Hereditarily α-universal Banach spaces

Pavlos Motakis
(joint work with Spiros A. Argyros)

Department of Mathematics
National Technical University of Athens
Athens, Greece

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During the present lecture, the notion of hereditary α-universality is going to be discussed. More precisely, the construction of a transfinite class of Banach spaces $\mathfrak{X}_\xi, \xi < \omega_1$ is going to be described. The spaces $\mathfrak{X}_\xi$ are reflexive HI spaces, the main property of which, is that every Schauder basic sequence ω^ξ-embeds into every subspace of $\mathfrak{X}_\xi$. The construction is based on a variant of the method of saturation under constraints, which was described in the previous lecture.

Hereditary universal finite representability

- In 1996 E. Odell and Th. Schlumprecht present a new reflexive HI space, having the remarkable and unexpected property, that any Banach space with a monotone basis is $1+\varepsilon$ block finitely representable in every block subspace.
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The rank of a tree

- The rank of a tree is an ordinal index, which, among others, determines the complexity of the finite representability of a Schauder basic sequence into an arbitrary Banach space X.
- For a well founded tree $\mathcal{T}$ with a root, denoted as $\varnothing$, the rank of $\mathcal{T}$ $\mathrm{rank}(\mathcal{T})$ is recursively defined.
- For s a maximal node of $\mathcal{T}$, set $\rho(s) = 0$.
- For s a non maximal node, set $\rho(s) = \sup\{\rho(t) + 1 : s < t\}$.
- Then the rank of $\mathcal{T}$ is defined as

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Rank of embedability of a Schauder basic sequence into a Banach space

- Given a Schauder basic sequence $\{e_k\}_k$ and a Banach space X, one would like to study, the complexity of the finite representation of $\{e_k\}_k$ into X.
- An approach to this problem, is to determine the rank of the Bourgain embedability tree of $\{e_k\}_k$ into X (J. Bourgain 1980), which is defined as follows.
- For a constant $C \geqslant 1$, we set $\mathcal{T}(\{e_k\}_k, C, X)$ to be the tree of all finite block sequences $\{x_k\}_{k=1}^m$, such that $\{x_k\}_{k=1}^m$ is C-equivalent to $\{e_k\}_{k=1}^m$.

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- We say that $\{e_k\}_k$ α-embeds into X, if $\mathrm{Emb}(\{e_k\}_k, X) \geqslant \alpha$

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Saturation under constraints

- In the previous lecture, the definitions of α-averages and the $(\theta, \mathcal{F}, \alpha)$ operation were discussed, leading to the notion of saturation under constraints.
- The advantage of saturation under constraints, is that it permits the space to admit many c_0 spreading models.
- In turn, c_0 spreading models allow the construction of vectors having specific properties, which are used to prove the existence of certain structures in the space.
- While saturation under constrains allows c_0 spreading models to appear everywhere, it completely rules out the existence of higher order c_0 spreading models.

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Higher order c_0 spreading models

- For a seminormalized Schauder basic sequence $\{x_k\}_k$ and ξ a countable ordinal, we say that $\{x_k\}_k$ generates a c_0^ξ spreading model, if there exists a constant $C > 0$, such that for every $F \in \mathcal{S}_\xi$

$$\|\sum_{k\in F} x_k\| \leqslant C$$

- The spaces $T_{0,1}$ and $\mathfrak{X}_{\text{ISP}}$, described in the previous lecture, do not admit a c_0^2 spreading model.

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Higher order c_0 spreading models

- Higher order c_0 spreading models are desirable, in order to obtain certain structures, which are of transfinite nature, for instance the ω^ξ-embedability of a sequence into a space.
- In order to achieve this, a variation of the method of saturation under constraints can be used.
- More precisely, instead of α-averages, α-special convex combinations (α-s.c.c), are used in the construction of the norming set.
- Special convex combinations are generalized averages of higher complexity. This higher complexity imposes the existence of higher order c_0 spreading models.

