# A review of 2nd order models for swarming

J. A. Carrillo

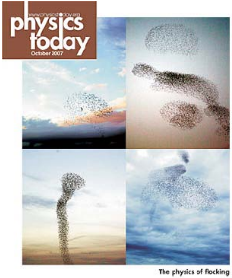
ICREA - Universitat Autònoma de Barcelona

BIRS, Banff, 2012

# Outline

# Outline

# Swarming by Nature or by design?




Fish schools and Birds flocks.

# Individual Based Models (Particle models)

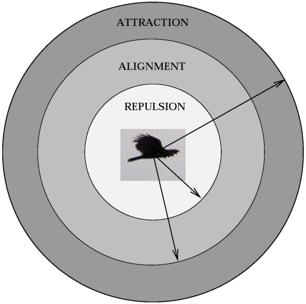
Swarming = Aggregation of agents of similar size and body type generally moving in a coordinated way.

Highly developed social organization: insects (locusts, ants, bees ...), fishes, birds, micro-organisms (myxo-bacteria, ...) and artificial robots for unmanned vehicle operation.

Interaction regions between individuals[a]

[a] Aoki, Helmerijk et al., Barbaro, Birnir et al.

- Repulsion Region: $R_k$.
- Attraction Region: $A_k$.
- Orientation Region: $O_k$.



# Individual Based Models (Particle models)

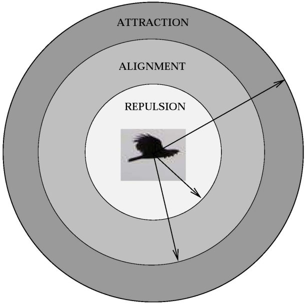
Swarming = Aggregation of agents of similar size and body type generally moving in a coordinated way.

Highly developed social organization: insects (locusts, ants, bees ...), fishes, birds, micro-organisms (myxo-bacteria, ...) and artificial robots for unmanned vehicle operation.

**Interaction regions between individuals**[a]

[a] Aoki, Helmerijk et al., Barbaro, Birnir et al.

- Repulsion Region: $R_k$.
- Attraction Region: $A_k$.
- Orientation Region: $O_k$.



# Individual Based Models (Particle models)

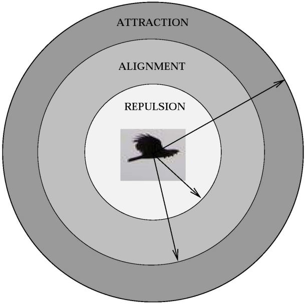
Swarming = Aggregation of agents of similar size and body type generally moving in a coordinated way.

Highly developed social organization: insects (locusts, ants, bees ...), fishes, birds, micro-organisms (myxo-bacteria, ...) and artificial robots for unmanned vehicle operation.

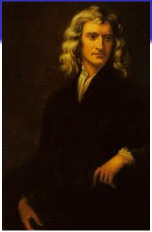
## Interaction regions between individuals[a]

[a] Aoki, Helmerijk et al., Barbaro, Birnir et al.

- Repulsion Region: $R_k$.
- Attraction Region: $A_k$.
- Orientation Region: $O_k$.



# 2nd Order Model: Newton's like equations



D'Orsogna, Bertozzi et al. model (PRL 2006):

$$\begin{cases} \dfrac{dx_i}{dt} = v_i, \\ m\dfrac{dv_i}{dt} = (\alpha - \beta\,|v_i|^2)v_i - \displaystyle\sum_{j\neq i} \nabla U(|x_i - x_j|). \end{cases}$$

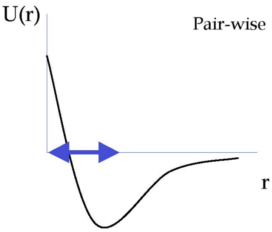
**Model assumptions:**

- Self-propulsion and friction terms determines an asymptotic speed of $\sqrt{\alpha/\beta}$.
- Attraction/Repulsion modeled by an effective pairwise potential $U(x)$.

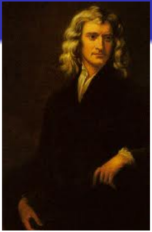
$$U(r) = -C_A e^{-r/\ell_A} + C_R e^{-r/\ell_R}.$$

One can also use Bessel functions in 2D and 3D to produce such a potential.

$C = C_R/C_A > 1$, $\ell = \ell_R/\ell_A < 1$ and $C\ell^2 < 1$:



# 2nd Order Model: Newton's like equations



D'Orsogna, Bertozzi et al. model (PRL 2006):

$$\begin{cases} \dfrac{dx_i}{dt} = v_i, \\ m\dfrac{dv_i}{dt} = (\alpha - \beta\,|v_i|^2)v_i - \displaystyle\sum_{j\neq i} \nabla U(|x_i - x_j|). \end{cases}$$

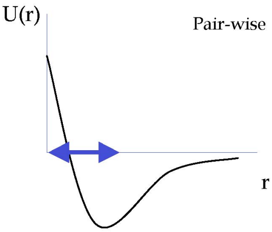
## Model assumptions:

- Self-propulsion and friction terms determines an asymptotic speed of $\sqrt{\alpha/\beta}$.
- Attraction/Repulsion modeled by an effective pairwise potential $U(x)$.

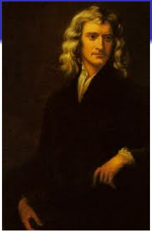
$$U(r) = -C_A e^{-r/\ell_A} + C_R e^{-r/\ell_R}.$$

One can also use Bessel functions in 2D and 3D to produce such a potential.

$C = C_R/C_A > 1$, $\ell = \ell_R/\ell_A < 1$ and $C\ell^2 < 1$:



# 2nd Order Model: Newton's like equations



D'Orsogna, Bertozzi et al. model (PRL 2006):

$$\begin{cases} \dfrac{dx_i}{dt} = v_i, \\ m\dfrac{dv_i}{dt} = (\alpha - \beta\,|v_i|^2)v_i - \displaystyle\sum_{j\neq i} \nabla U(|x_i - x_j|). \end{cases}$$

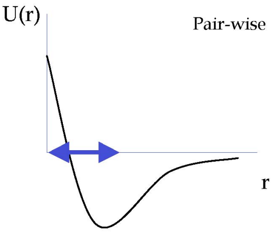
## Model assumptions:

- Self-propulsion and friction terms determines an asymptotic speed of $\sqrt{\alpha/\beta}$.
- Attraction/Repulsion modeled by an effective pairwise potential $U(x)$.

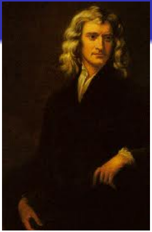
$$U(r) = -C_A e^{-r/\ell_A} + C_R e^{-r/\ell_R}.$$

One can also use Bessel functions in 2D and 3D to produce such a potential.

$C = C_R/C_A > 1$, $\ell = \ell_R/\ell_A < 1$ and $C\ell^2 < 1$:



# 2nd Order Model: Newton's like equations



D'Orsogna, Bertozzi et al. model (PRL 2006):

$$\begin{cases} \dfrac{dx_i}{dt} = v_i, \\ m\dfrac{dv_i}{dt} = (\alpha - \beta\,|v_i|^2)v_i - \displaystyle\sum_{j\neq i} \nabla U(|x_i - x_j|). \end{cases}$$

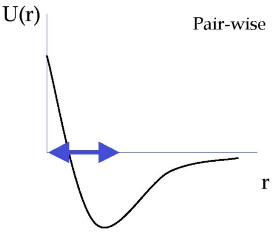
## Model assumptions:

- Self-propulsion and friction terms determines an asymptotic speed of $\sqrt{\alpha/\beta}$.
- Attraction/Repulsion modeled by an effective pairwise potential $U(x)$.

$$U(r) = -C_A e^{-r/\ell_A} + C_R e^{-r/\ell_R}.$$

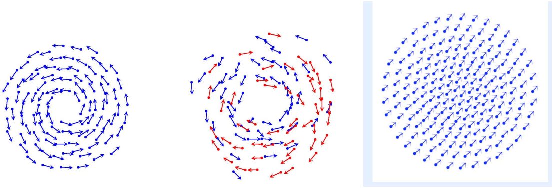
One can also use Bessel functions in 2D and 3D to produce such a potential.

$C = C_R/C_A > 1$, $\ell = \ell_R/\ell_A < 1$ and $C\ell^2 < 1$:



# Model with an asymptotic speed

Typical patterns: milling, double milling or flocking:



# Velocity consensus model

Cucker-Smale Model (IEEE Automatic Control 2007):

$$\begin{cases} \dfrac{dx_i}{dt} = v_i, \\ \dfrac{dv_i}{dt} = \displaystyle\sum_{j=1}^{N} a_{ij} \left( v_j - v_i \right), \end{cases}$$

with the communication rate, $\gamma \geq 0$:

$$a_{ij} = a(|x_i - x_j|) = \frac{1}{(1 + |x_i - x_j|^2)^{\gamma}}.$$

Asymptotic flocking: $\gamma < 1/2$; Cucker-Smale.

