Semiclassical resonances associated with an unstable equilibrium

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1. Introduction

We consider the semi-classical Schrödinger operator

$$P = -h^2\Delta + V(x),$$

where

$$\Delta = \sum_{j=1}^{n} \frac{\partial^2}{\partial x_j^2}, \quad V(x) \in C_0^\infty(\mathbb{R}^n;\mathbb{R}), \quad 0 < h << 1$$

Resonances

As operator on $L^2(\mathbb{R}^n)$, P is self-adjoint and $\sigma_{\mathrm{ess}}(P) = \mathbb{R}_+$. However, as operator $L^2_{\mathrm{comp}}(\mathbb{R}^n) \to L^2_{\mathrm{loc}}(\mathbb{R}^n)$, the resolvent $(z - P)^{-1}$ has meromorphic extension from $\mathbb{C}_+$ to $\mathbb{C}_-$ across $\mathbb{R}_+$. The poles are called "resonances".

Roughly speaking, resonances are characterized as complex numbers z s.t. there exists a non-trivial "outgoing" solution $u(x, h)$ (called "resonant state") to the equation

$$Pu = zu.$$

The imaginary part of resonances means the reciprocal of the exponential decay rate of the corresponding states for the evolution as time tends to $+\infty$.

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Classical mechanics

Let

$$p(x,\xi) = \xi^2 + V(x)$$

be the classical Hamiltonian, and

$$H_p = \nabla_\xi p \cdot \nabla_x - \nabla_x p \cdot \nabla_\xi$$

the Hamilton vector field on the phase space $\mathbb{R}^n_x \times \mathbb{R}^n_\xi$. The value $p(x,\xi)$ is invariant along the integral curve $\exp tH_p(x,\xi)$ starting from a point (x,ξ).

The "trapped trajectories" are defined as the set

$$K(z_0) := \{(x,\xi) \in p^{-1}(z_0); t \mapsto \exp tH_p(x,\xi) \text{ is bounded}\}$$

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Resonance free zone

Let z_0 be a positive energy and

$$\Omega(\epsilon,\delta)=\{z\in\mathbb{C};|\mathrm{Re}\,z-z_0|<\epsilon,\ -\delta<\Im z<0\}$$

Theorem :

(Martinez '03, cf : Sjöstrand '86)
Assume $K(z_0)=\emptyset$. Then $\exists\epsilon>0$ s.t. $\forall C>0$, there is no resonance in $\Omega(\epsilon, Ch|\log h|)$ for sufficiently small h.

- Given a geometry of non-empty $K(z_0)$, study the asymptotic (semi-classical) distribution of resonances in a complex neighborhood of z_0.

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Some known results and our problem

- In the case where $K(z_0)$ consists of a hyperbolic fixed point : $|\mathrm{Im}\, z| \sim \delta_1 h$ (Briet-Combes-Duclos '87, Sjöstrand '87)

- In the case where $K(z_0)$ consists of a hyperbolic periodic curve ($n \geq 2$) : $|\mathrm{Im}\, z| \sim \delta_2 h$ (Gérard-Sjöstrand '84)

- In the well in an island case : $|\mathrm{Im}\, z| \sim \exp(-S/h)$ where S is the Agmon distance from the well to the sea (Helffer-Sjöstrand '86).

- Our problem : the case where $K(z_0)$ consists of a hyperbolic fixed point and associated homoclinic trajectories

2. Results

We assume that $x = 0$ is a non-degenerate local maximum of $V(x)$ i.e.

$$V(x) = z_0 - \sum_{j=1}^{n} \frac{\lambda_j^2}{4} x_j^2 + \mathcal{O}(x^3) \quad \text{with } 0 < \lambda_1 \leq \ldots \leq \lambda_n.$$

The point $(x, \xi) = (0, 0)$ is a hyperbolic fixed point of H_p, and the "outgoing and incoming stable manifolds" $\Lambda_\pm$ are defined by

$$\Lambda_\pm := \{\rho := (x, \xi); \exp tH_p(\rho) \to (0,0) \text{ as } t \to \mp\infty\}$$

It turns out that for $\rho \in \Lambda_\pm$, $\exists \gamma(\rho)$ an eigenvector corresponding to the smallest eigenvalue λ_1 of the linearization of H_p s.t.

$$\exp tH_p(\rho) \sim e^{\pm\lambda_1 t}\gamma(\rho) \quad \text{as } t \to \mp\infty.$$

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We assume

(H1) $K(z_0)$ consists of $(0,0) \cup \mathcal{H}$, where $\mathcal{H} = \Lambda_+ \cap \Lambda_- \backslash (0,0)$ is the set of homoclinic trajectories

(H2) $g(\rho) \cdot g(\rho') \neq 0$ for $\forall \rho, \rho' \in \mathcal{H}$.

Theorem (BFRZ)

Assume (H1) and (H2). Then $\exists \delta > 0$ s.t. $\forall C > 0$, there is no resonance in $\Omega(Ch, \delta h)$ for sufficiently small h, if either (a) or (b) holds :

(a) The maximum at $x = 0$ is anisotropic, i.e. $\lambda_1 < \lambda_n$

(b) The intersection $\Lambda_+ \cap \Lambda_-$ is of finite order along $\mathcal{H}$.