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S.c.c. with respect to a sequence of regular families

- Let $\mathcal{F}_0 \subset \mathcal{F}_1 \subset \cdots \subset \mathcal{F}_j \subset \cdots$ be an increasing sequence of regular families of increasing complexity.

- ((j, ε) basic s.c.c). A convex combination $\sum_{i \in F} c_i e_i$ in c_{00} is said to be a

  $$(j, \varepsilon) \text{ b.s.c.c. with respect to } \{\mathcal{F}_j\}_{j=0}^{\infty}$$

  if $F \in \mathcal{F}_j$ and for $G \subset F$, $G \in \mathcal{F}_{j-1}$

  $$\sum_{i \in G} c_i < \varepsilon$$

- ((j, ε) s.c.c.) Let $x_1 < \cdots < x_m$ be vectors in c_{00} and $\psi(k) = \min \operatorname{supp} x_k$. Then $x = \sum_{k=1}^{m} c_k x_k$ is said to be a

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α-special convex combinations

- We fix $\mathcal{F}_0 \subset \mathcal{F}_1 \subset \cdots \subset \mathcal{F}_j \subset \cdots$ an increasing sequence of regular families of increasing complexity.
- A vector α in a norming set W is said to be an α-s.c.c. of size $s(\alpha) = j$, if there exist $f_1 < \cdots < f_k$ in W, such that

  $\alpha = \sum_{k=1}^{m} \lambda_k f_k$ is a $(j, \frac{1}{2^{j+1}})$ s.c.c. with respect to $\{\mathcal{F}_j\}_{j=0}^{\infty}$.
- A sequence $\alpha_1 < \alpha_2 < \cdots < \alpha_n < \cdots$ is very fast growing (v.f.g.), if for $n > 1$

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- A sequence $\alpha_1 < \alpha_2 < \cdots < \alpha_n < \cdots$ is very fast growing (v.f.g.), if for $n > 1$

$$s(\alpha_n) > 2^{\max \operatorname{supp} \alpha_{n-1}}$$

The $(\theta_j, \mathcal{F}_j, \alpha$-s.c.c.$)_j$ operations

- A norming set W is said to be closed under the $(\theta_j, \mathcal{F}_j, \alpha$-s.c.c.$)_j$ operations, if for every $\{\alpha_k\}_{k=1}^n$ $\mathcal{F}_j$-admissible and very fast growing family of α-s.c.c. in W, the functional

$$f = \theta_j \sum_{k=1}^n \alpha_k$$

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The space $\mathfrak{X}_{\xi_0}$

Theorem (S. A. Argyros, P.M.) For every $\xi_0 < \omega_1$ and bimonotone Schauder basic sequence $\{u_k\}_k$, there exists a reflexive space $\mathfrak{X}_{\xi_0}$ with a Schauder basis $\{e_k\}_k$ satisfying the following properties.

(i) The space $\mathfrak{X}_{\xi_0}$ is hereditarily indecomposable.

(ii) There exists $\xi \geqslant \xi_0$, such that every subspace of $\mathfrak{X}_{\xi_0}$ admits a c_0^ξ spreading model.

(iii) The sequence $\{u_k\}_k$ ω^{ξ_0}-embeds into every subspace of $\mathfrak{X}_{\xi_0}$.

In particular, there exists a uniform constant C, such that for every Y subspace of $\mathfrak{X}_{\xi_0}$

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The norming set W_{ξ_0}

- We fix $\{m_j\}_j, \{q_j\}_j$ strictly increasing sequences of naturals satisfying appropriate growth conditions.
- We choose

$$\mathcal{F}_0 \subset \mathcal{G}_1 \subset \mathcal{F}_1 \subset \cdots \subset \mathcal{G}_j \subset \mathcal{F}_j \subset \cdots$$

regular families satisfying the following

(i) If $\mathcal{F}_j^{(2)} = \{F \cup G : F, G \in \mathcal{F}_j\}$ and $\mathbb{N}_j = \{n : n \geqslant j\}$, then

$$\left((\mathcal{F}_j^{(2)})^{q_j} * G_{j+1}\right)[\mathbb{N}_j] \subset \mathcal{F}_{j+1}$$

i.e, for any $F \in (\mathcal{F}_j^{(2)})^{q_j} * G_{j+1}, j \leqslant \min F, F \in \mathcal{F}_{j+1}$