# Velocity consensus model

Cucker-Smale Model (IEEE Automatic Control 2007):

$$\begin{cases} \dfrac{dx_i}{dt} = v_i, \\ \dfrac{dv_i}{dt} = \displaystyle\sum_{j=1}^{N} a_{ij} \left( v_j - v_i \right), \end{cases}$$

with the communication rate, $\gamma \geq 0$:

$$a_{ij} = a(|x_i - x_j|) = \frac{1}{(1 + |x_i - x_j|^2)^{\gamma}}.$$

Asymptotic flocking: $\gamma < 1/2$; Cucker-Smale.

# Outline

# Leadership, Geometrical Constraints, and Cone of Influence

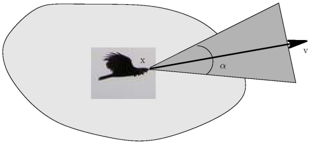
Cucker-Smale with local influence regions:

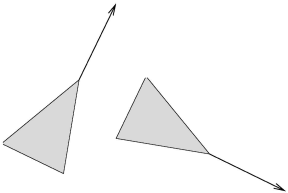
$$\begin{cases} \dfrac{dx_i}{dt} = v_i \ , \\ \dfrac{dv_i}{dt} = \displaystyle\sum_{j \in \Sigma_i(t)} a(|x_i - x_j|)(v_j - v_i) \ , \end{cases}$$

where $\Sigma_i(t) \subset \{1, \dots, N\}$ is the set of dependence, given by

$$\Sigma_i(t) := \left\{ 1 \leq \ell \leq N : \frac{(x_\ell - x_i) \cdot v_i}{|x_\ell - x_i||v_i|} \geq \alpha \right\}.$$

Cone of Vision:



# Leadership, Geometrical Constraints, and Cone of Influence

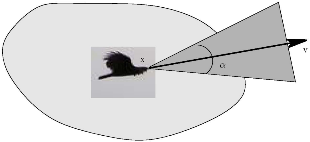
Cucker-Smale with local influence regions:

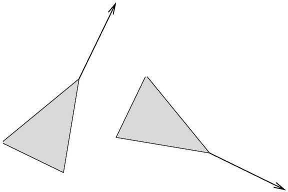
$$\begin{cases} \dfrac{dx_i}{dt} = v_i \ , \\ \dfrac{dv_i}{dt} = \displaystyle\sum_{j \in \Sigma_i(t)} a(|x_i - x_j|)(v_j - v_i) \ , \end{cases}$$

where $\Sigma_i(t) \subset \{1, \dots, N\}$ is the set of dependence, given by

$$\Sigma_i(t) := \left\{ 1 \leq \ell \leq N : \frac{(x_\ell - x_i) \cdot v_i}{|x_\ell - x_i||v_i|} \geq \alpha \right\}.$$

Cone of Vision:



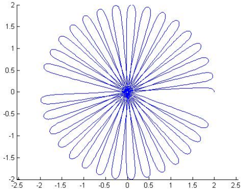
# Roosting Forces

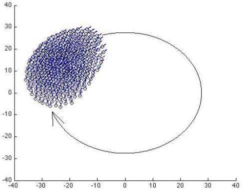
Adding a roosting area to the model:

$$\begin{cases} \dfrac{dx_i}{dt} = v_i, \\[2ex] \dfrac{dv_i}{dt} = (\alpha - \beta\,|v_i|^2)v_i - \displaystyle\sum_{j\neq i} \nabla U(|x_i - x_j|) - v_i^{\perp}\, \nabla_{x_i}\left[\phi(x_i)\cdot v_i^{\perp}\right]. \end{cases}$$

with the roosting potential $\phi$ given by $\phi(x) := \dfrac{b}{4}\left(\dfrac{|x|}{R_{\text{Roost}}}\right)^4$.

Roosting effect: milling flocks $N = 400, R_{\text{roost}} = 20$.



# Adding Noise

Self-Propelling/Friction/Interaction with Noise Particle Model:

$$\begin{cases} \dot{x}_i = v_i, \\ dv_i = \left[ (\alpha - \beta\, |v_i|^2) v_i - \nabla_{x_i} \sum_{j \neq i} U(|x_i - x_j|) \right] dt + \sqrt{2\sigma}\, d\Gamma_i(t) \,, \end{cases}$$

where $\Gamma_i(t)$ are $N$ independent copies of standard Wiener processes with values in $\mathbb{R}^d$ and $\sigma > 0$ is the noise strength. The Cucker–Smale Particle Model with Noise:

$$\begin{cases} dx_i = v_i dt \,, \\ dv_i = \sum_{j=1}^{N} a(|x_j - x_i|)(v_j - v_i)\, dt + \sqrt{2\sigma \sum_{j=1}^{m} a(|x_j - x_i|)}\, d\Gamma_i(t) \,. \end{cases}$$

# Adding Noise

Self-Propelling/Friction/Interaction with Noise Particle Model:

$$\begin{cases} \dot{x}_i = v_i, \\ dv_i = \left[ (\alpha - \beta\, |v_i|^2) v_i - \nabla_{x_i} \sum_{j \neq i} U(|x_i - x_j|) \right] dt + \sqrt{2\sigma}\, d\Gamma_i(t) \ , \end{cases}$$

where $\Gamma_i(t)$ are $N$ independent copies of standard Wiener processes with values in $\mathbb{R}^d$ and $\sigma > 0$ is the noise strength. The Cucker–Smale Particle Model with Noise:

$$\begin{cases} dx_i = v_i dt \ , \\ dv_i = \sum_{j=1}^{N} a(|x_j - x_i|)(v_j - v_i)\, dt + \sqrt{2\sigma \sum_{j=1}^{m} a(|x_j - x_i|)}\, d\Gamma_i(t) \ . \end{cases}$$

# Outline

# Vicsek's model

Assume $N$ particles moving at unit speed: reorientation & diffusion:

$$\begin{cases} dX_t^i = V_t^i\,dt, \\ dV_t^i = \sqrt{2}\,P(V_t^i)\circ dB_t^i - P(V_t^i)\left(\dfrac{1}{N}\displaystyle\sum_{j=1}^{N} K(X_t^i - X_t^j)(V_t^i - V_t^j)\right)dt. \end{cases}$$

Here $P(v)$ is the projection operator on the tangent space at $v/|v|$ to the unit sphere in $\mathbb{R}^d$, i.e.,

$$P(v) = I - \frac{v \otimes v}{|v|^2}\,.$$

Noise in the Stratatonovich sense: imposed by the rigorous construction of the Brownian motion on a manifold. Rigorous derivation: Bolley-Cañizo-Carrillo.

Main issue: phase transition? Degond-Liu-Frouvelle.

# Vicsek's model

Assume $N$ particles moving at unit speed: reorientation & diffusion:

$$\begin{cases} dX_t^i = V_t^i \, dt, \\ dV_t^i = \sqrt{2}\, P(V_t^i) \circ dB_t^i - P(V_t^i) \left( \dfrac{1}{N} \displaystyle\sum_{j=1}^{N} K(X_t^i - X_t^j)(V_t^i - V_t^j) \right) dt. \end{cases}$$

Here $P(v)$ is the projection operator on the tangent space at $v/|v|$ to the unit sphere in $\mathbb{R}^d$, i.e.,

$$P(v) = I - \frac{v \otimes v}{|v|^2}.$$

Noise in the Stratatonovich sense: imposed by the rigorous construction of the Brownian motion on a manifold. Rigorous derivation: Bolley-Cañizo-Carrillo.

Main issue: phase transition? Degond-Liu-Frouvelle.

# Vicsek's model

Assume $N$ particles moving at unit speed: reorientation & diffusion:

$$\begin{cases} dX_t^i = V_t^i \, dt, \\ dV_t^i = \sqrt{2}\, P(V_t^i) \circ dB_t^i - P(V_t^i) \left( \dfrac{1}{N} \displaystyle\sum_{j=1}^{N} K(X_t^i - X_t^j)(V_t^i - V_t^j) \right) dt. \end{cases}$$

Here $P(v)$ is the projection operator on the tangent space at $v/|v|$ to the unit sphere in $\mathbb{R}^d$, i.e.,

$$P(v) = I - \frac{v \otimes v}{|v|^2} .$$

Noise in the Stratatonovich sense: imposed by the rigorous construction of the Brownian motion on a manifold. Rigorous derivation: Bolley-Cañizo-Carrillo.

Main issue: phase transition? Degond-Liu-Frouvelle.

