# A phase transition in a kinetic Cucker-Smale model with friction

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Joint work with
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where

$$u_f(t, x) = \frac{\int vf(t, x, v)\, dv}{\int f(t, x, v)\, dv}$$

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- The first term encourages the velocity to align with the mean velocity
- The second term provides self-propulsion and friction, encouraging unit velocities
- The last term captures the influence of noise in the velocity

# The homogeneous problem

- We simplify by looking at the homogeneous problem:

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- In this case, we consider $f$ to be a function of only $v$, neglecting any effects of spatial inhomogeneity
- We work in the 1D case

# The stationary solutions

- We consider stationary solutions of the form:

$$f(v) = \frac{1}{Z} \exp\left( \frac{-1}{D} \left[ \alpha \frac{|v|^4}{4} + (1-\alpha)\frac{|v|^2}{2} - u_f \cdot v \right] \right)$$

[1] J. Tugaut. *Phase transitions of McKean-Vlasov processes in symmetric and asymmetric multi-wells landscape*, submitted 2011; S. Herrmann and J. Tugaut. *Non-uniqueness of stationary measures for self-stabilizing processes* Stochastic Processes and their Applications, 2010, vol. 120, issue 7, pages 1215-1246

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- We see that in order for the stationary solution to exist, $u_f$ must be a root of the equation:

$$H(u) = \frac{1}{Z} \int (v-u) \exp\left( \frac{\alpha}{D}\left(\frac{v^2}{2} - \frac{v^4}{4}\right) - \frac{v^2}{2D} \right) \exp\left(\frac{uv}{D}\right) dv$$

- We prove the following

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- We prove the following
  - There is a region of parameter space with only one such root, namely $u = 0$
  - There is another region of parameter space with at least three stationary solutions, $u = 0$ and $u = \pm C_{\alpha,D} \neq 0$
- This was independently proven by Julian Tugaut[1]

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# Our approach

- Our goal is to show that the number of stationary solutions depends on the values of $\alpha$ and $D$
- We can show that for any $\alpha > 0$:
  - in the large $D$ limit, there is only one stationary solution
  - as $D \to 0$, there are three stationary solutions

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- We can show that for any $\alpha > 0$:
  - in the large $D$ limit, there is only one stationary solution
  - as $D \to 0$, there are three stationary solutions
- We aim to numerically demonstrate that:
  - where the nonzero stationary solutions exist, they are stable while the zero solution is unstable
  - the zero solution is stable where it is the only solution

# Main idea of our proof

- Our proof hinges on the behavior of $H(u)$ as $D$ varies:
  - for small $D$, the slope of $H$ is positive at $u = 0$, while the slope is negative as $u \to \infty$

# Main idea of our proof

- Our proof hinges on the behavior of $H(u)$ as $D$ varies:
  - for small $D$, the slope of $H$ is positive at $u = 0$, while the slope is negative as $u \to \infty$
  - for large $D$, $\frac{dH}{du}$ is negative for all $u$.

# Main idea of our proof

- Our proof hinges on the behavior of $H(u)$ as $D$ varies:
  - for small $D$, the slope of $H$ is positive at $u = 0$, while the slope is negative as $u \to \infty$
  - for large $D$, $\frac{dH}{du}$ is negative for all $u$.
- Since we know that $u = 0$ is already a solution for all $D$, this shows that there are at least three roots of $H$ for small $D$, and only one root for large $D$

# The case of small $D$ at $u = 0$

Compute the derivative of $H$:

- Letting $P_u(v) = -\alpha(\frac{v^2}{2} - \frac{v^4}{4}) - \frac{v^2}{2D}$,

$$H(u) = \frac{1}{Z} \int (v - u) \exp\left(\frac{-1}{D} P_u(v)\right) dv$$

$$\Rightarrow \frac{dH}{du}(0) = \frac{1}{Z} \left[ - \int \exp\left(\frac{-1}{D} P_0(v)\right) dv + \frac{1}{D} \int v^2 \exp\left(\frac{-1}{D} P_0(v)\right) dv \right].$$

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- Define $I := - \int \exp\left(\frac{-1}{D} P_0(v)\right) dv, II := \int v^2 \exp\left(\frac{-1}{D} P_0(v)\right) dv$
- We use Laplace's method to show that as $D \to 0$, $I \geq 0$ and $II \to 1$, proving that $\frac{dH}{du}(0) > 0$

# The case of $u \to \infty$

- We derive an alternate expression for $H(u)$ using integration by parts:

$$H(u) = \int_{\mathbb{R}} (v-u) \exp\left(-\frac{|v|^2}{2D} + \frac{uv}{D}\right) \exp\left(-\alpha\left(\frac{|v|^4}{4D} - \frac{|v|^2}{2D}\right)\right) dv$$

$$= \alpha \int_{\mathbb{R}} (v - v^3) \exp\left(\frac{-\alpha}{D}\left(\frac{|v|^4}{4} - \frac{|v|^2}{2}\right) - \frac{|v|^2}{2D}\right) \exp\left(\frac{uv}{D}\right) dv$$