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(ii) For j and $j' \geqslant j$, every $F \in \mathcal{G}_j[\mathbb{N}_{j'}]$ maximal set supports a $(j, \frac{1}{2^{j'+2}})$ b.s.c.c. with respect to $\{\mathcal{F}_j\}_{j=0}^{\infty}$

(iii) The Cantor-Bendixson index of $\mathcal{F}_0$ is ω^{ξ_0} and there exists a strictly increasing sequence of countable ordinals $\{\xi_j\}_j$ with $\xi_0 < \xi_j$ such that the Cantor-Bendixson index of $\mathcal{F}_j$ is ω^{ξ_j}

From now on we will denote the ordinal $\sup_j \xi_j$ by ξ.

- We are now ready to define the norming set W_{ξ_0}. Note that all s.c.c. will be taken with respect to $\{\mathcal{F}_j\}_{j=0}^{\infty}$.

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The norming set W_{ξ_0}

- The norming set W_{ξ_0} is the minimal norming set satisfying the following properties.

(i) (Type I_α functionals) The set W_{ξ_0} is closed in the $(\frac{1}{m_j}, \mathcal{F}_j, \alpha\text{-s.c.c.})$ operations, for $j \geqslant 1$.
If f is of type I_α and is the result of $(\frac{1}{m_j}, \mathcal{F}_j, \alpha\text{-s.c.c.})$ operation, then the weight of f is $w(f) = j$.

(ii) (Type II functionals) The set W_{ξ_0} includes all $E\phi$, with E an interval of the naturals and $\phi = \frac{1}{2}\sum_{k=1}^{n} \lambda_k f_k$, where $f_1 < \cdots < f_n$ is an $\mathcal{F}_0$-admissible special family of type I_α special functionals (a special family satisfies the property, that for $k > 1$, $w(f_k)$ determines uniquely the sequence $\{f_i\}_{i=1}^{k-1}$.)

and $\{\lambda_k\}_k \subset [-1, 1] \cap \mathbb{Q}$ such that

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For $E_1\phi_1, E_2\phi_2$ functionals of type II and, we say that the weights of $E_1\phi_1, E_2\phi_2$ are incomparable, if there does not exist a functional ϕ of type II, such that

both $\hat{w}(\phi) \cap \hat{w}(\phi_1) \neq \varnothing$ and $\hat{w}(\phi) \cap \hat{w}(\phi_2) \neq \varnothing$.

(β-averages) A β-average is an average $\beta = \frac{1}{n}\sum_{k=1}^{n} E_k\phi_k$, where $E_k\phi_k$ are of type II with pairwise incomparable weights.
The size $s(\beta)$ and very fast growing sequences $(\beta_k)_k$ are defined in the same manner as for α-averages.

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(β-averages) A β-average is an average $\beta = \frac{1}{n}\sum_{k=1}^{n} E_k\phi_k$, where $E_k\phi_k$ are of type II with pairwise incomparable weights.
The size $s(\beta)$ and very fast growing sequences $(\beta_k)_k$ are defined in the same manner as for α-averages.

The norming set W_{ξ_0}

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(iii) (Type I_β functionals) The set W_{ξ_0} is closed in the $(\frac{1}{m_j}, \mathcal{F}_j, \beta)$ operations, for $j \geqslant 1$. If f is of type I_β and is the result of $(\frac{1}{m_j}, \mathcal{F}_j, \beta)$ operation, then the weight of f is $w(f) = j$.