# Vicsek's model

Assume $N$ particles moving at unit speed: reorientation & diffusion:

$$\begin{cases} dX_t^i = V_t^i \, dt, \\ dV_t^i = \sqrt{2}\, P(V_t^i) \circ dB_t^i - P(V_t^i) \left( \dfrac{1}{N} \displaystyle\sum_{j=1}^{N} K(X_t^i - X_t^j)(V_t^i - V_t^j) \right) dt. \end{cases}$$

Here $P(v)$ is the projection operator on the tangent space at $v/|v|$ to the unit sphere in $\mathbb{R}^d$, i.e.,

$$P(v) = I - \frac{v \otimes v}{|v|^2} \, .$$

Noise in the Stratatonovich sense: imposed by the rigorous construction of the Brownian motion on a manifold. Rigorous derivation: Bolley-Cañizo-Carrillo.

Main issue: phase transition? Degond-Liu-Frouvelle.

# Vicsek's model

Assume $N$ particles moving at unit speed: reorientation & diffusion:

$$\begin{cases} dX_t^i = V_t^i\,dt, \\ dV_t^i = \sqrt{2}\,P(V_t^i)\circ dB_t^i - P(V_t^i)\left(\dfrac{1}{N}\displaystyle\sum_{j=1}^{N} K(X_t^i - X_t^j)(V_t^i - V_t^j)\right)dt. \end{cases}$$

Here $P(v)$ is the projection operator on the tangent space at $v/|v|$ to the unit sphere in $\mathbb{R}^d$, i.e.,

$$P(v) = I - \frac{v\otimes v}{|v|^2}\,.$$

Noise in the Stratatonovich sense: imposed by the rigorous construction of the Brownian motion on a manifold. Rigorous derivation: Bolley-Cañizo-Carrillo.

Main issue: phase transition? Degond-Liu-Frouvelle.

# Particle-Particle Interaction

Assumption: agents interact binary (like molecules in a Boltzmann gas): Carlen-Degond-Wennberg.

CL model (choose the leader): each time that a interaction happens, with certain probability, one agent decides to follow the other instantaneously.

BDG model (Bertin-Droz-Grégoire): each time that a interaction happens, with certain probability, both agents decide to follow their average velocity instantaneously.

Propagation of chaos: finite versus infinite number of particles. In the $N \to \infty$ limit, they lead to Boltzmann like models.

# Particle-Particle Interaction

Assumption: agents interact binary (like molecules in a Boltzmann gas): Carlen-Degond-Wennberg.

CL model (choose the leader): each time that a interaction happens, with certain probability, one agent decides to follow the other instantaneously.

BDG model (Bertin-Droz-Grégoire): each time that a interaction happens, with certain probability, both agents decide to follow their average velocity instantaneously.

Propagation of chaos: finite versus infinite number of particles. In the $N \to \infty$ limit, they lead to Boltzmann like models.

# Particle-Particle Interaction

Assumption: agents interact binary (like molecules in a Boltzmann gas): Carlen-Degond-Wennberg.

CL model (choose the leader): each time that a interaction happens, with certain probability, one agent decides to follow the other instantaneously.

BDG model (Bertin-Droz-Grégoire): each time that a interaction happens, with certain probability, both agents decide to follow their average velocity instantaneously.

Propagation of chaos: finite versus infinite number of particles. In the $N \to \infty$ limit, they lead to Boltzmann like models.

# Particle-Particle Interaction

Assumption: agents interact binary (like molecules in a Boltzmann gas): Carlen-Degond-Wennberg.

CL model (choose the leader): each time that a interaction happens, with certain probability, one agent decides to follow the other instantaneously.

BDG model (Bertin-Droz-Grégoire): each time that a interaction happens, with certain probability, both agents decide to follow their average velocity instantaneously.

Propagation of chaos: finite versus infinite number of particles. In the $N \to \infty$ limit, they lead to Boltzmann like models.

# Outline

# Mesoscopic models

Model with asymptotic velocity + Attraction/Repulsion:

$$\frac{\partial f}{\partial t} + v \cdot \nabla_x f + \mathrm{div}_v[(\alpha - \beta|v|^2)vf] - \mathrm{div}_v\left[(\nabla_x U \star \rho)f\right] = 0.$$

Velocity consensus Model:

$$\frac{\partial f}{\partial t} + v \cdot \nabla_x f = \nabla_v \cdot \left[ \underbrace{\left( \int_{\mathbb{R}^{2d}} \frac{v-w}{(1+|x-y|^2)^\gamma} f(y,w,t)\, dy\, dw \right)}_{:=\xi(f)(x,v,t)} f(x,v,t) \right]$$

Orientation, Attraction and Repulsion:

$$\frac{\partial f}{\partial t} + v \cdot \nabla_x f - \mathrm{div}_v\left[(\nabla_x U \star \rho)f\right] = \nabla_v \cdot \left[\xi(f)(x,v,t) f(x,v,t)\right].$$

# Mesoscopic models

Model with asymptotic velocity + Attraction/Repulsion:

$$\frac{\partial f}{\partial t} + v \cdot \nabla_x f + \mathrm{div}_v[(\alpha - \beta|v|^2)vf] - \mathrm{div}_v\left[(\nabla_x U \star \rho)f\right] = 0.$$

Velocity consensus Model:

$$\frac{\partial f}{\partial t} + v \cdot \nabla_x f = \nabla_v \cdot \left[\underbrace{\left(\int_{\mathbb{R}^{2d}} \frac{v-w}{(1+|x-y|^2)^\gamma} f(y,w,t)\,dy\,dw\right)}_{:=\xi(f)(x,v,t)} f(x,v,t)\right]$$

Orientation, Attraction and Repulsion:

$$\frac{\partial f}{\partial t} + v \cdot \nabla_x f - \mathrm{div}_v\left[(\nabla_x U \star \rho)f\right] = \nabla_v \cdot \left[\xi(f)(x,v,t) f(x,v,t)\right].$$

# Mesoscopic models

Model with asymptotic velocity + Attraction/Repulsion:

$$\frac{\partial f}{\partial t} + v \cdot \nabla_x f + \mathrm{div}_v[(\alpha - \beta|v|^2)vf] - \mathrm{div}_v\left[(\nabla_x U \star \rho)f\right] = 0.$$

Velocity consensus Model:

$$\frac{\partial f}{\partial t} + v \cdot \nabla_x f = \nabla_v \cdot \Bigg[ \underbrace{\left( \int_{\mathbb{R}^{2d}} \frac{v-w}{(1+|x-y|^2)^\gamma} f(y,w,t)\, dy\, dw \right)}_{:=\xi(f)(x,v,t)} f(x,v,t) \Bigg]$$

Orientation, Attraction and Repulsion:

$$\frac{\partial f}{\partial t} + v \cdot \nabla_x f - \mathrm{div}_v\left[(\nabla_x U \star \rho)f\right] = \nabla_v \cdot \left[\xi(f)(x,v,t) f(x,v,t)\right].$$

# Definition of the distance

## Transporting measures:

Given $T : \mathbb{R}^d \longrightarrow \mathbb{R}^d$ mesurable, we say that $\nu = T\#\mu$, if $\nu[K] := \mu[T^{-1}(K)]$ for all mesurable sets $K \subset \mathbb{R}^d$, equivalently

$$\int_{\mathbb{R}^d} \varphi \, d\nu = \int_{\mathbb{R}^d} (\varphi \circ T) \, d\mu \qquad \text{for all } \varphi \in C_o(\mathbb{R}^d) \, .$$

## Random variables:

Say that $X$ is a random variable with law given by $\mu$, is to say $X : (\Omega, \mathcal{A}, P) \longrightarrow (\mathbb{R}^d, \mathcal{B}_d)$ is a mesurable map such that $X\#P = \mu$, i.e.,

$$\int_{\mathbb{R}^d} \varphi(x) \, d\mu = \int_{\Omega} (\varphi \circ X) \, dP = \mathbb{E}\left[\varphi(X)\right] .$$

## Kantorovich-Rubinstein-Wasserstein Distance $p = 1, 2$:

$$W_p^p(\mu, \nu) = \inf_{(X,Y)} \left\{ \mathbb{E}\left[|X - Y|^p\right] \right\}$$

where $(X, Y)$ are all possible couples of random variables with $\mu$ and $\nu$ as respective laws.

