Shift Invariant Preduals of $\ell_1(\mathbb{Z})$

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Preliminary Definition: A concrete algebraic predual of $\ell_1(\mathbb{Z})$ is a closed subspace E of $\ell_\infty(\mathbb{Z})$, so that E is shiftinvariant and E^* is isomorphic to $\ell_1(\mathbb{Z})$.

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Definition

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2. *Operator algebras*: Closed subalgebras of $L(X)$, for example C^*-algebras,
3. *Convolution algebras*. G locally compact group, μ Haar measure. (a) $M(G)$ space of finite Radon measure (b) $L_1(\mu)$, both with convolution.

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Then we say that $\mathcal{A}$ is a Dual Algebra and call $\mathcal{A}_*$ **a concrete** Predual of $\mathcal{A}$.

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1. *(1) simply means that as a Banach $\mathcal{A}$ is isomorphic to the dual of a Banach space X. Indeed if X is Banach space and $T : \mathcal{A} \to X^*$ is onto isomorphism, then consider $T^* : X^{**} \to \mathcal{A}^*$ and define*

$$\mathcal{A}_* := T^*(\iota(X)) \subset \mathcal{A}^*, \text{ with } \iota : X \hookrightarrow X^{**} \text{ canonical embedding,}$$

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2. *Assuming $\mathcal{A}_*$ satisfies (1). Then property (2) is equivalent with*

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3. *If $\mathcal{A}$ is a trivial Banach algebra (2) is vacuous, and, thus a trivial dual algebra is simply a Banach space which is isomorphic to a dual space. Thus, in that case, preduals are in general* **Not** *unique.*

The case of C^*-algebras

Sakai (1956): If $\mathcal{A}$ is a C^* algebra then (1) implies that $\mathcal{A}$ is a *von Neuman algebra* and (2) is automatically satisfied for any concrete predual. Moreover the predual is unique, up to isometry (but not up to isomorphism: $\ell_\infty \simeq L_\infty[0,1]$).

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Daws, Pham and White (2009): If $\mathcal{A}$ is a von Neuman algebra then $\mathcal{A}$ (literally!) has a unique concrete algebraic predual, meaning any two closed $\mathcal{A}$-submoduls $\mathcal{A}_*^{(1)}$ and $\mathcal{A}_*^{(2)}$ of $\mathcal{A}^*$ whose duals are (canonically) isomorphic to $\mathcal{A}$, are equal as vector spaces.

Formulation of Main Question

Consider on $\ell_1(\mathbb{Z})$ the *convolution* $* : \ell_1(\mathbb{Z}) \times \ell_1(\mathbb{Z}) \to \ell_1(\mathbb{Z})$

$$f * g = \Big(\sum_{k\in\mathbb{Z}} f(n-k)g(k) : n \in \mathbb{N} \Big) = \Big(\sum_{k\in\mathbb{N}} f(k)g(n-k) : n \in \mathbb{N} \Big),$$

for $f = (f(n))_{n\in\mathbb{Z}}$ and $g = (g(n))_{n\in\mathbb{Z}}$ in $\ell_1(\mathbb{Z})$.

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2. *Does it follow that E is isomorphic $c_0(\mathbb{Z})$?*

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1. Construction of a concrete algebraic predual $H_\lambda \subset \ell_\infty(\mathbb{Z})$ of $\ell_1(\mathbb{Z})$, for every $\lambda \in \mathbb{C}$, $|\lambda| > 0$, of $\ell_1(\mathbb{Z})$ not equal to $c_0(\mathbb{Z})$, not even isometric to c_0, but isomorphic to c_0.

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2. Characterization of all algebraic preduals of $\ell_1(\mathbb{Z})$ as certain quotients of $C(\mathcal{S})$, where $\mathcal{S}$ is a semi-topological semi-group compactification of $\mathbb{Z}$.
3. Construction of an algebraic predual E of $\ell_1(\mathbb{Z})$ which is not isomorphic to $c_0(\mathbb{Z})$.

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For simplicity we set $\lambda = 2$ and $H = H_2$, $E = E_2$.

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Then both, E and H, satisfy (1) and (2), and the canonical operators

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are injective and surjective, and thus, by the *Closed Graph Theorem* isomorphism. Thus E and H are both concrete algebraic preduals. Since $H \subset E$, an application of the Hahn Banach Theorem shows that both spaces are equal.

