n)_{\ell^1}$.

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$$\mathcal{G}^i(X,Y) = \{T \in L(X,Y) : \exists \text{ embedding } J : Y \to \ell^\infty, JT \in \overline{\mathcal{G}_{\ell^1}(X,\ell^\infty)}\}.$$

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Preliminaries

$T : X \to Y$, $\varepsilon > 0$.

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- $T = A_1 + A_2 + A_3$, $A_i \in L(X, Y)$. $\varepsilon > 0$. Then

$$\mathrm{Fac}_1^{3\varepsilon}(T) \le 3 \max_{1 \le i \le 3} \mathrm{Fac}_1^{\varepsilon}(A_i).$$

If $\|A_i - S_i R_i\| \le \varepsilon$, let $R : X \to \ell^1 \oplus_1 \ell^1 \oplus_1 \ell^1$ be $Rx = R_1 x \oplus R_2 x \oplus R_3 x$, and $S : \ell^1 \oplus_1 \ell^1 \oplus_1 \ell^1 \to Y$ be $S(z_1 \oplus z_2 \oplus z_3) = S_1 z_1 + S_2 z_2 + S_3 z_3$.

$E = (\oplus E_n)_{\ell^1}$. $T \in L(E)$. $T_n = TP_n$.

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1. If $\sup_n \mathrm{Fac}_1^{\varepsilon}(T_n) < \infty$, then $d(T, \mathcal{G}_{\ell^1}(E)) \le \varepsilon$.
2. If there exists an embedding $J : Y \to \infty$ such that $\sup_n \mathrm{Fac}_1^{\varepsilon}(JT_n) < \infty$, then $d(JT, \mathcal{G}_{\ell^1}(E, \ell^\infty)) \le \varepsilon$.

$T : X \to Y$, $\varepsilon > 0$.

$$\mathrm{Fac}_1^\varepsilon(T) = \inf\{\|R\|\,\|S\| : R \in L(X,\ell^1), S \in L(\ell^1, Y), \|T - SR\| \le \varepsilon\}.$$

- $T = A_1 + A_2 + A_3$, $A_i \in L(X,Y)$. $\varepsilon > 0$. Then

$$\mathrm{Fac}_1^{3\varepsilon}(T) \le 3 \max_{1\le i\le 3} \mathrm{Fac}_1^\varepsilon(A_i).$$

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Beginning of proof

$E = (\oplus E_n)_{\ell^1}$. $F = (\oplus E'_n)_{c_0}$. $T \in L(E)$. $T_n = TP_n$.

$\mathcal{G}^i(X, Y) = \{T \in L(X, Y) : \exists$ embedding $J : Y \to \ell^\infty, JT \in \overline{\mathcal{G}_{\ell^1}(X, \ell^\infty)}\}$.

Suppose that $T \notin \mathcal{G}^i(E)$.
Take quotient $Q : \ell^1 \to F$ and $J = Q'$. There exists $\varepsilon > 0$ such that $\sup_n \mathrm{Fac}_1^{3\varepsilon}(JT_n) = \infty$.
We obtain a blocking of T, $W_j = P_{I_j} T P_{n_j}$, $n_1 < n_2 < \cdots$, $I_1 < I_2 < \cdots$, such that $\sup_j \mathrm{Fac}_1^{\varepsilon}(JW_j) = \infty$.

	n_1	n_2	$\cdots$
I_1	W_1		
I_2		W_2	
$\vdots$			

Then $\sup_j \mathrm{Fac}_0^{\varepsilon}(W'_j Q) = \infty$.
So $1_{\ell^1(k)}$ uniformly factor through W'_j. Thus $1_{\ell^\infty(k)}$ uniformly factor through W_j.

Beginning of proof

$E = (\oplus E_n)_{\ell^1}$. $F = (\oplus E_n')_{c_0}$. $T \in L(E)$. $T_n = TP_n$.
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Then $\sup_j \mathrm{Fac}_0^{\varepsilon}(W_j' Q) = \infty$.
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Beginning of proof

$E = (\oplus E_n)_{\ell^1}$. $F = (\oplus E_n')_{c_0}$. $T \in L(E)$. $T_n = TP_n$.
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	n_1	n_2	$\dots$
I_1	W_1		
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Beginning of proof

$E = (\bigoplus E_n)_{\ell^1}$. $F = (\bigoplus E'_n)_{c_0}$. $T \in L(E)$. $T_n = TP_n$.
$\mathcal{G}^i(X,Y) = \{T \in L(X,Y) : \exists$ embedding $J : Y \to \ell^\infty, JT \in \overline{\mathcal{G}_{\ell^1}(X,\ell^\infty)}\}$.

Suppose that $T \notin \mathcal{G}^i(E)$.
Take quotient $Q : \ell^1 \to F$ and $J = Q'$. There exists $\varepsilon > 0$ such that $\sup_n \operatorname{Fac}_1^{3\varepsilon}(JT_n) = \infty$.
We obtain a blocking of T, $W_j = P_{I_j} T P_{n_j}$, $n_1 < n_2 < \cdots$, $I_1 < I_2 < \cdots$, such that $\sup_j \operatorname{Fac}_1^{\varepsilon}(JW_j) = \infty$.