# Definition of the distance

## Transporting measures:

Given $T : \mathbb{R}^d \longrightarrow \mathbb{R}^d$ mesurable, we say that $\nu = T\#\mu$, if $\nu[K] := \mu[T^{-1}(K)]$ for all mesurable sets $K \subset \mathbb{R}^d$, equivalently

$$\int_{\mathbb{R}^d} \varphi \, d\nu = \int_{\mathbb{R}^d} (\varphi \circ T) \, d\mu \qquad \text{for all } \varphi \in C_o(\mathbb{R}^d) \, .$$

## Random variables:

Say that $X$ is a random variable with law given by $\mu$, is to say $X : (\Omega, \mathcal{A}, P) \longrightarrow (\mathbb{R}^d, \mathcal{B}_d)$ is a mesurable map such that $X\#P = \mu$, i.e.,

$$\int_{\mathbb{R}^d} \varphi(x) \, d\mu = \int_{\Omega} (\varphi \circ X) \, dP = \mathbb{E}\left[\varphi(X)\right] .$$

## Kantorovich-Rubinstein-Wasserstein Distance $p = 1, 2$:

$$W_p^p(\mu, \nu) = \inf_{(X,Y)} \left\{ \mathbb{E}\left[ |X - Y|^p \right] \right\}$$

where $(X, Y)$ are all possible couples of random variables with $\mu$ and $\nu$ as respective laws.

# Definition of the distance

## Transporting measures:

Given $T : \mathbb{R}^d \longrightarrow \mathbb{R}^d$ mesurable, we say that $\nu = T\#\mu$, if $\nu[K] := \mu[T^{-1}(K)]$ for all mesurable sets $K \subset \mathbb{R}^d$, equivalently

$$\int_{\mathbb{R}^d} \varphi \, d\nu = \int_{\mathbb{R}^d} (\varphi \circ T) \, d\mu \qquad \text{for all } \varphi \in C_o(\mathbb{R}^d)\,.$$

## Random variables:

Say that $X$ is a random variable with law given by $\mu$, is to say $X : (\Omega, \mathcal{A}, P) \longrightarrow (\mathbb{R}^d, \mathcal{B}_d)$ is a mesurable map such that $X\#P = \mu$, i.e.,

$$\int_{\mathbb{R}^d} \varphi(x) \, d\mu = \int_{\Omega} (\varphi \circ X) \, dP = \mathbb{E}\left[\varphi(X)\right].$$

## Kantorovich-Rubinstein-Wasserstein Distance $p = 1, 2$:

$$W_p^p(\mu, \nu) = \inf_{(X,Y)} \left\{ \mathbb{E}\left[|X - Y|^p\right] \right\}$$

where $(X, Y)$ are all possible couples of random variables with $\mu$ and $\nu$ as respective laws.

# Definition of the distance

## Transporting measures:

Given $T : \mathbb{R}^d \longrightarrow \mathbb{R}^d$ mesurable, we say that $\nu = T\#\mu$, if $\nu[K] := \mu[T^{-1}(K)]$ for all mesurable sets $K \subset \mathbb{R}^d$, equivalently

$$\int_{\mathbb{R}^d} \varphi \, d\nu = \int_{\mathbb{R}^d} (\varphi \circ T) \, d\mu \qquad \text{for all } \varphi \in C_o(\mathbb{R}^d)\,.$$

## Random variables:

Say that $X$ is a random variable with law given by $\mu$, is to say $X : (\Omega, \mathcal{A}, P) \longrightarrow (\mathbb{R}^d, \mathcal{B}_d)$ is a mesurable map such that $X\#P = \mu$, i.e.,

$$\int_{\mathbb{R}^d} \varphi(x) \, d\mu = \int_{\Omega} (\varphi \circ X) \, dP = \mathbb{E}\left[\varphi(X)\right].$$

## Kantorovich-Rubinstein-Wasserstein Distance $p = 1, 2$:

$$W_p^p(\mu, \nu) = \inf_{(X,Y)} \left\{ \mathbb{E}\left[|X - Y|^p\right] \right\}$$

where $(X, Y)$ are all possible couples of random variables with $\mu$ and $\nu$ as respective laws.

# Definition of the distance

## Transporting measures:

Given $T : \mathbb{R}^d \longrightarrow \mathbb{R}^d$ mesurable, we say that $\nu = T\#\mu$, if $\nu[K] := \mu[T^{-1}(K)]$ for all mesurable sets $K \subset \mathbb{R}^d$, equivalently

$$\int_{\mathbb{R}^d} \varphi \, d\nu = \int_{\mathbb{R}^d} (\varphi \circ T) \, d\mu \qquad \text{for all } \varphi \in C_o(\mathbb{R}^d)\,.$$

## Random variables:

Say that $X$ is a random variable with law given by $\mu$, is to say $X : (\Omega, \mathcal{A}, P) \longrightarrow (\mathbb{R}^d, \mathcal{B}_d)$ is a mesurable map such that $X\#P = \mu$, i.e.,

$$\int_{\mathbb{R}^d} \varphi(x) \, d\mu = \int_{\Omega} (\varphi \circ X) \, dP = \mathbb{E}\left[\varphi(X)\right].$$

## Kantorovich-Rubinstein-Wasserstein Distance $p = 1, 2$:

$$W_p^p(\mu, \nu) = \inf_{(X,Y)} \left\{\mathbb{E}\left[|X - Y|^p\right]\right\}$$

where $(X, Y)$ are all possible couples of random variables with $\mu$ and $\nu$ as respective laws.

# Well-posedness in probability measures[1]

## Existence, uniqueness and stability

Take a potential $U \in \mathcal{C}_b^2(\mathbb{R}^d)$, and $f_0$ a measure on $\mathbb{R}^d \times \mathbb{R}^d$ with compact support. There exists a solution $f \in \mathcal{C}([0,+\infty); \mathcal{P}_1(\mathbb{R}^d))$ in the sense of solving the equation through the characteristics: $f_t := P^t \# f_0$ with $P^t$ the flow map associated to the equation.

Moreover, the solutions remains compactly supported for all time with a possibly growing in time support.

Moreover, given any two solutions $f$ and $g$ with initial data $f_0$ and $g_0$, there is an increasing function depending on the size of the support of the solutions and the parameters, such that

$$W_1(f_t, g_t) \le \alpha(t)\, W_1(f_0, g_0)$$

Hauray-Jabin 2011: mean field limit for Vlasov with potentials such that $|\nabla U| \le r^{-\alpha}$, with $\alpha < 1$.

[1] Dobrushin-Hepp-Neunzert, 1977-79 for the Vlasov.

# Well-posedness in probability measures[1]

## Existence, uniqueness and stability

Take a potential $U \in \mathcal{C}_b^2(\mathbb{R}^d)$, and $f_0$ a measure on $\mathbb{R}^d \times \mathbb{R}^d$ with compact support. There exists a solution $f \in \mathcal{C}([0,+\infty); \mathcal{P}_1(\mathbb{R}^d))$ in the sense of solving the equation through the characteristics: $f_t := P^t \# f_0$ with $P^t$ the flow map associated to the equation.

Moreover, the solutions remains compactly supported for all time with a possibly growing in time support.

Moreover, given any two solutions $f$ and $g$ with initial data $f_0$ and $g_0$, there is an increasing function depending on the size of the support of the solutions and the parameters, such that

$$W_1(f_t, g_t) \le \alpha(t)\, W_1(f_0, g_0)$$

Hauray-Jabin 2011: mean field limit for Vlasov with potentials such that $|\nabla U| \le r^{-\alpha}$, with $\alpha < 1$.

[1] Dobrushin-Hepp-Neunzert, 1977-79 for the Vlasov.

# Well-posedness in probability measures[1]

## Existence, uniqueness and stability

Take a potential $U \in \mathcal{C}_b^2(\mathbb{R}^d)$, and $f_0$ a measure on $\mathbb{R}^d \times \mathbb{R}^d$ with compact support. There exists a solution $f \in \mathcal{C}([0,+\infty);\mathcal{P}_1(\mathbb{R}^d))$ in the sense of solving the equation through the characteristics: $f_t := P^t \# f_0$ with $P^t$ the flow map associated to the equation.

Moreover, the solutions remains compactly supported for all time with a possibly growing in time support.

Moreover, given any two solutions $f$ and $g$ with initial data $f_0$ and $g_0$, there is an increasing function depending on the size of the support of the solutions and the parameters, such that

$$W_1(f_t, g_t) \le \alpha(t)\, W_1(f_0, g_0)$$

Hauray-Jabin 2011: mean field limit for Vlasov with potentials such that $|\nabla U| \le r^{-\alpha}$, with $\alpha < 1$.

[1] Dobrushin-Hepp-Neunzert, 1977-79 for the Vlasov.

# Well-posedness in probability measures[1]

## Existence, uniqueness and stability

Take a potential $U \in \mathcal{C}_b^2(\mathbb{R}^d)$, and $f_0$ a measure on $\mathbb{R}^d \times \mathbb{R}^d$ with compact support. There exists a solution $f \in \mathcal{C}([0,+\infty);\mathcal{P}_1(\mathbb{R}^d))$ in the sense of solving the equation through the characteristics: $f_t := P^t \# f_0$ with $P^t$ the flow map associated to the equation.

Moreover, the solutions remains compactly supported for all time with a possibly growing in time support.

