Viscosity-induced instability for Euler and averaged Euler equations in a circular domain

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Motivation

- The Navier-Stokes' and second grade fluid equations play a role in many fluids applications.
- The zero-viscosity limits are Euler's and averaged Euler's equations. Both have a Hamiltonian structure with a Poisson structure map. An important associated Casimir is the (potential) enstrophy.
- Constrained critical points of the energy on level sets of the Casimir form families of stationary solutions for (averaged) Euler's equations. If they are minima, the stationary solutions are Lyapunov stable.

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Question: Do viscous solutions stay close to the minimal families for small viscosity?

Answer: It depends on the boundary conditions!

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Known convergence/stability results in 2D

2D Navier-Stokes' (NS) and Euler's equations:

- Boundaryless manifold: solutions of NS equation converge to the solutions of Euler's equations with the same initial condition [Ebin & Marsden (70)].
- Free boundary condition: the Euler families are invariant under NS equations and they shadow solutions of the NS equations starting nearby [van Groesen (88); Derks & Ratiu (98)].
- For a fixed time interval $[0, T]$, solutions of the NS equation converge to solutions of Euler's equation with the same initial condition for various boundary conditions: free boundary condition [Lions (69); Temam (77)]; no-slip in disk [Bona & Wu (02)]; Navier-slip [Clopeau, Mikelić & Robert (98)].
- General belief: the no-slip boundary condition causes a turbulent boundary layer in the zero viscosity limit.

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2D Navier Stokes and Euler

In vorticity (ω) formulation, Navier-Stokes equation for divergence free vector fields in a bounded domain $D \subset \mathbb{R}^2$ is

$$\omega_t = \mathbf{u} \cdot \nabla\omega + \nu\Delta\omega, \quad \text{with} \quad \mathbf{u} = \nabla^{\perp}\psi, \quad \omega = -\Delta\psi.$$

We will discuss two boundary conditions:

- Free: $\psi = 0 = \omega$ on ∂D; or
- No-slip: $\nabla\psi = 0$ $(\mathbf{u} = 0)$ on ∂D;

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The viscosity is ν and Euler's equation follows from setting $\nu = 0$ and reducing the boundary condition to $\psi = 0$ (free) or $\mathbf{u} \cdot \mathbf{n} = 0$ (no-slip).

The energy is $H(\omega) = \dfrac{1}{2}\displaystyle\int_D |\mathbf{u}|^2 = \dfrac{1}{2}\displaystyle\int_D \psi\omega$

and the enstrophy $C(\omega) = \dfrac{1}{2}\displaystyle\int_D \omega^2$ is a Casimir.

Energy-Casimir manifolds

The critical points of the enstrophy on level sets of the energy give a family of stationary solutions of the Euler's equation. These solutions are eigenfunctions of the spectral problem

$$(\omega =)\ -\Delta\psi = \gamma\psi, \quad \text{plus reduced free or no-slip BC.}$$

- For any bounded, simply connected domain D, this has spectral problem has eigenvalues $0 < \gamma_0 < \gamma_1 \leq \ldots$ and the eigenfunctions span $L^2(D)$.
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Lemma *Define* $\omega_0 = -\Delta\chi_0$*; the family* $\mathcal{E} = \{c\omega_0 \mid c \in \mathbb{R}\}$ *and the distance* $d(\omega, \mathcal{E}) = \inf_{c\in\mathbb{R}} \|\omega - c\omega_0\|_{L^2}$.

Energy-Casimir method: The family is orbitally Lyapunov stable for solutions $\omega(t)$ *of Euler's equations, i.e., for all* $\varepsilon > 0$, *there is some* $\delta > 0$ *such that for all* $t \geq 0$

$$d(\omega(0), \mathcal{E}) < \delta \Rightarrow d(\omega(t), \mathcal{E}) < \varepsilon.$$

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NS with free boundary condition

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- Solutions starting nearby $\mathcal{E}$ can be shadowed by a curve on $\mathcal{E}$: Let $\Omega(t;\widehat{\omega},\nu)$ be a solution of the NS equation with viscosity ν, starting at $\widehat{\omega}$. Define the shadowing curve $\Omega_0(t;\widehat{\omega},\nu) = \sqrt{2\gamma_0 H(\Omega(t;\widehat{\omega},\nu))}\,\omega_0$. Then for all $\widehat{\omega}$ with $d(\widehat{\omega},\mathcal{E}) < 2(\gamma_1 - \gamma_0)$, there exists an $M > 0$ such that
$$\forall_{\nu\geq 0}\forall_{t\geq 0}\left[d(\Omega(t;\widehat{\omega},\nu),\mathcal{E}) \leq M\,\|\Omega_0(t;\widehat{\omega},\nu)\|_{L^2}\, e^{-2\nu(\gamma_1-\gamma_0)t}\right]$$

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- The family $\mathcal{E}$ is stable under the NS evolution: There is some $\varepsilon_0 > 0$ and $K > 0$ such that
$$\forall_{c\in\mathbb{R}}\forall_{0\leq\varepsilon<\varepsilon_0}\forall_{\nu>0}\forall_{t\geq 0}\,[\|\widehat{\omega} - c\omega_0\|_{L^2} < \varepsilon \Rightarrow d(\Omega(t;\widehat{\omega},\nu),\mathcal{E}) < K\varepsilon]$$

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NS with no-slip boundary condition in disk

Consider the Navier-Stokes (NS) equation with the *no-slip* boundary condition in a circular disk. Denote the solution of the NS equation with viscosity ν and starting vorticity $\widehat{\omega}$ by $\Omega(t; \widehat{\omega}, \nu)$.

