# On the Continuum Hamiltonian Hopf Bifurcation II (Vlasov-Poisson)

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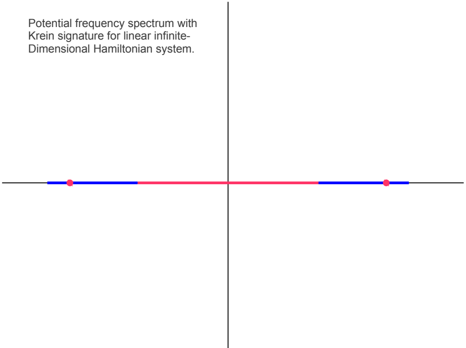
# Review: Linearize Vlasov-Poisson around $f_0(v)$

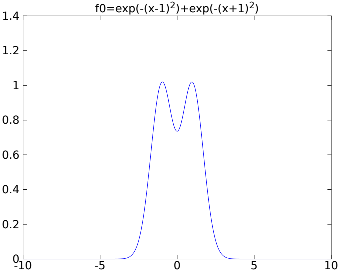
- The Vlasov-Poisson equation has a rich family of equilibria, simplest are $f_0 = f_0(v)$.
- Linearize, put in $k$ space:

$$\frac{\partial \hat{f}_k}{\partial t} + ikv\hat{f}_k - \frac{i}{k} f_0' \int_{\mathbb{R}} dv \hat{f}_k = 0$$

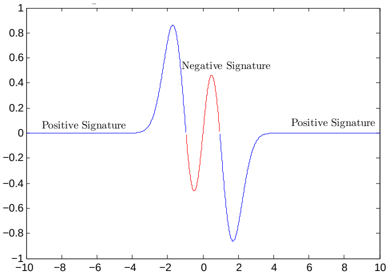
- Continuous spectrum of time evolution operator $T$ is $i\mathbb{R}$, has a signature given by $sgn(-uf_0'(u))$, $u = \omega/k$ $i\omega \in \sigma_T$.



# Signature



f0=exp(-(x-1)2)+exp(-(x+1)2)

Negative Signature
Positive Signature
Positive Signature

# Overview

- Study bifurcations to instability of the linearized Vlasov equation through changes in $f_0$.
- **We show that all $f_0$ are infinitesimally close to instability in the $W_{1,1}$ norm.**
- **If perturbations to $f_0$ are restricted to be dynamically accessible, then $f_0$ is only close to instability if it has a signature change.**
- These results and more in arxiv.org/abs/1002.1039

# Perturbations to the time evolution operator

- Time evolution operator $T = ikv\hat{f}_k - \frac{i}{k} f_0' \int_{\mathbb{R}} dv \hat{f}_k$.
- **How does the spectrum change when we change $f_0$?**
- Perturbation of $T$ is $-\frac{i}{k}\delta f_0' \int_{\mathbb{R}} dv \hat{f}_k$.
- Place ourselves in the Banach space $W_{1,1}(\mathbb{R})$ and use operator norm.
- Then $\|\delta T\|$ is proportional to $\|\delta f_0'\|$.

# Stablility

- For eigenvalues there is a dispersion relation:

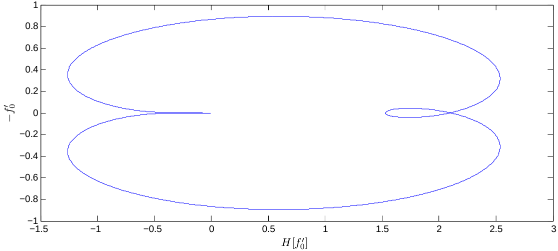
$$\epsilon(k,u) \equiv 1 - \frac{1}{k^2}\int_{\mathbb{R}} dv \frac{f_0'}{v-u} = 0$$

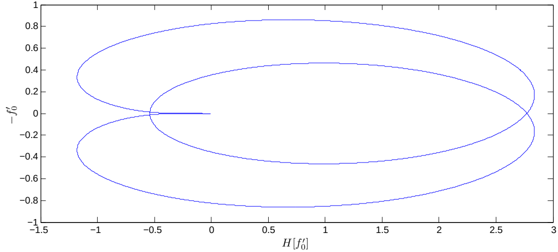
- Analytic function in upper half plane, use the Nyquist Method.

$$\epsilon(k,u) = 1 - \frac{1}{k^2}\mathbf{PV}\int_{\mathbb{R}} dv \frac{f_0'}{v-u} - \pi i f_0'(u)$$

- The number of unstable eigenvalues is the winding number of image of the real line under this map.