Moreover, given any two solutions $f$ and $g$ with initial data $f_0$ and $g_0$, there is an increasing function depending on the size of the support of the solutions and the parameters, such that

$$W_1(f_t, g_t) \le \alpha(t)\, W_1(f_0, g_0)$$

Hauray-Jabin 2011: mean field limit for Vlasov with potentials such that $|\nabla U| \le r^{-\alpha}$, with $\alpha < 1$.

[1] Dobrushin-Hepp-Neunzert, 1977-79 for the Vlasov.

# Convergence of the particle method

- Empirical measures: if $x_i, v_i : [0,T) \to \mathbb{R}^d$, for $i = 1, \dots, N$, is a solution to the ODE system,

$$\begin{cases} \dfrac{dx_i}{dt} = v_i, \\[2ex] \dfrac{dv_i}{dt} = \overbrace{(\alpha - \beta |v_i|^2) v_i}^{\text{propulsion-friction}} - \overbrace{\sum_{j \neq i} m_j \nabla U(|x_i - x_j|)}^{\text{attraction-repulsion}} + \overbrace{\sum_{j=1}^{N} m_j a_{ij} (v_j - v_i)}^{\text{orientation}} . \end{cases}$$

then the $f : [0,T) \to \mathcal{P}_1(\mathbb{R}^d)$ given by

$$f_N(t) := \sum_{i=1}^{N} m_i \delta_{(x_i(t), v_i(t))}$$

is the solution corresponding to initial atomic measures.

- Convergence of approximations of measures by particles due to the stability at any given time $T$ as an alternative derivation of the kinetic models.

# Convergence of the particle method

- Empirical measures: if $x_i, v_i : [0,T) \to \mathbb{R}^d$, for $i = 1, \ldots, N$, is a solution to the ODE system,

$$\begin{cases} \dfrac{dx_i}{dt} = v_i, \\ \dfrac{dv_i}{dt} = \overbrace{(\alpha - \beta\,|v_i|^2)v_i}^{\text{propulsion-friction}} - \overbrace{\sum_{j \neq i} m_j \nabla U(|x_i - x_j|)}^{\text{attraction-repulsion}} + \overbrace{\sum_{j=1}^{N} m_j a_{ij}\,(v_j - v_i)}^{\text{orientation}}. \end{cases}$$

  then the $f : [0,T) \to \mathcal{P}_1(\mathbb{R}^d)$ given by

$$f_N(t) := \sum_{i=1}^{N} m_i \delta_{(x_i(t), v_i(t))}$$

  is the solution corresponding to initial atomic measures.

- Convergence of approximations of measures by particles due to the stability at any given time $T$ as an alternative derivation of the kinetic models.

# Convergence of the particle method

- Empirical measures: if $x_i, v_i : [0,T) \to \mathbb{R}^d$, for $i = 1, \ldots, N$, is a solution to the ODE system,

$$\begin{cases} \dfrac{dx_i}{dt} = v_i, \\ \dfrac{dv_i}{dt} = \overbrace{(\alpha - \beta\,|v_i|^2)v_i}^{\text{propulsion-friction}} - \overbrace{\sum_{j \neq i} m_j \nabla U(|x_i - x_j|)}^{\text{attraction-repulsion}} + \overbrace{\sum_{j=1}^{N} m_j a_{ij}\,(v_j - v_i)}^{\text{orientation}}. \end{cases}$$

  then the $f : [0,T) \to \mathcal{P}_1(\mathbb{R}^d)$ given by

$$f_N(t) := \sum_{i=1}^{N} m_i \delta_{(x_i(t), v_i(t))}$$

  is the solution corresponding to initial atomic measures.

- Convergence of approximations of measures by particles due to the stability at any given time $T$ as an alternative derivation of the kinetic models.

# Convergence of the particle method

- Empirical measures: if $x_i, v_i : [0,T) \to \mathbb{R}^d$, for $i = 1, \ldots, N$, is a solution to the ODE system,

$$\begin{cases} \dfrac{dx_i}{dt} = v_i, \\ \dfrac{dv_i}{dt} = \overbrace{(\alpha - \beta\,|v_i|^2)v_i}^{\text{propulsion-friction}} - \overbrace{\sum_{j\neq i} m_j \nabla U(|x_i - x_j|)}^{\text{attraction-repulsion}} + \overbrace{\sum_{j=1}^{N} m_j a_{ij}\,(v_j - v_i)}^{\text{orientation}}. \end{cases}$$

  then the $f : [0,T) \to \mathcal{P}_1(\mathbb{R}^d)$ given by

$$f_N(t) := \sum_{i=1}^{N} m_i \delta_{(x_i(t), v_i(t))}$$

  is the solution corresponding to initial atomic measures.

- Convergence of approximations of measures by particles due to the stability at any given time $T$ as an alternative derivation of the kinetic models.

# Convergence of the particle method

- Empirical measures: if $x_i, v_i : [0,T) \to \mathbb{R}^d$, for $i = 1, \ldots, N$, is a solution to the ODE system,

$$\begin{cases} \dfrac{dx_i}{dt} = v_i, \\ \dfrac{dv_i}{dt} = \overbrace{(\alpha - \beta\,|v_i|^2)v_i}^{\text{propulsion-friction}} - \overbrace{\sum_{j\neq i} m_j \nabla U(|x_i - x_j|)}^{\text{attraction-repulsion}} + \overbrace{\sum_{j=1}^{N} m_j a_{ij}\,(v_j - v_i)}^{\text{orientation}}. \end{cases}$$

  then the $f : [0,T) \to \mathcal{P}_1(\mathbb{R}^d)$ given by

$$f_N(t) := \sum_{i=1}^{N} m_i \delta_{(x_i(t), v_i(t))}$$

  is the solution corresponding to initial atomic measures.

- Convergence of approximations of measures by particles due to the stability at any given time $T$ as an alternative derivation of the kinetic models.

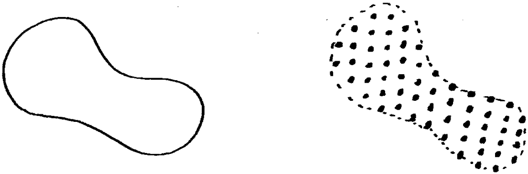
# Mean-Field Limit

Just take as many particles as needed in order to have

$$W_1(f_t, f_t^N) \leq \alpha(t)\, W_1(f_0, f_0^N) \to 0 \qquad \text{as } N \to \infty$$

by sampling the initial data in a suitable way.

The sequences of particle solutions becomes a Cauchy sequence with the distance $W_1$ converging to the solution of the kinetic equation.



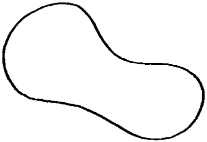
# Mean-Field Limit

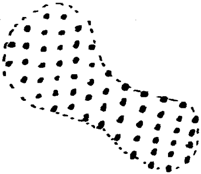
Just take as many particles as needed in order to have

$$W_1(f_t, f_t^N) \leq \alpha(t)\, W_1(f_0, f_0^N) \to 0 \qquad \text{as } N \to \infty$$

by sampling the initial data in a suitable way.

The sequences of particle solutions becomes a Cauchy sequence with the distance $W_1$ converging to the solution of the kinetic equation.





# Outline

# Stochastic Particle System

## General Interacting Particle System with Noise:

$N$ interacting $\mathbb{R}^{2d}$-valued processes $(X_t^i, V_t^i)_{t\geq 0}$ with $1 \leq i \leq N$ solution of

$$\begin{cases} dX_t^i = V_t^i dt, \\ dV_t^i = \sqrt{2} dB_t^i - F(X_t^i, V_t^i) dt - \dfrac{1}{N} \displaystyle\sum_{j=1}^{N} H(X_t^i - X_t^j, V_t^i - V_t^j) dt, \end{cases}$$

with independent and commonly distributed initial data $(X_0^i, V_0^i)$ with $1 \leq i \leq N$.

Empirical Measure:

$$\hat{f}_t^N = \frac{1}{N} \sum_{i=1}^{N} \delta_{(X_t^i, V_t^i)}$$

# Stochastic Particle System

**General Interacting Particle System with Noise:**

$N$ interacting $\mathbb{R}^{2d}$-valued processes $(X_t^i, V_t^i)_{t\geq 0}$ with $1 \leq i \leq N$ solution of

$$\begin{cases} dX_t^i = V_t^i dt, \\ dV_t^i = \sqrt{2}dB_t^i - F(X_t^i, V_t^i)dt - \dfrac{1}{N}\displaystyle\sum_{j=1}^{N} H(X_t^i - X_t^j, V_t^i - V_t^j)dt, \end{cases}$$

with independent and commonly distributed initial data $(X_0^i, V_0^i)$ with $1 \leq i \leq N$.