- When $n = 1$, neither (a) nor (b) holds. In this case, the precise location of resonances is known (F-Ramond '97) :

$$|\mathrm{Im} z| \sim \frac{\log 2}{2} \lambda_1 \frac{h}{|\log h|}$$

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3. Method

Let z be a resonance and $u(x,h)$ a corresponding resonant state.

- Step 1 : Using the fact that u is outgoing (Bony-Michel '03), we show that u is microlocally 0 outside Λ_+ : i.e. the global FBI transform of u

$$(Tu)(x,\xi,h) := \int_{\mathbb{R}^n} e^{i(x-y)\cdot\xi/h-(x-y)^2/(2h)}u(y)dy$$

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- Step 2 : Continue u microlocally along $\mathcal{H}$ and show that, if $z \in \Omega(Ch, \delta h)$, its amplitude becomes smaller after a tour :

$$|u_{\mathrm{final}}| \lesssim h^{\alpha} |u_{\mathrm{initial}}| \quad \text{with} \;\; \alpha = \alpha(\delta) > 0,$$

for small h microlocally at a point on $\mathcal{H}$, which is a contradiction to the single-valuedness of u.

Microlocal continuation of the solution

- along $\mathcal{H}$: Maslov theory on WKB solutions
  $\rightarrow$ no decay in power of h.
- through $(0,0)$: Following theorem by BFRZ.

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Propagation of singularities through a hyperbolic fixed point

Let $\rho_- := (x_-, \xi_-) \in \Lambda_-$ with $|x_-| = \epsilon$ small, and $S_\epsilon = \{(x,\xi) \in \Lambda_-; |x| = \epsilon\}$. Consider a microlocal Cauchy problem

$$(\mathrm{MCP}) \begin{cases} Pu &= zu \quad \text{microlocally near } (0,0), \\ u &= u_- \quad \text{microlocally near } S_\epsilon \end{cases}$$

where the data u_- with $||u_-|| \leq 1$ satisfies

$$\begin{cases} Pu_- &= zu_- \quad \text{microlocally near } S_\epsilon, \\ u_- &= 0 \quad \text{microlocally near } S_\epsilon \backslash \{\rho_-\} \end{cases}$$

Theorem (BFRZ '07)

There exists $\delta' > 0$ such that, for $z \in \Omega(Ch, \delta' h)$, (MCP) has a unique solution u with $||u|| = \mathcal{O}(h^{-C})$. Moreover, microlocally near a point $\rho_+ \in \Lambda_+$ satisfying $g(\rho_-) \cdot g(\rho_+) \neq 0$, $u(x; h)$ is given by

$$h^{\sum \frac{\lambda_j - \lambda_1}{2\lambda_1} - i\frac{z - z_0}{h\lambda_1}} \int e^{i(\phi_+(x) - \phi_-(y))/h} d(x, y; h) u_-(y) dy.$$

Here $\phi_\pm(x)$ are generating functions of $\Lambda_\pm$, and $d(x, y; h)$ is an elliptic symbol of order 0 (explicitly computed at the principal level).

Sketch the step 2

- u_{initial} is of WKB form $u_{\text{initial}}(y,h) = e^{i\phi_+(y)/h}b(y;h)$ on $\mathcal{H} \cap \Lambda_+$, and so is its continuation to $\mathcal{H} \cap \Lambda_-$ along $\mathcal{H}$: $u_-(y,h) = e^{i\tilde{\phi}_+(y)/h}\tilde{b}(y;h)$, where $\tilde{\phi}_+(y)$ is a generating function of the evolution of Λ_+.

- Applying the pervious theorem, we obtain, for $-\delta h < \operatorname{Im} z < 0$,

$$|u_{\text{final}}| = \left| h^{\sum \frac{\lambda_j - \lambda_1}{2\lambda_1} - i\frac{z - z_0}{h\lambda_1}} \right| \left| \int e^{i(\phi_+(x) - \phi_-(y))/h} d(x,y;h) u_-(y) dy \right|$$

$$\leq h^{(\frac{1}{2}\sum(\lambda_j - \lambda_1) - \delta)/\lambda_1} \left| \int e^{i(\tilde{\phi}_+(y) - \phi_-(y))/h} d(x,y;h) b(y,h) dy \right|.$$

By the stationary phase method, the integral in the RHS is of $\mathcal{O}(h^\beta)$ for some $\beta > 0$ if Λ_+ and Λ_- intersects in finite order along $\mathcal{H}$.

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Hence

$$||u_{\text{final}}|| \lesssim h^{\alpha} ||u_{\text{initial}}||$$

with

$$\alpha = \frac{\frac{1}{2}\sum(\lambda_j - \lambda_1) + \lambda_1\beta - \delta}{\lambda_1},$$

and obvoiusly $\alpha > 0$ if either (a) or (b) holds and $\delta < \frac{1}{2}\sum(\lambda_j - \lambda_1) + \lambda_1\beta$.