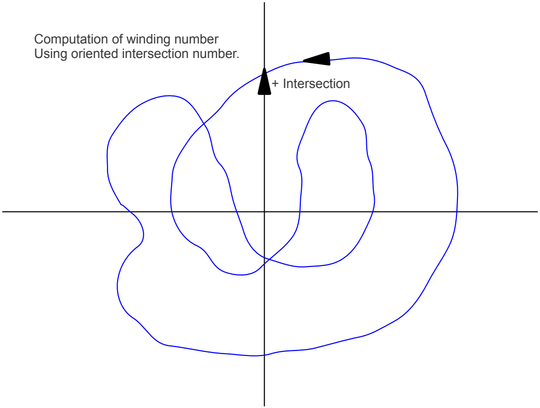
# Stable and Unstable Penrose Plots



# Determining the Winding Number

- The winding number is the oriented intersection number of the curve of interest with any line segment from the origin to the point at infinity.



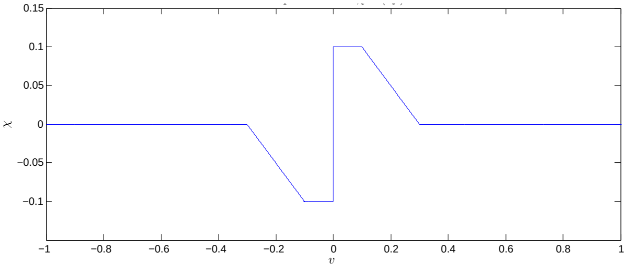
# Increase the Winding Number With a Small Perturbation

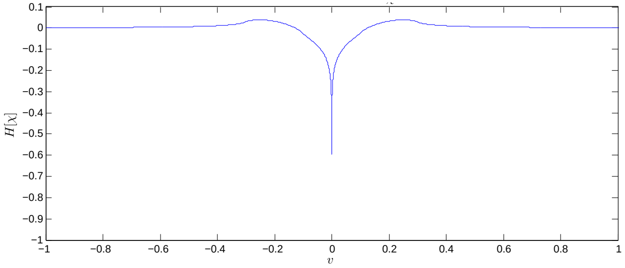
- The Hilbert transform of a $W_{1,1}$ function is not necessarily continuous (thus results depend on choice of norm).
- Construct infinitesimal perturbation that 'shifts' the crossing in the Penrose plot by a unit amount.

$$\begin{aligned} \chi &= \frac{hv}{\epsilon} & |v| < \epsilon \\ &= hsgn(v) & \epsilon < |v| < d + \epsilon \\ &= h + d + \epsilon - v & h + d + \epsilon > v > d + \epsilon \\ &= -h - d - \epsilon - v & h + d + \epsilon > -v > d + \epsilon \\ &= 0 & |v| > h + d + \epsilon \end{aligned}$$

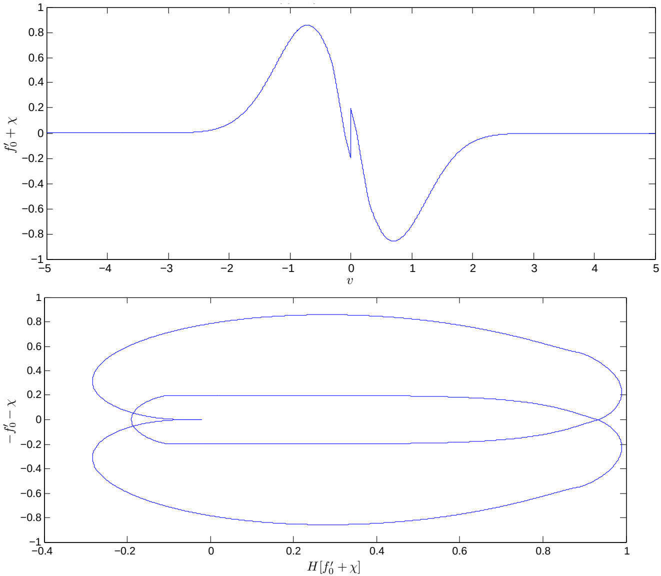
- $\|\chi(v, h, d, \epsilon)\| = h^2 + 2hd + h\epsilon$ and if $\epsilon = O(e^{-1/h})$ then $H\chi(0) = O(1)$.