**Empirical Measure:**

$$\hat{f}_t^N = \frac{1}{N}\sum_{i=1}^{N} \delta_{(X_t^i, V_t^i)}$$

# Stochastic Particle System

**General Interacting Particle System with Noise:**

$N$ interacting $\mathbb{R}^{2d}$-valued processes $(X_t^i, V_t^i)_{t\geq 0}$ with $1 \leq i \leq N$ solution of

$$\begin{cases} dX_t^i = V_t^i dt, \\ dV_t^i = \sqrt{2} dB_t^i - F(X_t^i, V_t^i)dt - \dfrac{1}{N}\displaystyle\sum_{j=1}^{N} H(X_t^i - X_t^j, V_t^i - V_t^j)dt, \end{cases}$$

with independent and commonly distributed initial data $(X_0^i, V_0^i)$ with $1 \leq i \leq N$.

Empirical Measure:

$$\hat{f}_t^N = \frac{1}{N}\sum_{i=1}^{N} \delta_{(X_t^i, V_t^i)}$$

# Coupling Method 1

**Stochastic Particle System Associated to PDE:**

$N$ interacting processes $(\overline{X}_t^i, \overline{V}_t^i)_{t\geq 0}$ solutions of the kinetic McKean-Vlasov type equation on $\mathbb{R}^{2d}$:

$$\begin{cases} d\overline{X}_t^i = \overline{V}_t^i \, dt \\ d\overline{V}_t^i = \sqrt{2}\, dB_t^i - F(\overline{X}_t^i, \overline{V}_t^i)dt - H * f_t(\overline{X}_t^i, \overline{V}_t^i)dt, \\ (\overline{X}_0^i, \overline{V}_0^i) = (X_0^i, V_0^i), \quad f_t = \text{law}(\overline{X}_t^i, \overline{V}_t^i). \end{cases}$$

The stochastic processes are independent and identically distributed according to

$$\partial_t f_t + v \cdot \nabla_x f_t = \Delta_v f_t + \nabla_v \cdot ((F + H * f_t) f_t), \quad t > 0, x, v \in \mathbb{R}^d.$$

# Coupling Method 1

### Stochastic Particle System Associated to PDE:

$N$ interacting processes $(\overline{X}_t^i, \overline{V}_t^i)_{t\geq 0}$ solutions of the kinetic McKean-Vlasov type equation on $\mathbb{R}^{2d}$:

$$\begin{cases} d\overline{X}_t^i = \overline{V}_t^i \, dt \\ d\overline{V}_t^i = \sqrt{2}\, dB_t^i - F(\overline{X}_t^i, \overline{V}_t^i)dt - H * f_t(\overline{X}_t^i, \overline{V}_t^i)dt, \\ (\overline{X}_0^i, \overline{V}_0^i) = (X_0^i, V_0^i), \quad f_t = \text{law}(\overline{X}_t^i, \overline{V}_t^i). \end{cases}$$

The stochastic processes are independent and identically distributed according to

$$\partial_t f_t + v \cdot \nabla_x f_t = \Delta_v f_t + \nabla_v \cdot ((F + H * f_t) f_t), \quad t > 0, x, v \in \mathbb{R}^d.$$

# Coupling Method 1

### Stochastic Particle System Associated to PDE:

$N$ interacting processes $(\overline{X}_t^i, \overline{V}_t^i)_{t\geq 0}$ solutions of the kinetic McKean-Vlasov type equation on $\mathbb{R}^{2d}$:

$$\begin{cases} d\overline{X}_t^i = \overline{V}_t^i \, dt \\ d\overline{V}_t^i = \sqrt{2}\, dB_t^i - F(\overline{X}_t^i, \overline{V}_t^i)dt - H * f_t(\overline{X}_t^i, \overline{V}_t^i)dt, \\ (\overline{X}_0^i, \overline{V}_0^i) = (X_0^i, V_0^i), \quad f_t = \mathrm{law}(\overline{X}_t^i, \overline{V}_t^i). \end{cases}$$

The stochastic processes are independent and identically distributed according to

$$\partial_t f_t + v \cdot \nabla_x f_t = \Delta_v f_t + \nabla_v \cdot ((F + H * f_t) f_t), \quad t > 0, x, v \in \mathbb{R}^d.$$

# Coupling Method 2

Conjecture: The $N$ interacting processes $(X_t^i, V_t^i)_{t\geq 0}$ behave as $N \to \infty$ like the processes $(\overline{X}_t^i, \overline{V}_t^i)_{t\geq 0}$ associated to the PDE.

More precisely, the objective is to estimate the convergence as $N \to \infty$ of

$$\mathbb{E}\left[|X_t^i - \overline{X}_t^i|^2 + |V_t^i - \overline{V}_t^i|^2\right] \leq \varepsilon(N)$$

**Consequences**

1. $f_t^{(1)}$ of any of the particles $X_t^i$ at time $t$ converges to $f_t$ as $N$ goes to infinity:

$$W_2^2(f_t^{(1)}, f_t) \leq \mathbb{E}\left[|X_t^i - \overline{X}_t^i|^2 + |V_t^i - \overline{V}_t^i|^2\right] \leq \varepsilon(N)\,.$$

# Coupling Method 2

Conjecture: The $N$ interacting processes $(X_t^i, V_t^i)_{t\geq 0}$ behave as $N \to \infty$ like the processes $(\overline{X}_t^i, \overline{V}_t^i)_{t\geq 0}$ associated to the PDE.

More precisely, the objective is to estimate the convergence as $N \to \infty$ of

$$\mathbb{E}\left[|X_t^i - \overline{X}_t^i|^2 + |V_t^i - \overline{V}_t^i|^2\right] \leq \varepsilon(N)$$

## Consequences

1. $f_t^{(1)}$ of any of the particles $X_t^i$ at time $t$ converges to $f_t$ as $N$ goes to infinity:

$$W_2^2(f_t^{(1)}, f_t) \leq \mathbb{E}\left[|X_t^i - \overline{X}_t^i|^2 + |V_t^i - \overline{V}_t^i|^2\right] \leq \varepsilon(N) \,.$$

# Coupling Method 2

Conjecture: The $N$ interacting processes $(X_t^i, V_t^i)_{t\geq 0}$ behave as $N \to \infty$ like the processes $(\overline{X}_t^i, \overline{V}_t^i)_{t\geq 0}$ associated to the PDE.

More precisely, the objective is to estimate the convergence as $N \to \infty$ of

$$\mathbb{E}\left[|X_t^i - \overline{X}_t^i|^2 + |V_t^i - \overline{V}_t^i|^2\right] \leq \varepsilon(N)$$

## Consequences

1. $f_t^{(1)}$ of any of the particles $X_t^i$ at time $t$ converges to $f_t$ as $N$ goes to infinity:

$$W_2^2(f_t^{(1)}, f_t) \leq \mathbb{E}\left[|X_t^i - \overline{X}_t^i|^2 + |V_t^i - \overline{V}_t^i|^2\right] \leq \varepsilon(N)\,.$$

# Coupling Method 3

## Consequences

2. Propagation of chaos: The law $f_t^{(k)}$ of any $k$ particles $(X_t^i, V_t^i)$ converges to the tensor product $f_t^{\otimes k}$ as $N$ goes to infinity:

$$W_2^2(f_t^{(k)}, f_t^{\otimes k}) \le k\varepsilon(N).$$

3. Convergence of the empirical measure $\hat{f}_t^N$ to $f_t$: if $\varphi$ is a Lipschitz map on $\mathbb{R}^{2d}$, then

$$\begin{aligned}
&\mathbb{E}\left[\left|\frac{1}{N}\sum_{i=1}^{N}\varphi(X_t^i, V_t^i) - \int_{\mathbb{R}^{2d}}\varphi\, df_t\right|^2\right] \\
&\le 2\,\mathbb{E}\left[|\varphi(X_t^i, V_t^i) - \varphi(\overline{X}_t^i, \overline{V}_t^i)|^2 + \left|\frac{1}{N}\sum_{i=1}^{N}\varphi(\overline{X}_t^i, \overline{V}_t^i) - \int_{\mathbb{R}^{2d}}\varphi\, df_t\right|^2\right] \\
&\le \varepsilon(N) + \frac{C}{N}
\end{aligned}$$

# Coupling Method 3

## Consequences

2. Propagation of chaos: The law $f_t^{(k)}$ of any $k$ particles $(X_t^i, V_t^i)$ converges to the tensor product $f_t^{\otimes k}$ as $N$ goes to infinity:

$$W_2^2(f_t^{(k)}, f_t^{\otimes k}) \leq k\varepsilon(N).$$

3. Convergence of the empirical measure $\hat{f}_t^N$ to $f_t$: if $\varphi$ is a Lipschitz map on $\mathbb{R}^{2d}$, then

$$\mathbb{E}\left[\left|\frac{1}{N}\sum_{i=1}^N \varphi(X_t^i, V_t^i) - \int_{\mathbb{R}^{2d}} \varphi\, df_t\right|^2\right]$$
$$\leq 2\,\mathbb{E}\left[|\varphi(X_t^i, V_t^i) - \varphi(\overline{X}_t^i, \overline{V}_t^i)|^2 + \left|\frac{1}{N}\sum_{i=1}^N \varphi(\overline{X}_t^i, \overline{V}_t^i) - \int_{\mathbb{R}^{2d}} \varphi\, df_t\right|^2\right]$$
$$\leq \varepsilon(N) + \frac{C}{N}$$

# Main Result

Previous Results: If the functions involved $F$ and $H$ are globally Lipschitz then there are classical results by Snitzman and Meleard, implying that

$$\varepsilon(N) = O\left(\frac{1}{N}\right)$$

The typical $F$ and $H$ in our Cucker-Smale and D'Orsogna etal model are not globally Lipschitz.

