Coherent Pattern Prediction in Swarms of Delay-Coupled Agents

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Introduction

- Collective motion of systems have long been observed in biological populations.
- Mathematical studies of swarming behavior:
  - Provided examples of biological patterns.
  - Led to intelligent design and control of man-made vehicles.

  - Two main types of models:
    - Continuum-based.
    - Individual-based (Can be deterministic or stochastic).
- Emergence of ordered swarm states from initial disordered state:
  - Translational or rotational in motion.
  - Spatially distributed or localized in clusters.
- Effect of time delayed interactions (finite communication times).
  - Detailed bifurcation study.
  - Effect of noise on the system bifurcations.

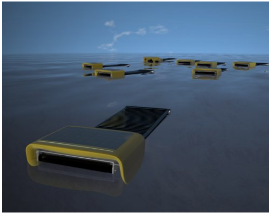
Swarms

Peter Scoones / Getty Images

Christian Ziegler / Agentur Focus

Reuters

StarFlag Project



MIT - Senseable City Lab

SwarmRobot.org

The Swarm Model

$$\dot{\mathbf{r}}_i = \mathbf{v}_i$$

$$\dot{\mathbf{v}}_i = (1 - |\mathbf{v}_i|^2)\mathbf{v}_i - \frac{a}{N}\sum_{\substack{j=1 \\ i\neq j}}^{N}(\mathbf{r}_i(t) - \mathbf{r}_j(t-\tau)) + \boldsymbol{\eta}_i(t)$$

$$\boldsymbol{\eta}_i(t) = (\eta_i^{(1)}, \eta_i^{(2)})$$

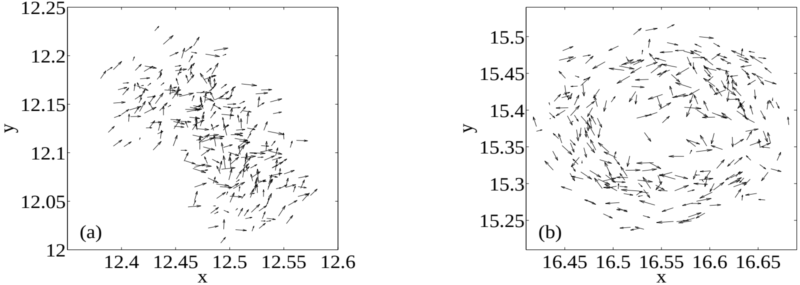
$$\langle \eta_i^{(\ell)}(t) \rangle = 0 \quad \text{and} \quad \langle \eta_i^{(\ell)}(t)\eta_j^{(k)}(t') \rangle = 2D\delta(t-t')\delta_{ij}\delta_{\ell k}$$

$$D = \sigma^2/2 \qquad i,j = 1,2,\ldots,N \quad \ell,k = 1,2$$

a - particle interaction coupling parameter
N - number of particles
τ - constant communication time delay
D - intensity of noise

Noise Induced Transition (No Time Delay)

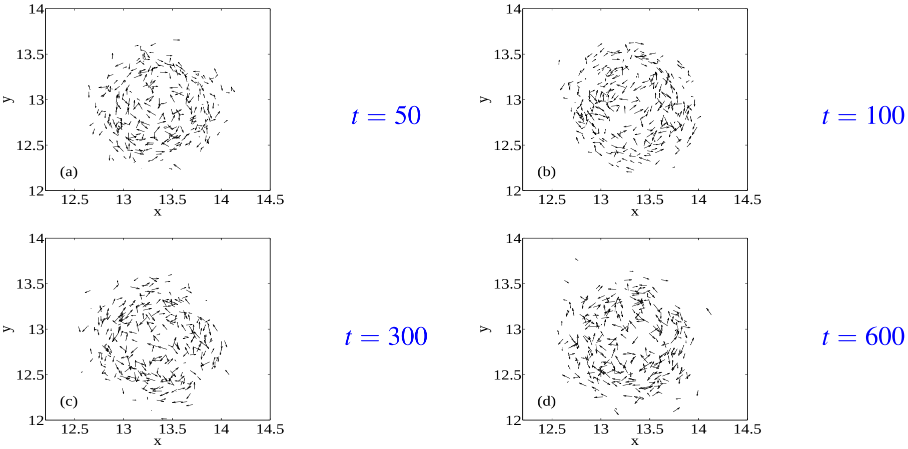
Erdmann, *et al.* (2005), Forgoston and Schwartz (2008).