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- If $\nu > 0$, then solutions starting in $\mathcal{E}$ have a ν-independent deviation away from $\mathcal{E}$ before returning to the zero state:

$$\forall_{c\in\mathbb{R}}\exists_{M>0}\forall\nu > 0\exists_{t>0} \left[\, d(\, \Omega(t; c\omega_0, \nu)\, ,\, \mathcal{E}) > M \,\right].$$

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Note: this doesn't contradict [Bona & Wu (02)] as it takes $\mathcal{O}(1/\nu)$ time to get order 1 away from the manifold.

2D second grade and averaged Euler

The second grade fluid equation in a bounded domain $D \subset \mathbb{R}^2$ is

$$\partial_t q = -\mathbf{u} \cdot \nabla q + \nu \Delta \omega, \quad \text{with} \quad \mathbf{u} = \nabla^\perp \psi, \quad \omega = -\Delta \psi,$$

and potential vorticity $q = L\psi = -\Delta(1 - \alpha\Delta)\psi$ (*not atmospheric PV*).

- We consider the Navier-slip boundary condition: tangential component of viscous stress is proportial to the tangential velocity: $\psi = 0$ and $-\Delta\psi = 2\kappa \nabla\psi \cdot \mathbf{n}$ on ∂D with κ curvature and $\mathbf{n}$ the normal of ∂D.

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- *Note*: the 2nd grade fluid equation is a regular perturbation of averaged Euler.

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Consider the second grade fluid equation with the Navier-slip boundary condition in a circular disk. Denote the stream function of the solution of the second grade equation with viscosity ν and starting stream function $\widehat{\psi}$ by $\Psi(t;\widehat{\psi},\nu)$.

- The family $\mathcal{E}_{AE}$ is not invariant under the evolution of the second grade fluid equation.
- If $\nu > 0$, then solutions starting in $\mathcal{E}_{AE}$ have a ν-independent deviation away from $\mathcal{E}_{AE}$ before returning to the zero state:

$$\forall_{\psi_0\in\mathcal{E}_{AE}}\exists_{M>0}\forall\nu > 0\exists_{t>0}[d_{H_0^1}(\,\Psi(t;\psi_0,\nu)\,,\,\mathcal{E}_{AE}) > M].$$

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Viscosity induced instability: Without viscosity ($\nu = 0$, averaged Euler), the family $\mathcal{E}_{AE}$ is Lyapunov stable. But with viscosity ($\nu > 0$, second grade fluid), the solutions move away from $\mathcal{E}_{AE}$ in a viscosity-independent way.

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Radially symmetric solutions

Some observations for radially symmetric stream functions

- Both the Navier-Stokes equation and second grade fluid equation are radially equivariant in a disk, as well as all boundary conditions considered. So the set of all radially symmetric stream functions is invariant.
- In a disk, the sets $\mathcal{E}$ and $\mathcal{E}_{AE}$ consist of radially symmetric functions.
- In polar coordinates (r, ϕ), the nonlinear term $\mathbf{u} \cdot \nabla$ becomes

$$\mathbf{u} \cdot \nabla \equiv \frac{1}{r} \left[\psi_\phi \frac{\partial}{\partial r} - \psi_r \frac{\partial}{\partial \phi} \right].$$

This vanishes for a radially symmetric stream function. Thus on a disk, the Navier-Stokes equation and second grade fluid equation are linear for radially symmetric stream functions.

2nd grade fluid equation w. radial symm.

The second grade fluid equation with Navier-slip boundary condition for radially symmetric stream functions is

$$\partial_t L\psi = -\nu\Delta^2\psi,\ 0 \le r < R,\ t > 0 \quad \text{and} \quad \psi(R,t) = 0 = \psi_{rr}(R,t),\ t \ge 0. \tag{1}$$

The eigenvalue problem

$$L\psi = \lambda\Delta^2\psi,\ 0 \le r < R \quad \text{and} \quad \psi(R) = 0 = \psi_{rr}(R) \tag{2}$$

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- Eigenvalues are $0 < \lambda_0 \le \lambda_1 \le \ldots$ and the eigenfuntions ψ_n form an orthonormal basis in $H_0^{1,\mathrm{rad}}$.
- Define $\beta_n = \frac{\lambda_n}{1+\alpha\lambda_n}$, then $\psi(t) = e^{-\beta_n\nu t}\psi_n$ solves (1).
- The average Euler function χ_0 is not an eigenfunction of (2)

Starting at $\mathcal{E}_{AE}$

- Write $\chi_0 = \sum_{n=0}^{\infty} a_n \psi_n$, then $\Psi(t;\chi_0,\nu) = \sum_{n=0}^{\infty} a_n e^{-\beta_n \nu t} \psi_n$ and

$$d_{H_0^1}(\Psi(t;\chi_0,\nu),\mathcal{E}_{AE})^2 = \sum_{m=0}^{\infty} a_m^2 \left(e^{-\beta_m \nu t} - \sum_{n=0}^{\infty} a_n^2 e^{-\beta_n \nu t} \right)^2, \quad t \geq 0.$$

- Maximal deviation of the solution curve $\{\Psi(t;\chi_0,\nu) \mid t \geq 0\}$ and the family $\mathcal{E}_{AE}$ is independent of ν and given by

$$\max_{\tau \geq 0} \sum_{m=0}^{\infty} a_m^2 \left(e^{-\beta_m \tau} - \sum_{n=0}^{\infty} a_n^2 e^{-\beta_n \tau} \right)^2 > 0$$

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Note: For the free boundary condition, χ_0 would be the solution of the eigenvalue problem.

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Questions or suggestions?

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