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Then, after some computations, we obtain

$$\tau(x_0) = \Big(1 - \frac{\sigma^2}{4}\Big)^{-1}\Big(1 - \frac{\sigma}{2}\Big)(x_0) = \sum_{j=0}^{\infty} \frac{\sigma^{2j}}{4^j}\Big(1 - \frac{\sigma}{2}\Big)(x_0) \in H.$$

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meaning that $\tau|_E$ is an operator on E (into E)

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$$\tau(x_0) = \Big(1 - \frac{\sigma^2}{4}\Big)^{-1}\Big(1 - \frac{\sigma}{2}\Big)(x_0) = \sum_{j=0}^{\infty} \frac{\sigma^{2j}}{4^j}\Big(1 - \frac{\sigma}{2}\Big)(x_0) \in H.$$

meaning that $\tau|_E$ is an operator on E (into E)

Thus

$$\langle \sigma^{m_0} \circ \tau^k(x_0), f \rangle = \langle \tau^k(x_0), \sigma^{-m_0}(f) \rangle \to_{k\to\infty} x_0(0) f(m_0) = f(m_0) \neq 0.$$

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It follows for $x \in E \subset C(\beta\mathbb{Z})$

$$\langle \mu, x\rangle = \langle \tilde{\mu}, x\rangle = \sum_{t\in\mathbb{Z}} f(t)\tilde{\mu}(\{t\}) + \sum_{t\in\mathbb{Z}}\sum_{k\in\mathbb{Z}} \int_{X_t^{(k)}} f(\mathcal{U})\, d\tilde{\mu}(\mathcal{U}) = \langle f, x\rangle.$$

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Lemma

Assume $y \in \ell_\infty(\mathbb{Z})$ has finite support. Then there is an $x \in H_\lambda$, so that

$$x|_{supp(y)} = y|_{supp(y)} \text{ and } \|x|_{\mathbb{Z}\setminus supp(y)}\|_\infty \le \lambda^{-1}\|y\|_\infty.$$

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(Meaning: $\mathcal{S}$ is a compact space containing $\mathbb{Z}$ as a dense subset, admitting an operation $+$, which extends $+$ on $\mathbb{Z}$, so that $(\mathcal{S}, +)$ is a semigroup, and which is separately continuous)

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$$\Theta : M(\mathcal{S}) \to \ell_1(\mathbb{Z}),$$

so that $Ker(\Theta)$ is w^*-closed ($w^* = \sigma(M(\mathcal{S}), C(\mathcal{S}))$ and

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Moreover in that case the pair $(\mathcal{S}, \Theta)$ can be chosen to be minimal, meaning that

$$\mathcal{S} \to \ell_1(\mathbb{Z}), \quad s \mapsto \Theta(\delta_s) \text{ is injective.}$$

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Injectivity: if $a \in \ell_1(\mathbb{Z})$ with $\langle a, x \rangle = 0$ for all $x \in E$, and thus $a \in ({}^{\perp}\mathrm{Ker}(\Theta))^{\perp}$. Since $\mathrm{Ker}(\Theta)$ is $\sigma(M(\mathcal{S}), C(\mathcal{S}))$-closed it follows that $a \in ({}^{\perp}\mathrm{Ker}(\Theta))^{\perp} = \mathrm{Ker}(\Theta)$.

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Surjectvity: if $\mu \in E^*$, extend μ to $\tilde{\mu} \in M(\mathcal{S})$, then $\tilde{\mu} - \Theta(\tilde{\mu}) \in \text{Ker}(\Theta)$, and thus for $x \in E =^\perp \text{Ker}(\Theta)$

$$\iota_E(\Theta(\tilde{\mu}))(x) = \Theta(\tilde{\mu})(x) = \tilde{\mu}(x) = \mu(x).$$

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If $\lim_{n\to\infty}\|a^n\|_\infty = 0$, then, regardless of the compact Hausdorff topology on $\mathcal{S}$, it follows that Ker(Θ) is $\sigma(\ell_1(\mathcal{S}), C(\mathcal{S}))$-closed in $\ell_1(\mathcal{S})$.

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It is enough to define a local compact topology on $\mathcal{T} = \mathbb{Z} \times \mathbb{N}_0$*, which turns* $\mathcal{T}$ *to a semi topological semi group. Then the one-point compactification on* $\mathcal{S} = \mathcal{T} \cup \{\infty\}$ *is also a semi-topological semi-group.*

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Important property: J is additively sparse:

$\forall s \neq t \in \mathbb{N} : (s + J) \cap (t + J)$ is finite.

We will define topology on $\mathcal{T}$ (and thus on $\mathcal{S}$) so that

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- $a \in \ell_1(\mathbb{Z})$ so that $1 = \|a\|_1 = \|a^n\|$, but on the other hand we still have $\lim_{n\to\infty} \|a^n\|_\infty = 0$ (needed to ensure that $\mathrm{Ker}(\Theta)$ is w^*-closed) then it follows that Szlenk index of E is ω^2.

We still need to choose $a := \Theta(e) \in \ell_1(\mathbb{Z})$.

- If $a = 0$ then $E = c_0(\mathbb{Z})$,
- If $a = \frac{1}{\lambda}\delta_0, |\lambda| < 1$, then we recapture first example H_λ,

But if we choose

- $a \in \ell_1(\mathbb{Z})$ so that $1 = \|a\|_1 = \|a^n\|$, but on the other hand we still have $\lim_{n\to\infty} \|a^n\|_\infty = 0$ (needed to ensure that $\mathrm{Ker}(\Theta)$ is w^*-closed) then it follows that Szlenk index of E is ω^2. Thus choose for example $a = \frac{1}{2}\delta_0 + \frac{1}{2}\delta_1$, in order to get algebraic predual of ℓ_1 which is not isomorphic to c_0.