$t = 18$ $t = 40$

$a = 100$, $N = 300$, $\tau = 0$, and $D = 0.08$. Noise switched on at $t = 10$.

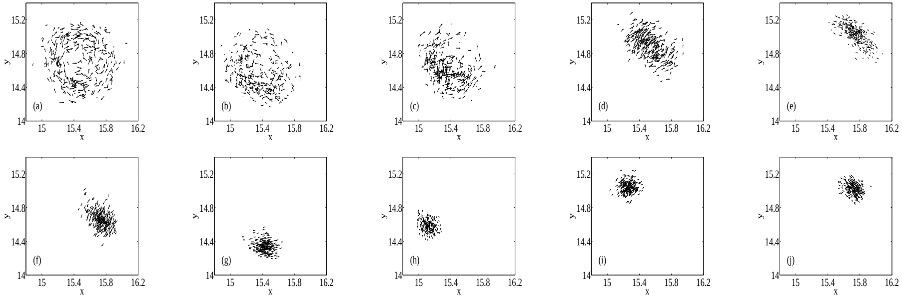
Time Delay Induced Transition (No Transition)



$a = 2$, $N = 300$, $\tau = 1$, and $D = 0.08$.

Noise switched on at $t = 10$. Time delay switched on at $t = 40$.

Time Delay Induced Transition

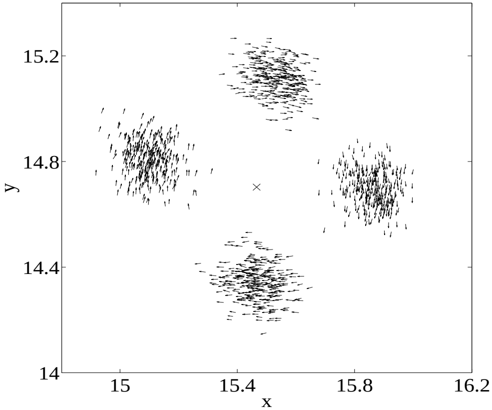


$t = 50$, $t = 60$, $t = 62$, $t = 64$, $\ldots$, $t = 72$, $t = 74$, $t = 76$.

$a = 4$, $N = 300$, $\tau = 1$, and $D = 0.08$.
Noise switched on at $t = 10$. Time delay switched on at $t = 40$.

Forgoston and Schwartz (2008).

Compact, Rotating Aligned Swarm State



$t = 90.2$ (left)

$t = 90.6$ (top)

$t = 91.0$ (right)

$t = 91.4$ (bottom)

Center of mass defined as

$\mathbf{R}(t) = (1/N) \sum_i \mathbf{r}_i(t)$

$a = 4$, $N = 300$, $\tau = 1$, and $D = 0.08$.

Mean Field Equation for Center of Mass

Center of mass: $\mathbf{R}(t) = \frac{1}{N}\sum_{i=1}^{N} \mathbf{r}_i(t)$

Decompose position of i^{th} particle: $\mathbf{r}_i = \mathbf{R} + \delta\mathbf{r}_i, \quad i = 1, 2, \ldots, N$

Substitute $\mathbf{r}_i$ into the governing equations with $D = 0$, sum all i of these equations, and neglect all fluctuation terms $\Longrightarrow$

$$\ddot{\mathbf{R}} = \left(1 - |\dot{\mathbf{R}}|^2\right)\dot{\mathbf{R}} - a\left(\mathbf{R}(t) - \mathbf{R}(t-\tau)\right)$$

where we approximate $a\frac{N-1}{N} \approx a$ since we consider the thermodynamic limit.