Hypotheses:

Assume that $F$ and $H$ with $H(-x,-v) = -H(x,v)$, satisfy

$$-(v-w)\cdot(F(x,v)-F(x,w)) \le A\,|v-w|^2$$
$$|F(x,v)-F(y,v)| \le L\min\{|x-y|,1\}(1+|v|^p)$$

for all $x, y, v, w$ in $\mathbb{R}^d$, and analogously for $H$ instead of $F$.

# Main Result

Previous Results: If the functions involved $F$ and $H$ are globally Lipschitz then there are classical results by Snitzman and Meleard, implying that

$$\varepsilon(N) = O\left(\frac{1}{N}\right)$$

The typical $F$ and $H$ in our Cucker-Smale and D'Orsogna etal model are not globally Lipschitz.

Hypotheses:

Assume that $F$ and $H$ with $H(-x,-v) = -H(x,v)$, satisfy

$$-(v-w)\cdot(F(x,v)-F(x,w)) \le A\,|v-w|^2$$
$$|F(x,v)-F(y,v)| \le L\min\{|x-y|,1\}(1+|v|^p)$$

for all $x, y, v, w$ in $\mathbb{R}^d$, and analogously for $H$ instead of $F$.

# Main Result

Previous Results: If the functions involved $F$ and $H$ are globally Lipschitz then there are classical results by Snitzman and Meleard, implying that

$$\varepsilon(N) = O\left(\frac{1}{N}\right)$$

The typical $F$ and $H$ in our Cucker-Smale and D'Orsogna etal model are not globally Lipschitz.

Hypotheses:

Assume that $F$ and $H$ with $H(-x,-v) = -H(x,v)$, satisfy

$$-(v-w)\cdot(F(x,v)-F(x,w)) \le A\,|v-w|^2$$
$$|F(x,v)-F(y,v)| \le L\min\{|x-y|,1\}(1+|v|^p)$$

for all $x, y, v, w$ in $\mathbb{R}^d$, and analogously for $H$ instead of $F$.

# Main Result 2

## Properties of the Stochastic Processes and PDE:

Assume that the particle system and the processes have global solutions on $[0, T]$ with initial data $(X_0^i, V_0^i)$ such that the uniform moment condition holds:

$$\sup_{0\le t\le T}\left\{\int_{\mathbb{R}^{4d}}|H(x-y, v-w)|^2 df_t(x,v)df_t(y,w) + \int_{\mathbb{R}^{2d}}(|x|^2 + e^{a|v|^{p'}})df_t(x,v)\right\} < +\infty$$

with $f_t = law(\overline{X}_t^i, \overline{V}_t^i)$ and some $p' > p$.

## Result:

For all $0 < \epsilon < 1$ there exists a constant $C$ such that

$$\mathbb{E}\left[|X_t^i - \overline{X}_t^i|^2 + |V_t^i - \overline{V}_t^i|^2\right] \le \frac{C}{N^{1-\epsilon}}$$

for all $0 \le t \le T$ and $N \ge 1$.

# Main Result 2

### Properties of the Stochastic Processes and PDE:

Assume that the particle system and the processes have global solutions on $[0, T]$ with initial data $(X_0^i, V_0^i)$ such that the uniform moment condition holds:

$$\sup_{0\le t\le T}\left\{\int_{\mathbb{R}^{4d}} |H(x-y, v-w)|^2 df_t(x,v) df_t(y,w) + \int_{\mathbb{R}^{2d}} (|x|^2 + e^{a|v|^{p'}}) df_t(x,v)\right\} < +\infty$$

with $f_t = law(\overline{X}_t^i, \overline{V}_t^i)$ and some $p' > p$.

### Result:

For all $0 < \epsilon < 1$ there exists a constant $C$ such that

$$\mathbb{E}\left[|X_t^i - \overline{X}_t^i|^2 + |V_t^i - \overline{V}_t^i|^2\right] \le \frac{C}{N^{1-\epsilon}}$$

for all $0 \le t \le T$ and $N \ge 1$.

# Outline

# Asymptotic Flocking

Let us consider the $N_p$-particle system:

$$\begin{cases} \dfrac{dx_i}{dt} = v_i & , \; x_i(0) = x_i^0 \\ \dfrac{dv_i}{dt} = \displaystyle\sum_{j=1}^{N_p} m_j a(|x_i - x_j|)(v_j - v_i) & , \; v_i(0) = v_i^0, \end{cases} .$$

Due to translation invariancy, w.l.o.g. the mean velocity is zero and thus the center of mass is preserved along the evolution, i.e.,

$$\sum_{i=1}^{N_p} m_i v_i(t) = 0 \quad \text{and} \quad \sum_{i=1}^{N_p} m_i x_i(t) = x_c$$

for all $t \geq 0$ and $x_c \in \mathbb{R}^d$.

# Asymptotic Flocking

Let us consider the $N_p$-particle system:

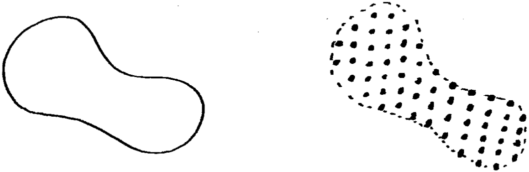
$$\begin{cases} \dfrac{dx_i}{dt} = v_i & , \; x_i(0) = x_i^0 \\ \dfrac{dv_i}{dt} = \displaystyle\sum_{j=1}^{N_p} m_j a(|x_i - x_j|)\,(v_j - v_i) & , \; v_i(0) = v_i^0, \end{cases} .$$

Due to translation invariancy, w.l.o.g. the mean velocity is zero and thus the center of mass is preserved along the evolution, i.e.,

$$\sum_{i=1}^{N_p} m_i v_i(t) = 0 \qquad \text{and} \qquad \sum_{i=1}^{N_p} m_i x_i(t) = x_c$$

for all $t \geq 0$ and $x_c \in \mathbb{R}^d$.

# Asymptotic Flocking



Find a bound independent of the number of particles for the time it takes for all the particles to travel at the mean velocity.

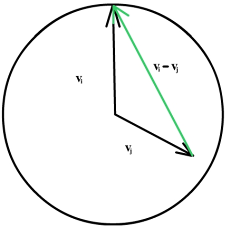
# Asymptotic Flocking

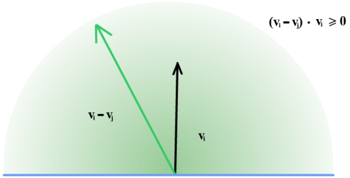
## Unconditional Non-universal Asymptotic Flocking: C.-Fornasier-Rosado-Toscani

Given $\mu_0 \in \mathcal{M}(\mathbb{R}^{2d})$ compactly supported, then the unique measure-valued solution to the CS kinetic model with $\gamma \leq 1/2$, satisfies the following bounds on their supports:

$$\text{supp}\ \mu(t) \subset B(x_c(0) + mt, R^x(t)) \times B(m, R^v(t))$$

for all $t \geq 0$, with $R^x(t) \leq \bar{R}$ and $R^v(t) \leq R_0\, e^{-\lambda t}$ with $\bar{R}^x$ depending only on the initial support radius.



# Outline

# Macroscopic equations

## Monokinetic Solutions

Assuming that there is a deterministic velocity for each position and time, $f(x,v,t) = \rho(x,t)\,\delta(v - u(x,t))$ is a distributional solution if and only if,

$$\begin{cases} \dfrac{\partial \rho}{\partial t} + \mathrm{div}_x(\rho u) = 0, \\ \rho\,\dfrac{\partial u}{\partial t} + \rho\,(u\cdot\nabla_x)u = \rho\,(\alpha - \beta|u|^2)u - \rho\,(\nabla_x U \star \rho). \end{cases}$$

# Particular solutions

Let us look for stationary solutions with an asymptotic speed value $\beta|u(x,t)|^2 = \alpha$.