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ω^{ξ_0}-embedability of the sequence $\{u_k\}_k$

- Let Y be a block subspace of $\mathfrak{X}_{\xi_0}$ and assume that there exists a normalized block sequence $\{y_k\}_k$ in Y generating a c_0^ξ spreading model. For any $j \geqslant 1$, one may find a subset F of the naturals, such that $j \leqslant \{\min \operatorname{supp} y_k : k \in F\}$ is a maximal $\mathcal{G}_j$ set and $\|\sum_{k \in F} y_k\|$ is bounded by a universal constant K.
- Thus there exists an α-s.c.c. α of size $s(\alpha) = j$, such that $\alpha(\sum_{k \in F} y_i) > 1 - \varepsilon$.
- For $j \geqslant 1$ By taking $F_1 < \cdots < F_n$ such that

(i) If $z_k = \sum_{i \in F_k} y_i$, then $\{\min \operatorname{supp} z_k : k = 1, \ldots, n\}$ is a maximal $\mathcal{G}_j$ set.

(ii) $j_k \leqslant \{\min \operatorname{supp} y_i : i \in F_k\}$ is j_k admissible with $j_k > 2^{\max \operatorname{supp} y_{k-1}}$ for $k > 1$

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- Combining the above, we conclude that there exists $\{\alpha_k\}_{k=1}^n$ an $\mathcal{F}_j$ admissible v.f.g. sequence of α-s.c.c. with $\alpha(y_k) > 1 - \varepsilon$, thus $f = \frac{1}{m_j} \sum_{k=1}^n \alpha_k$ is a functional of type I_α in W_{ξ_0}.
- Then, for $\{c_k\}_{k=1}^n$, such that $w = \sum_{k=1}^n c_k z_k$ is a (j, δ) s.c.c. we have that $f(w) > \frac{1-\varepsilon}{m_j}$.
- Moreover, $\{y_k\}_{k=1}^n$ is RIS, therefore setting $x = \frac{m_j}{f(w)} w$, we conclude that $\{x, f\}$ is a j-exact pair, where f is a functional of type I_α. Moreover $1 \leqslant \|x\| \leqslant M$, for a universal constant M.

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- Therefore, a dependent sequence $\{x_k, f_k\}_{k=1}^m$ can be constructed, i.e.
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- Fix $\{\mu_k\}_{k=1}^m \subset \mathbb{R}$. Since for any $\{\lambda_k\}_{k=1}^m \subset [-1,1] \cap \mathbb{Q}$, such that $\|\sum_{k=1}^m \lambda_k u_k^*\| \leqslant 1$ we have that $\frac{1}{2}\sum_{k=1}^m \lambda_k f_k$ is a functional of type II in W_{ξ_0}, we conclude the following.

$$\|\sum_{k=1}^m \mu_k x_k\|_{\xi_0} \geqslant \frac{1}{2}\|\sum_{k=1}^m \mu_k u_k\|$$

- With some effort, it can be proven by induction on the tree complexity of the functionals in W_{ξ_0}, that the action of any functional not directly associated to $\{f_k\}_{k=1}^m$ is neutralized. This yields that there exists a universal constant C, such that

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- Using the fact that special sequences are $\mathcal{F}_0$-admissible and the Cantor-Bendixson index of $\mathcal{F}_0$ is ω^{ξ_0} and an inductive construction, it is shown that

$$\text{rank}\left(\mathcal{T}(\{u_k\}_k, C, Y)\right) \geqslant \omega^{\xi_0}.$$

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The α and β indices

- As in the case of the space $\mathfrak{X}_{\text{ISP}}$, we associate the behaviour of the α-s.c.c. and β-averages on a sequence, to the spreading models generated by it, by introducing the transfinite hierarchy of $\alpha_\zeta, \beta_\zeta$ indices, $\zeta < \xi$.
- Let $\{x_k\}_k$ be a block sequence in $\mathfrak{X}_{\xi_0}$ and $\zeta < \xi$ such that the following is satisfied.

  For any j, for any very fast growing sequence $\{\alpha_q\}_q$ of α-s.c.c. in W_{ξ_0} and for any $\{F_k\}_k$ increasing sequence of subsets of the naturals, such that $\{\alpha_q\}_{q\in F_k}$ is $\mathcal{F}_j$-admissible, the following holds.