Pitchfork Bifurcation

Write in component form with $\mathbf{R} = (X, Y)$ and $\dot{\mathbf{R}} = (U, V)$.

$$\begin{aligned}
\dot{X} &= U \\
\dot{U} &= (1 - U^2 - V^2)U - a(X - X(t - \tau)) \\
\dot{Y} &= V \\
\dot{V} &= (1 - U^2 - V^2)V - a(Y - Y(t - \tau))
\end{aligned}$$

Translationally invariant stationary solutions:

$X = X_0, \quad U = 0, \quad Y = Y_0, \quad V = 0$

Uniformly translating solutions:

$X = U_0 t + X_0, \quad U = U_0, \quad Y = V_0 t + Y_0, \quad V = V_0$

Requires $U_0^2 + V_0^2 = 1 - a\tau$ (exists only for $a\tau < 1$).

The hyperbola $a\tau = 1$ is a pitchfork bifurcation curve on which the uniformly translating states are born from the stationary state $(X_0, 0, Y_0, 0)$.

Hopf Bifurcation

Linearize about the stationary state.

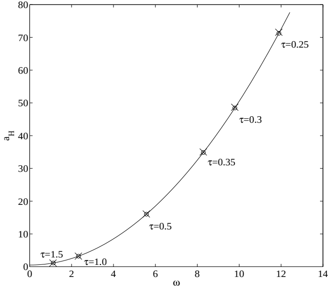
Characteristic equation: $\left(a(1-e^{-\lambda\tau})-\lambda+\lambda^2\right)^2=0$

To identify Hopf bifurcations, let $\lambda = i\omega$. Substitution $\implies$

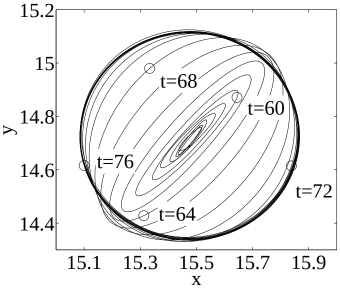
$$a_H(\omega)=\frac{1+\omega^2}{2}$$

$$\tau_{H_n}(\omega)=\frac{1}{\omega}\left(\arctan\left(\frac{2\omega}{1-\omega^2}\right)+2n\pi\right), \quad n=0,1,\ldots$$

Hopf Bifurcation



$\tau = 1, \quad a_H \approx 3.2$ (analytical)



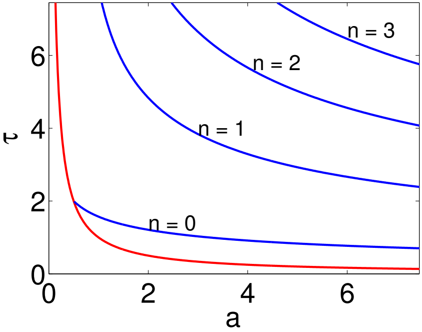
$t = 45$ to $t = 90$

(delay switched on at $t = 40$)

Pitchfork and Hopf Bifurcation

$$\tau_P(a) = \frac{1}{a}$$

$$\tau_{H_n}(a) = \frac{1}{\sqrt{2a-1}}\left(\arctan\left(\frac{\sqrt{2a-1}}{1-a}\right) + 2n\pi\right), \quad n = 0, 1, \ldots$$

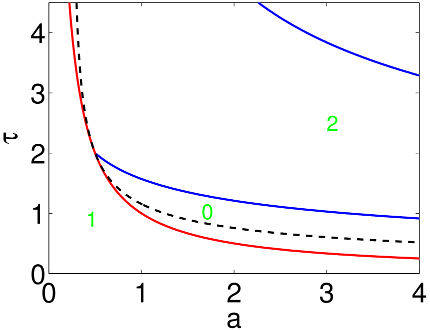


Pitchfork bifurcation

Hopf bifurcation

Bogdanov-Takens point (double zero eigenvalue) at $(a = 1/2, \tau = 2)$.