# The Perturbation $f_p'$

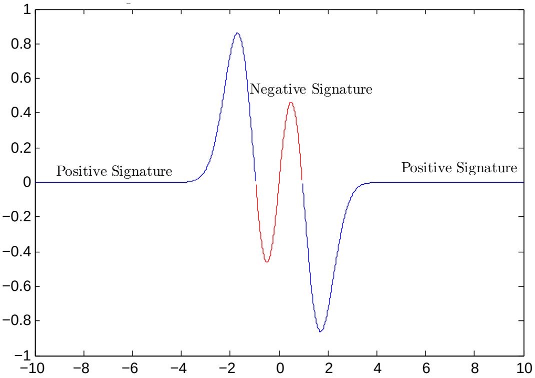


# Destabilization of a Maxwellian Distribution



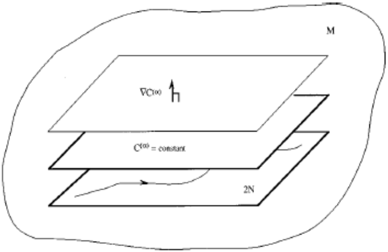
# Role of Signature?




- Maxwellian distribution function has entirely positive signature, thus no role for signature.
- Did something go wrong?

# Dynamical Accessibility

- Reason: Bracket $\{,\}$ depended on $f_0'$, thus perturbations change the bracket as well as the Hamiltonian. Non-canonical system require more care.
- Dynamics of the full nonlinear Vlasov-Poisson equation is an area preserving rearrangement, even under outside forcing.
- New question what if we restrict to a single symplectic leaf?



# Perturbation by Rearrangement

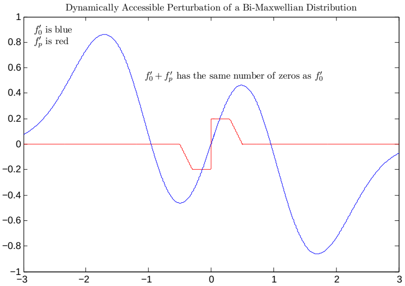
- Consider $(x, v) \to (X, V)$
- Symplectic maps satisfy $[X, V] = \frac{\partial X}{\partial x}\frac{\partial V}{\partial v} - \frac{\partial X}{\partial v}\frac{\partial V}{\partial x} = 1$
- Need $f_0 \circ (X, V)$ to be homogeneous.
- $V(v)$ monotonic, $X(x, v) = x/V'(v)$

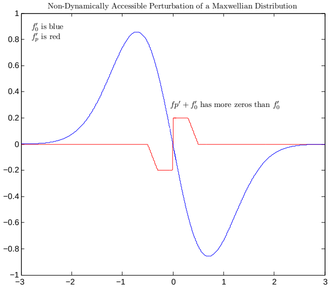
# Positive Signature Implies Structural Stability

- Composition with $V(v)$ preserves critical points of $f_0$ by chain rule.
- Unstable Penrose plots all have more than one critical point.
- When the signature is only positive, there is one critical point.
- **No rearrangement can cause a bifurcation to instability in the positive signature case.**

# Topology of Critical Points

- If there is negative signature, perturbation $\chi$ may not increase the number of critical points.



# Destabilizing Rearrangement

- Find a $V(v)$ such that $V(v)'f_0' \circ V$ is approximately $f_0' + \chi$.
- If there is negative signature, construct such a rearrangement directly using Morse's Lemma.
- If the family $f_0$ is restricted to a single leaf, then there are only bifurcations if $f_0$ has negative signature.
- The bifurcation point occurs only at the 'valleys' of the distribution function, i.e. where $f_0'' > 0$. Bifurcations do not come from every signature change in our distribution function.

# Conclusions

- Under nondynamically accessible perturbations, every distribution function is structurally unstable.
- Under dynamically accessible perturbations, we recover an analogue of the Krein-Moser theorem.
- What does this say about the properties of the full Vlasov-Poisson Equation, and can we draw any conclusions from the linear theory?
- Remaining open problems: More general perturbations, other noncanonical systems. 2D Euler may be particularly easy as it is closely related.