  For any $\{G_k\}_k$ increasing sequence of $\mathcal{S}_\zeta$ sets, we have that

$$\lim_k \sum_{q\in F_k} |\alpha_q(\sum_{i\in G_k} x_i)| = 0.$$

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For any j, for any very fast growing sequence $\{\alpha_q\}_q$ of α-s.c.c. in W_{ξ_0} and for any $\{F_k\}_k$ increasing sequence of subsets of the naturals, such that $\{\alpha_q\}_{q\in F_k}$ is $\mathcal{F}_j$-admissible, the following holds.

For any $\{G_k\}_k$ increasing sequence of $\mathcal{S}_\zeta$ sets, we have that

$$\lim_k \sum_{q\in F_k} |\alpha_q(\sum_{i\in G_k} x_i)| = 0.$$

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The α and β indices

Then we say that the α_ζ-index of $\{x_k\}_k$ is zero and write

$$\alpha_\zeta(\{x_k\}_k) = 0.$$

The β_ζ indices are similarly defined.

- The α, β indices provide the following criterion for sequences generating higher order c_0 spreading models.
- Let $\{x_k\}_k$ be a seminormalized block sequence in $\mathfrak{X}_{\xi_0}$ and $\zeta \leqslant \xi$, such that $\alpha_\eta(\{x_k\}_k) = 0$ and $\beta_\eta(\{x_k\}_k) = 0$ for all $\eta < \zeta$. Then, passing if necessary, to a subsequence, the following holds.

(i) The sequence $\{x_k\}_k$ generates a c_0^ζ spreading model.

(ii) If $\zeta > 0$, then the sequence $\{x_k\}_k$ is $\mathcal{S}_\zeta$-RIS, i.e. for any $\{G_k\}$ increasing sequence of $\mathcal{S}_\zeta$ sets, if $y_k = \sum_{i \in G_k} x_i$, then $\{y_k\}_k$ is RIS.

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c_0^ξ spreading models in subspaces of $\mathfrak{X}_{\xi_0}$

- Beginning with an arbitrary normalized block sequence $\{x_k\}_k$, we are now going to describe how we may pass to a further normalized block sequence generating a c_0^ξ spreading model.

- Case 1: For every $\zeta < \xi$, for any $N \in [\mathbb{N}]$ there exists $L \in [N]$ with $\alpha_\zeta(\{x_k\}_{k\in L}) = 0$ and $\beta_\zeta(\{x_k\}_{k\in L}) = 0$
  Then passing, if necessary, to a subsequence, we have that $\alpha_\zeta(\{x_k\}_k) = 0$ and $\beta_\zeta(\{x_k\}_k) = 0$ for all $\zeta < \xi$. Hence we achieve the desired result.

- Case 2: If the above does not hold, set
  $\zeta_1 = \min\{\zeta : \text{there exists } N \in [\mathbb{N}] \text{ with } \alpha_\zeta(\{x_k\}_{k\in L}) \neq 0, \text{ for all } L \in [N]\}$
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- Beginning with an arbitrary normalized block sequence $\{x_k\}_k$, we are now going to describe how we may pass to a further normalized block sequence generating a c_0^ξ spreading model.

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c_0^ξ spreading models in subspaces of $\mathfrak{X}_{\xi_0}$

- Beginning with an arbitrary normalized block sequence $\{x_k\}_k$, we are now going to describe how we may pass to a further normalized block sequence generating a c_0^ξ spreading model.
- Case 1: For every $\zeta < \xi$, for any $N \in [\mathbb{N}]$ there exists $L \in [N]$ with $\alpha_\zeta(\{x_k\}_{k\in L}) = 0$ and $\beta_\zeta(\{x_k\}_{k\in L}) = 0$
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c_0^ξ spreading models in subspaces of $\mathfrak{X}_{\xi_0}$

- Beginning with an arbitrary normalized block sequence $\{x_k\}_k$, we are now going to describe how we may pass to a further normalized block sequence generating a c_0^ξ spreading model.