### Flocking

Traveling wave case, $u =$ const such that $\beta|\mathbf{u}(\mathbf{x},t)|^2 = \alpha$, then $\rho(x,t) = \tilde{\rho}(x - ut)$, and the density is determined by

$$\tilde{\rho}\left(\nabla_{\mathbf{x}} U \star \tilde{\rho}\right) = 0,$$

from which

$$U \star \tilde{\rho} = C, \quad \tilde{\rho} \neq 0,$$

in the support of $\tilde{\rho}$ if the support has not empty interior.

Complete set of solutions depending on regularity of the potential and stability are open problems.

# Particular solutions

Let us look for stationary solutions with an asymptotic speed value $\beta|u(x,t)|^2 = \alpha$.

### Flocking

Traveling wave case, $u =$ const such that $\beta|\mathbf{u}(\mathbf{x},t)|^2 = \alpha$, then $\rho(x,t) = \tilde{\rho}(x - ut)$, and the density is determined by

$$\tilde{\rho}\left(\nabla_{\mathbf{x}} U \star \tilde{\rho}\right) = 0,$$

from which

$$U \star \tilde{\rho} = C, \quad \tilde{\rho} \neq 0,$$

in the support of $\tilde{\rho}$ if the support has not empty interior.

Complete set of solutions depending on regularity of the potential and stability are open problems.

# Particular solutions

Let us look for stationary solutions with an asymptotic speed value $\beta|u(x,t)|^2 = \alpha$.

## Flocking

Traveling wave case, $u =$ const such that $\beta|\mathbf{u}(\mathbf{x},t)|^2 = \alpha$, then $\rho(x,t) = \tilde{\rho}(x - ut)$, and the density is determined by

$$\tilde{\rho}\left(\nabla_{\mathbf{x}} U \star \tilde{\rho}\right) = 0,$$

from which

$$U \star \tilde{\rho} = C, \quad \tilde{\rho} \neq 0,$$

in the support of $\tilde{\rho}$ if the support has not empty interior.

Complete set of solutions depending on regularity of the potential and stability are open problems.

# Particular solutions

Let us look for stationary solutions with an asymptotic speed value $\beta|u(x,t)|^2 = \alpha$.

## Milling

we set **u** in a rotatory state,

$$u = \pm\sqrt{\frac{\alpha}{\beta}}\,\frac{x^\perp}{|x|},$$

where $x = (x_1, x_2)$, $x^\perp = (-x_2, x_1)$, and look for $\rho = \rho(|x|)$ radial, then

$$U \star \rho = D + \frac{\alpha}{\beta}\log|x|, \quad \text{whenever} \;\; \rho \neq 0.$$

A special family of singular solutions are given by $\rho(r) = c\,\delta(r - r_0)$.

Complete set of solutions depending on regularity of the potential and stability are open problems.

# Particular solutions

Let us look for stationary solutions with an asymptotic speed value $\beta|u(x,t)|^2 = \alpha$.

## Milling

we set **u** in a rotatory state,

$$u = \pm\sqrt{\frac{\alpha}{\beta}}\,\frac{x^\perp}{|x|},$$

where $x = (x_1, x_2)$, $x^\perp = (-x_2, x_1)$, and look for $\rho = \rho(|x|)$ radial, then

$$U \star \rho = D + \frac{\alpha}{\beta}\log|x|, \quad \text{whenever} \ \ \rho \neq 0.$$

A special family of singular solutions are given by $\rho(r) = c\,\delta(r - r_0)$.

Complete set of solutions depending on regularity of the potential and stability are open problems.

# Particular solutions

Let us look for stationary solutions with an asymptotic speed value $\beta|u(x,t)|^2 = \alpha$.

## Milling

we set **u** in a rotatory state,

$$u = \pm\sqrt{\frac{\alpha}{\beta}}\,\frac{x^\perp}{|x|},$$

where $x = (x_1, x_2)$, $x^\perp = (-x_2, x_1)$, and look for $\rho = \rho(|x|)$ radial, then

$$U \star \rho = D + \frac{\alpha}{\beta}\log|x|, \quad \text{whenever} \;\; \rho \neq 0.$$

A special family of singular solutions are given by $\rho(r) = c\,\delta(r - r_0)$.

Complete set of solutions depending on regularity of the potential and stability are open problems.

# Conclusions & Open Problems

- Simple modelling of the three main mechanisms leads to complicated patterns. More information from particular species should be included to make more realistic models (Helmelrijk & collaborators, ...)
- Millings can be understood as kinetic measure solutions concentrated on certain velocities. Geometric constraints: velocities on a sphere. Stability of these patterns?
- Phase transition from ordered to disordered state driven by noise: (Liu-Frouvelle, 2011) (Barbaro-Cañizo-C.-Degond, work in preparation).
- References:


1. C.-D'Orsogna-Panferov (KRM 2008).
2. C.-Fornasier-Rosado-Toscani (SIMA 2010).
3. C.-Fornasier-Toscani-Vecil (Birkhäuser 2011)
4. C.-Klar-Martin-Tiwari (M3AS 2010).
5. Cañizo-C.-Rosado (M3AS 2011).
6. Bolley-Cañizo-C. (M3AS 2011 & AML 2011).

# Conclusions & Open Problems

- Simple modelling of the three main mechanisms leads to complicated patterns. More information from particular species should be included to make more realistic models (Helmelrijk & collaborators, ...)
- Millings can be understood as kinetic measure solutions concentrated on certain velocities. Geometric constraints: velocities on a sphere. Stability of these patterns?
- Phase transition from ordered to disordered state driven by noise: (Liu-Frouvelle, 2011) (Barbaro-Cañizo-C.-Degond, work in preparation).
- References:


1. C.-D'Orsogna-Panferov (KRM 2008).
2. C.-Fornasier-Rosado-Toscani (SIMA 2010).
3. C.-Fornasier-Toscani-Vecil (Birkhäuser 2011)
4. C.-Klar-Martin-Tiwari (M3AS 2010).
5. Cañizo-C.-Rosado (M3AS 2011).
6. Bolley-Cañizo-C. (M3AS 2011 & AML 2011).

# Conclusions & Open Problems

- Simple modelling of the three main mechanisms leads to complicated patterns. More information from particular species should be included to make more realistic models (Helmelrijk & collaborators, ...)
- Millings can be understood as kinetic measure solutions concentrated on certain velocities. Geometric constraints: velocities on a sphere. Stability of these patterns?
- Phase transition from ordered to disordered state driven by noise: (Liu-Frouvelle, 2011) (Barbaro-Cañizo-C.-Degond, work in preparation).
- References:


1. C.-D'Orsogna-Panferov (KRM 2008).
2. C.-Fornasier-Rosado-Toscani (SIMA 2010).
3. C.-Fornasier-Toscani-Vecil (Birkhäuser 2011)
4. C.-Klar-Martin-Tiwari (M3AS 2010).
5. Cañizo-C.-Rosado (M3AS 2011).
6. Bolley-Cañizo-C. (M3AS 2011 & AML 2011).

# Conclusions & Open Problems

- Simple modelling of the three main mechanisms leads to complicated patterns. More information from particular species should be included to make more realistic models (Helmelrijk & collaborators, ...)
- Millings can be understood as kinetic measure solutions concentrated on certain velocities. Geometric constraints: velocities on a sphere. Stability of these patterns?
- Phase transition from ordered to disordered state driven by noise: (Liu-Frouvelle, 2011) (Barbaro-Cañizo-C.-Degond, work in preparation).
- References:


1. C.-D'Orsogna-Panferov (KRM 2008).
2. C.-Fornasier-Rosado-Toscani (SIMA 2010).
3. C.-Fornasier-Toscani-Vecil (Birkhäuser 2011)
4. C.-Klar-Martin-Tiwari (M3AS 2010).
5. Cañizo-C.-Rosado (M3AS 2011).
6. Bolley-Cañizo-C. (M3AS 2011 & AML 2011).

# Conclusions & Open Problems

- Simple modelling of the three main mechanisms leads to complicated patterns. More information from particular species should be included to make more realistic models (Helmelrijk & collaborators, ...)
- Millings can be understood as kinetic measure solutions concentrated on certain velocities. Geometric constraints: velocities on a sphere. Stability of these patterns?
- Phase transition from ordered to disordered state driven by noise: (Liu-Frouvelle, 2011) (Barbaro-Cañizo-C.-Degond, work in preparation).
- References:


1. C.-D'Orsogna-Panferov (KRM 2008).
2. C.-Fornasier-Rosado-Toscani (SIMA 2010).
3. C.-Fornasier-Toscani-Vecil (Birkhäuser 2011)
4. C.-Klar-Martin-Tiwari (M3AS 2010).
5. Cañizo-C.-Rosado (M3AS 2011).
6. Bolley-Cañizo-C. (M3AS 2011 & AML 2011).