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- We then divide the integral into four pieces
- For all $D$, as $u \to \infty$, the negative pieces compensate for the positive, showing that $H(u) \to -\infty$ as $u \to \infty$

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- We show that $H$ is strictly decreasing for $D \to \infty$
- We similarly split the derivative into three pieces and show that the negative pieces compensate for the positive
- This shows that $H$ can have at most one zero for large $D$

# Varying $\alpha$ and $D$

- We have proven analytically that for small $D$, there are three stationary solutions, while for large $D$, there is only one

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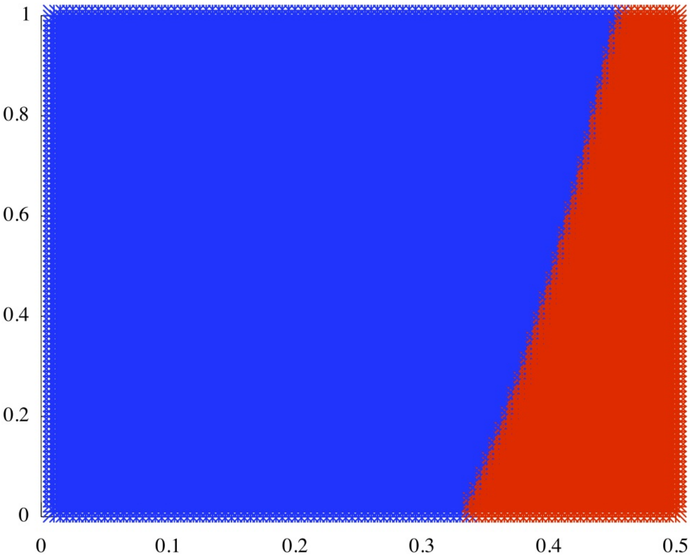
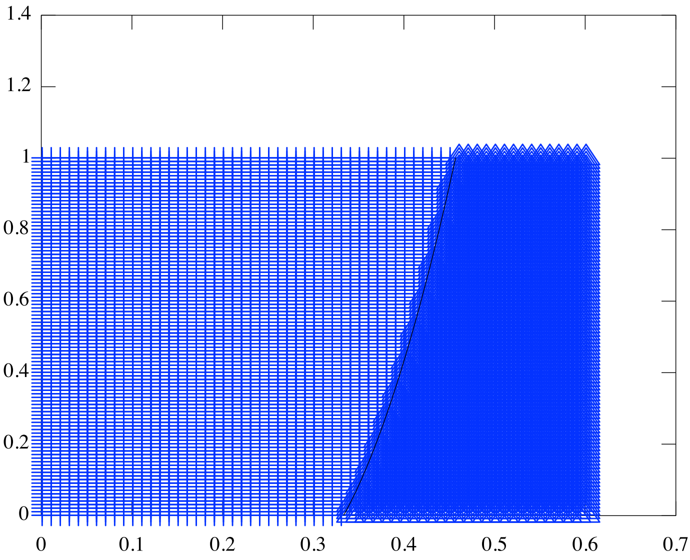
- We have proven analytically that for small $D$, there are three stationary solutions, while for large $D$, there is only one
- We next consider where in parameter space each of these situations occur
- We vary $\alpha$ and $D$ and count the number of roots of $H$

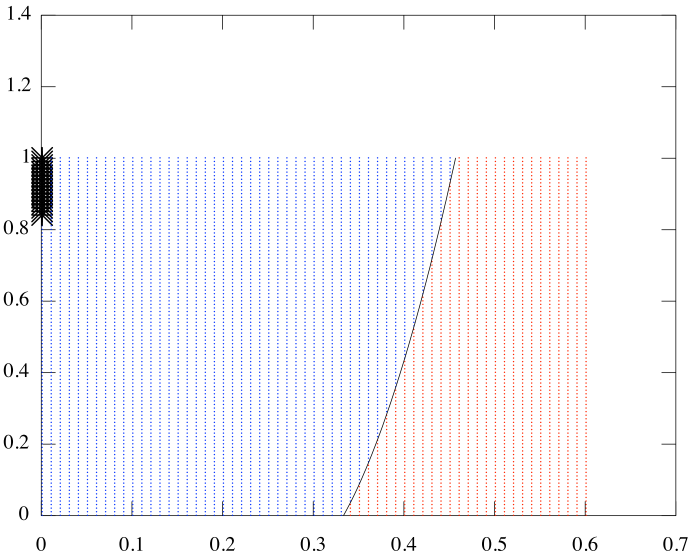
# Numerical exploration of the number of roots



# Numerically exploring the sign of the derivative



# The continuation method with the number of roots



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  - In the case of small noise, we expect the nonzero velocities to be stable
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  - We expect that stability to shift to the zero velocity once the sign of the derivative of $H$ changes
  - This result will show that the transition from three to one stationary solution is indeed a phase transition
- Explore the inhomogeneous case numerically
- We have an entropy for this problem
  - Can numerically compute this entropy
  - Are working to analytically prove the stability of the stationary solutions