- Case 1: For every $\zeta < \xi$, for any $N \in [\mathbb{N}]$ there exists $L \in [N]$ with $\alpha_\zeta\left(\{x_k\}_{k\in L}\right) = 0$ and $\beta_\zeta\left(\{x_k\}_{k\in L}\right) = 0$
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c_0^ξ spreading models in subspaces of $\mathfrak{X}_{\xi_0}$

- Set $\zeta_0 = \min\{\zeta_1, \zeta_2\}$. We distinguish two further subcases.
- Subcase 1: $\zeta_0 > 0$. In this case, passing to a subsequence, $\{x_k\}_k$ generates a $c_0^{\zeta_0}$ spreading model and is $\mathcal{S}_{\zeta_0}$-RIS.

  Moreover, since $\alpha_{\zeta_0}(\{x_k\}_k) \neq 0$, or $\beta_{\zeta_0}(\{x_k\}_k) \neq 0$, we may construct a sequence of exact pairs $\{z_k, f_k\}_k$, such that the f_k are either of type I_α or of type I_β, such that for any k and any ϕ of type II, $w(f_k) \notin \hat{w}(\phi)$.

  It is proven that such a sequence admits a c_0^ξ spreading model.
- Subcase 2: $\zeta_0 = 0$. In this case, using classical techniques, we construct a normalized RIS sequence $\{y_k\}_k$

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The hereditarily ω^{ξ_0}-universal version of $\mathfrak{X}_{\xi_0}$

- By using the universal basis of Pełczyński $\{u_k\}_k$ when defining the norming set W_{ξ_0}, we obtain a reflexive HI space, such that $\{u_k\}_k$ ω^{ξ_0}-embeds into every subspace of $\mathfrak{X}_{\xi_0}$.
- The following fact can be easily proven.
- If α is a limit ordinal and rank $\left(\mathcal{T}(\{e_k\}_k, C, X)\right) \geqslant \alpha$, then for every $\{e_{k_n}\}_n$ subsequence of $\{e_k\}_k$, we have that rank $\left(\mathcal{T}(\{e_{k_n}\}_n, C, X)\right) \geqslant \alpha$.
- This yields, that the space $\mathfrak{X}_{\xi_0}$ is hereditarily ω^{ξ_0}-universal.

- There seems to be no obstacle, in defining an unconditional version of the space $\mathfrak{X}_{\xi_0}$, i.e. a space that has an unconditional basis and is hereditarily ω^{ξ_0}-universal for the unconditional basic sequences.

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- If α is a limit ordinal and rank $\big(\mathcal{T}(\{e_k\}_k, C, X)\big) \geqslant \alpha$, then for every $\{e_{k_n}\}_n$ subsequence of $\{e_k\}_k$, we have that rank $\big(\mathcal{T}(\{e_{k_n}\}_n, C, X)\big) \geqslant \alpha$.
- This yields, that the space $\mathfrak{X}_{\xi_0}$ is hereditarily ω^{ξ_0}-universal.
- There seems to be no obstacle, in defining an unconditional version of the space $\mathfrak{X}_{\xi_0}$, i.e. a space that has an unconditional basis and is hereditarily ω^{ξ_0}-universal for the unconditional basic sequences.

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HI α-minimal spaces.

- The following definition is due to C. Rodendal.
- Let α be countable ordinal. A Banach X space with a Schauder basis is said to be α-minimal, if any block sequence α-embeds into every subspace Y of X.
- Being hereditarily ω^{ξ_0}-universal, the space $\mathfrak{X}_{\xi_0}$ is also ω^{ξ_0}-minimal.
- Therefore, for every $\alpha < \omega_1$, there exists an α-minimal HI space $\mathfrak{X}_\alpha$.
- We would like to add, that there is no method known to us, of constructing an α-minimal HI space, without using the hereditary α-universality.

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