Eigenvalue Structure Around Bogdanov-Takens Point



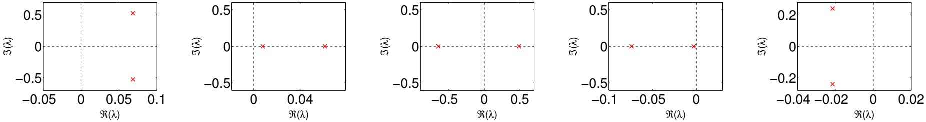
Pitchfork bifurcation

Hopf bifurcation

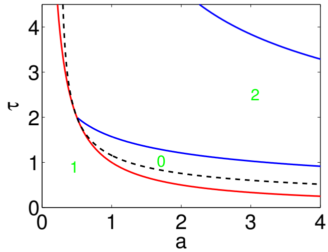
Node to Focus transition

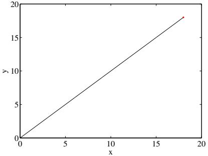
$$\tau = \frac{1}{\sqrt{a - 1/4}}$$

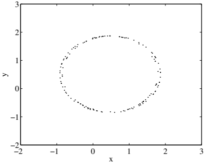
Number of eigenvalues with positive real parts.



Swarm Structure in the (a, τ) Plane



1:


0:


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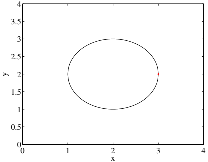


1: Swarm translates along line with unit speed. Direction depends on IC.

0: Swarm particles are distributed on a ring.

2: Swarm collapses to a point and either rotates in a circular orbit or oscillates back and forth along a line. The state depends on IC.

Stationary Center of Mass: the Ring State

$\mathbf{R} = \text{const.}$ satisfies the mean field equation with $\sum_{i=1}^{N} \delta\dot{\mathbf{r}}_i^2 \delta\dot{\mathbf{r}}_i = 0$.

$\delta\ddot{\mathbf{r}}_i = \left(1 - \delta\dot{\mathbf{r}}_i^2\right) \delta\dot{\mathbf{r}}_i - a\frac{(N-1)}{N}\delta\mathbf{r}_i - \frac{a}{N}\delta\mathbf{r}_i\left(t-\tau\right)$

Thermodynamic limit $\implies \delta\ddot{\mathbf{r}}_i = \left(1 - \delta\dot{\mathbf{r}}_i^2\right) \delta\dot{\mathbf{r}}_i - a\delta\mathbf{r}_i$

$\delta\mathbf{r}_j = (\delta x_j, \delta y_j)$ and define $\delta z_j = \delta x_j + i\delta y_j \implies$

$$\delta\ddot{z}_j = \left(1 - |\delta\dot{z}_j|^2\right) \delta\dot{z}_j - a\delta z_j$$

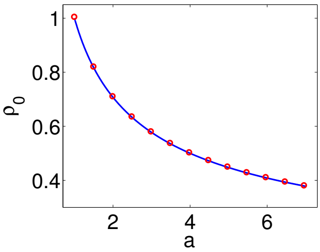
Write in polar form $\delta z_j = \rho_j e^{i\theta_j}$, and separate into Re and Im parts.

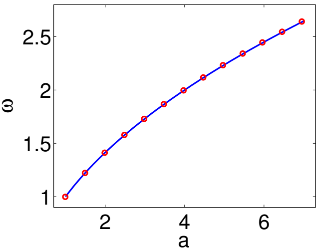
Ring state solution: $\rho_j = \frac{1}{\sqrt{a}}, \quad \dot{\theta}_j = \pm\sqrt{a}$

Stationary Center of Mass: the Ring State

Analytical ring state solution: $\rho_0 = \frac{1}{\sqrt{a}}, \quad \omega = \sqrt{a}$

Numerical simulation $\circ$





Moving Center of Mass: the Circular Orbit

After Hopf bifurcation, particles collapse to a point so that $\delta\mathbf{r}_i = 0$.

$$\ddot{\mathbf{R}} = \left(1 - \dot{\mathbf{R}}^2\right)\dot{\mathbf{R}} - a\left(\mathbf{R}(t) - \mathbf{R}(t-\tau)\right)$$

$\mathbf{R} = (X, Y)$ and define $Z = X + iY \implies$

$$\ddot{Z} = \left(1 - |\dot{Z}|^2\right)\dot{Z} - \alpha(Z(t) - Z(t-\tau))$$

Write in polar form $Z = \rho e^{i\theta}$, and separate into Re and Im parts.