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$E = (\oplus E_n)_{\ell^1}$. $F = (\oplus E_n')_{c_0}$. $T \in L(E)$. $T_n = TP_n$.
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	n_1	n_2	$\dots$
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More of the proof

We want to show that $1_{(\oplus \ell^\infty(n))_{\ell^1}}$ factors through T.
A submatrix of T factors through T, but a block diagonal may not.
The columns of T are summable in operator norm (but not necessarily the rows). For simplicity, assume that T has the form

$$T = \begin{pmatrix} \boxed{W_1} & \cdots & \cdots & V_1 & \cdots \\ 0 & \boxed{W_2} & \cdots & V_2 & \cdots \\ \vdots & 0 & \ddots & & \end{pmatrix},$$

where $W_n : E_{I_n} \to E_{I_n}$, $V_n : (\oplus_{m > I_n} E_m)_{\ell^1} \to E_{I_n}$. $1_{E_n} \prec W_n$.
Let $E_n = \ell^\infty(n)$. We have $E_n \overset{i_n}{\hookrightarrow} E_{n+1} \overset{q_n}{\to} E_n$, $q_n i_n = 1_{E_n}$.
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$$T = \begin{pmatrix} W_1 + V_1 \\ W_2 + V_2 \\ \vdots \end{pmatrix}, W_n : E_{I_n} \to E_{I_n}, V_n : (\oplus_{m > I_n} E_m)_{\ell^1} \to E_{I_n}.$$

$$1_{E_n} = R_n W_n S_n, \|R_n : E_{I_n} \to E_n\| \le 1, \|S_n : E_n \to E_{I_n}\| \le C.$$

Let $V_{m,n,j} = V_j S_n i_{mn} : E_m \to E_{I_j}, j, m < n$.
Up to small perturbation, we may assume $V_{m,n,j} = V_{m,n',j}$ if $j, m < n, n'$.

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Consider the matrix equation

$$\begin{pmatrix} 0 & R_2 & 0 & R_4 & \cdots \end{pmatrix} \begin{pmatrix} 0 \\ W_2 + V_2 \\ 0 \\ W_4 + V_4 \\ 0 \\ W_6 + V_6 \\ \vdots \end{pmatrix} \begin{pmatrix} 0 \\ S_2 i_{12} x_1 \\ -S_3 i_{23} x_2 \\ S_4 i_{24} x_2 \\ -S_5 i_{35} x_3 \\ S_6 i_{36} x_3 \\ \vdots \end{pmatrix} = \begin{pmatrix} 0 \\ R_2 W_2 S_2 i_{12} x_1 \\ 0 \\ R_4 W_4 S_4 i_{24} x_2 \\ \vdots \end{pmatrix}$$

$$= 0 \oplus i_{12}x_1 \oplus 0 \oplus i_{24}x_2 \oplus \cdots .$$

$i_{12}E_1 \overset{c}{\hookrightarrow} E_2$, $i_{24}E_2 \overset{c}{\hookrightarrow} E_4, \ldots$ with uniform constants.
It follows that $1_{(\oplus E_n)_{\ell 1}} \prec$ submatrix of $T \prec T$.

$1_{E_n} = R_n W_n S_n$.
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$V_{m,n,j} = V_{m,n',j}$ if $j, m < n, n'$. $V_{m,n,j} = V_j S_n i_{mn}$.
Consider the matrix equation

$$\begin{pmatrix} 0 & R_2 & 0 & R_4 & \cdots \end{pmatrix} \begin{pmatrix} 0 \\ W_2 + V_2 \\ 0 \\ W_4 + V_4 \\ 0 \\ W_6 + V_6 \\ \vdots \end{pmatrix} \begin{pmatrix} 0 \\ S_2 i_{12} x_1 \\ -S_3 i_{23} x_2 \\ S_4 i_{24} x_2 \\ -S_5 i_{35} x_3 \\ S_6 i_{36} x_3 \\ \vdots \end{pmatrix} = \begin{pmatrix} 0 \\ R_2 W_2 S_2 i_{12} x_1 \\ 0 \\ R_4 W_4 S_4 i_{24} x_2 \\ \vdots \end{pmatrix}$$

$$= 0 \oplus i_{12} x_1 \oplus 0 \oplus i_{24} x_2 \oplus \cdots .$$

$i_{12} E_1 \overset{c}{\hookrightarrow} E_2$, $i_{24} E_2 \overset{c}{\hookrightarrow} E_4, \ldots$ with uniform constants.
It follows that $1_{(\oplus E_n)_{\ell^1}} \prec$ submatrix of $T \prec T$.

$1_{E_n} = R_n W_n S_n$.
$V_{m,n,j} = V_{m,n',j}$ if $j, m < n, n'$. $V_{m,n,j} = V_j S_n i_{mn}$.
Consider the matrix equation

$$\begin{pmatrix} 0 & R_2 & 0 & R_4 & \cdots \end{pmatrix} \begin{pmatrix} 0 \\ W_2 + V_2 \\ 0 \\ W_4 + V_4 \\ 0 \\ W_6 + V_6 \\ \vdots \end{pmatrix} \begin{pmatrix} 0 \\ S_2 i_{12} x_1 \\ -S_3 i_{23} x_2 \\ S_4 i_{24} x_2 \\ -S_5 i_{35} x_3 \\ S_6 i_{36} x_3 \\ \vdots \end{pmatrix} = \begin{pmatrix} 0 \\ R_2 W_2 S_2 i_{12} x_1 \\ 0 \\ R_4 W_4 S_4 i_{24} x_2 \\ \vdots \end{pmatrix}$$

$$= 0 \oplus i_{12} x_1 \oplus 0 \oplus i_{24} x_2 \oplus \cdots .$$

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Thank you