Almost invariant subspaces of operators in Banach spaces

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Joint work with: L. Marcoux and H. Radjavi.

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Voiculescu ('76): In fact, can do

$$T = \begin{bmatrix} * & K_1 \\ K_2 & * \end{bmatrix}, \quad \text{where } K_1, K_2 \in \mathcal{K}(\mathcal{H}) \text{ and } \|K_1\|, \|K_2\| \leqslant \varepsilon.$$

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In particular: D has no invariant half-spaces.

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Remark. D is a compact quasinilpotent operator without eigenvalues.

Theorem (Androulakis, P., Tcaciuc, Troitsky, '09)

Let $T \in L(X)$ is such that

1. T *has no eigenvalues;*
2. *for some $\varepsilon > 0$, the unbounded component of the resolvent set contains $\{0 < |z| < \varepsilon\}$;*
3. *there is $e \in X$ such that $T^n e \notin \overline{span}\{T^k e \ : \ k \neq n\}$ for all $n \in \mathbb{N}$.*

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Recall: $T \in L(X)$ is ***triangularizable*** if there is a chain $\mathcal{C}$ of subspaces in X such that

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Let $T \in L(X)$ be quasinilpotent, triangularizable and injective. Then T has an almost invariant half-space.
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This shows in particular: For the class of quasinilpotent operators on the reflexive spaces, the problem of existence of almost invariant half-spaces is a *weakening* of the Invariant Subspace Problem.

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Definition

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Davidson, Herrero ('90): T is bitriangular $\iff$ T is quasisimilar to its Jordan form $J(T)$,

$$J(T) = \bigoplus_{n \geqslant 1} \Big(\bigoplus_{k \geqslant 1} (\lambda_n I_k + J_k)^{\alpha_{n,k}} \Big), \quad \text{where } (\lambda_n)_{n \geqslant 1} = \sigma_p(T).$$

Theorem (Marcoux, P., Radjavi)

If $T \in \mathcal{B}(\mathcal{H})$ is bitriangular then either $T = \lambda I + F$ with $F \in \mathcal{F}(\mathcal{H})$ or T has a hyperinvariant half-space. In both cases, T admits an invariant half-space.

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Theorem (P., '10)

Let $\mathcal{A}$ be norm closed. The defects for a (fixed) $\mathcal{A}$-almost invariant half-space corresponding to different $T \in \mathcal{A}$ are uniformly bounded.

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Theorem (P., '10)

Let $\mathcal{A}$ be norm-closed, finitely generated, commutative. If $\mathcal{A}$ has an almost invariant half-space then $\mathcal{A}$ has an invariant half-space.

Theorem (Marcoux, P., Radjavi)

Let $\mathcal{A} \subseteq L(X)$ be norm-closed. If $\mathcal{A}$ has an almost invariant half-space that is complemented in X then $\mathcal{A}$ has an invariant half-space.

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Example (Marcoux, P., Radjavi)

There exists an operator $T \in \mathcal{B}(\mathcal{H})$ without reducing subspaces such that the norm-closed algebra $\mathcal{A}(T)$ generated by T has plenty of almost reducing half-spaces.

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Let $\mathcal{M}$ be a masa and $T \in \mathcal{B}(\mathcal{H})$ be such that $TP - PT \in \mathcal{F}(\mathcal{H})$ for all projections $P \in \mathcal{M}$. Then $T = D + F$ for some $D \in \mathcal{M}$ and $F \in \mathcal{F}(\mathcal{H})$.

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Corollary (Marcoux, P., Radjavi)

If $T \in \mathcal{B}(\mathcal{H})$ is such that every half-space in $\mathcal{H}$ is T-almost invariant then $T = \lambda I + F$ where $F \in \mathcal{F}(\mathcal{H})$.