Circular orbit solution: $\rho = \rho_0$ and $\theta = \omega t + \theta_0$ satisfying

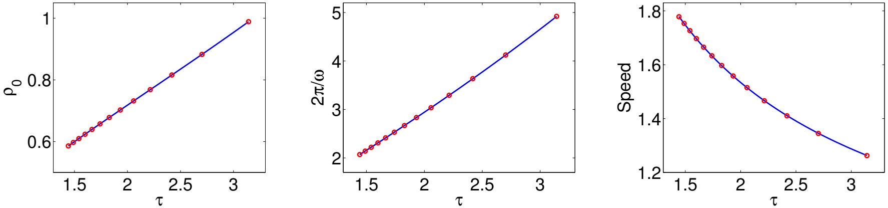
$$\omega^2 = a \cdot (1 - \cos\omega\tau),$$
$$\rho_0 = \frac{1}{|\omega|}\sqrt{1 - a\frac{\sin\omega\tau}{\omega}}$$

Moving Center of Mass: the Circular Orbit

Analytical circular orbit solution:

$$\omega^2 = a \cdot (1 - \cos \omega\tau), \quad \rho_0 = \frac{1}{|\omega|}\sqrt{1 - a\frac{\sin\omega\tau}{\omega}}$$

Numerical simulation ∘

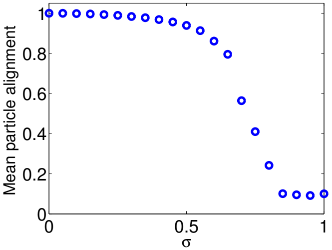


In absence of coupling, particles move with speed one.

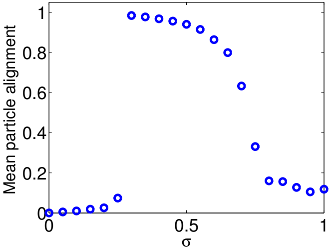
The period of oscillation $2\pi/\omega > \tau \implies \mathbf{R}(t-\tau)$ is ahead of $\mathbf{R}(t)$.

The attraction that a particle feels to the delayed position of the rest forces the whole system to move faster.

Interplay of Delay, Noise, and Initial Conditions $(\tau = 2, a = 2)$



Particles initially aligned with unit x and y speeds. They settle into rotating state with swarm moving with high density. As noise increases, the large scale rotation stops.



Particles initially distributed uniformly over unit square and at rest. For noise $\sigma < 0.25$ particles distributed on ring. For higher noise $\sigma > 0.25$, ring transitions into rotating state which breaks up as noise increases further ($\sigma > 0.8$).

Conclusions

- Considered model of self-propelling particles interacting through pairwise force in presence of noise and time delay.
- Used bifurcation analysis to identify different coherent structures of the swarm in different regions of (a, τ) space.
- Analyzed the coherent structures and derived relations that govern their spatio-temporal length scales.
- Showed how stochasticity modifies the coherent structures and attractor sensitivity.

Future Work

- Generalize to other inter-agent potential functions.
- Consider internal delays, random delays, swarm control.

E. Forgoston and I.B. Schwartz, "Delay-induced instabilities in self-propelling swarms," Phys. Rev. E **77**, 035203(R) (2008).

L. Mier-y-Teran-Romero, E. Forgoston and I.B. Schwartz, "Noise, Bifurcations, and Modeling of Interacting Particle Systems," 2011 IEEE/RSJ Conference on Intelligent Robots and Systems (IROS 2011).

L. Mier-y-Teran-Romero, E. Forgoston and I.B. Schwartz, "Coherent Pattern Prediction in Swarms of Delay-Coupled Agents," Submitted.