Lagrangian approach to weakly and strongly nonlinear stability analyses of fluid models

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in collaboration with

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1. Introduction

Fluid models (PDEs)

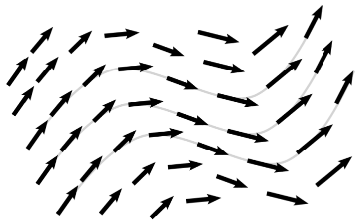
Geophysical fluids
Astrophysical plasmas
Gravitational many-body systems
etc.

Direct manipulations
straightforward, but cumbersome ...

no dissipation

(Noncanonical) Hamiltonian system

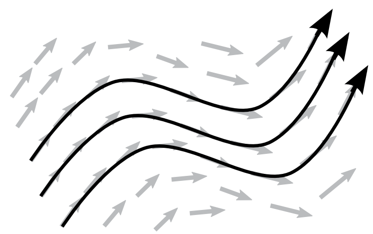
Velocity $\boldsymbol{v}$, Density ρ, Pressure p etc.



reduction

(Canonical) Lagrangian system

Flow map $\varphi_t : \mathbb{R}^n \to \mathbb{R}^n$



Linear & Nonlinear stability analysis

Powerful techniques
- variational principle
- normal forms
- ...

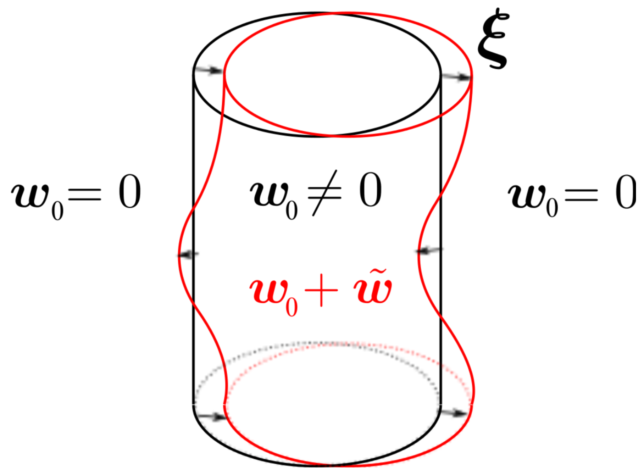
Lagrangian approach is advantageous to analyze complicated fluid models.

Many (conserved) variables, 3D & non-Euclidean space, free boundary etc.

Example 1. Continuity equation

$$\partial_t \rho + \nabla \cdot (\rho \boldsymbol{v}) = 0 \quad \Rightarrow \quad \varphi_t^* (\rho d^3 x) = \rho_0 d^3 x_0$$ solved! (in terms of φ_t)

Example 2. Vortex tube dynamics ($\boldsymbol{w}$: vorticity)



$$\partial_t \boldsymbol{w} = \nabla \times (\boldsymbol{v} \times \boldsymbol{w})$$

Isovortical perturbation $\tilde{\boldsymbol{w}} = \nabla \times (\boldsymbol{\xi} \times \boldsymbol{w}_0)$
[Arnold (1966)]

Only the deformation of the tube can be discussed.
[Next talk by Fukumoto]

Example 3. Ideal magnetohydrodynamic stability [Bernstein *et al.* (1958), Newcomb (1962)]

$$\rho \frac{D\boldsymbol{v}}{Dt} = -\nabla p + (\nabla \times \boldsymbol{B}) \times \boldsymbol{B}$$
$$\frac{\partial \boldsymbol{B}}{\partial t} = \nabla \times (\boldsymbol{v} \times \boldsymbol{B})$$
$$\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \boldsymbol{v}) = 0$$
$$\frac{\partial s}{\partial t} + \boldsymbol{v} \cdot \nabla s = 0$$

($\boldsymbol{v}$: velocity, $\boldsymbol{B}$: magnetic field, ρ: density, s: specific entropy, $p(\rho, s)$: pressure)

$\Longrightarrow$ Linearization

$$\tilde{\boldsymbol{v}} = \partial_t \boldsymbol{\xi} + \boldsymbol{v}_0 \cdot \nabla \boldsymbol{\xi} - \boldsymbol{\xi} \cdot \nabla \boldsymbol{v}_0,$$
$$\tilde{\boldsymbol{B}} = \nabla \times (\boldsymbol{\xi} \times \boldsymbol{B}_0),$$
$$\tilde{\rho} = -\nabla \cdot (\rho_0 \boldsymbol{\xi}),$$
$$\tilde{s} = -\boldsymbol{\xi} \cdot \nabla s_0,$$

$\Downarrow$

Frieman-Rotenberg equation(1960)

$$\rho_0 \frac{\partial^2 \boldsymbol{\xi}}{\partial t^2} + 2\rho_0 \boldsymbol{v}_0 \cdot \nabla \frac{\partial \boldsymbol{\xi}}{\partial t} = \mathcal{F} \boldsymbol{\xi}$$

Gyroscopic system ($\rho_0 \boldsymbol{v}_0 \cdot \nabla$: anti-Hermitian, $\mathcal{F}$: Hermitian)
$\Rightarrow$ Hamiltonian Hopf bifurcation occurs only in the presense of basic flow $\boldsymbol{v}_0$.

Outline of this talk

1. Introduction

2. Action-angle representation of linear perturbation $(\cdots$ *Linear regime*$)$

   Krein signature for eigenmode and continuum mode

3. Formulation of weakly nonlinear mode coupling

   $(\cdots$ *Weakly nonlinear regime*$)$

   Reduction to normal forms using Lagrangian

4. Lagrangian approach to explosive instability

   $(\cdots$ *Strongly nonlinear regime*$)$

   Boudary layer problem

2. Action-angle representation of linear perturbation

Action-angle variables for eigenmodes

In linearized Hamiltonian system, each periodic eigenmode ($\propto e^{-i\omega t}$) satisfies

$$\boxed{\text{Modal energy } (E)} = \boxed{\text{Frequency } (\omega)} \times \boxed{\text{action } (\mu)}$$

$\mathrm{sgn}(\mu)$: Krein signature

- (noncanonical) Hamiltonian formulation

Linearized system: $\partial_t u = \mathcal{J}\mathcal{H}u$ for $u = (\tilde{\boldsymbol{v}}, \tilde{\boldsymbol{B}}, \tilde{\rho}, \tilde{s})$

($\mathcal{J}$: anti-Hermitian, $\mathcal{H}$: Hermitian)

Dynamically accesible perturbation: $u = \mathcal{J}u^{\dagger}$, $u^{\dagger} = (\boldsymbol{\xi}, \boldsymbol{\eta}, \alpha, \beta)$

For $u = \hat{u}e^{-i\omega t}$, $E = (u, \mathcal{H}u) = i\omega(\hat{u}^{\dagger}, \hat{u})$ $\Rightarrow$ Action $\mu = (\hat{u}^{\dagger}, i\mathcal{J}\hat{u}^{\dagger})$ (*)

- (canonical) Lagrangian formulation

F-R eq. $\Rightarrow$ Canonical variables $(\boldsymbol{q}, \boldsymbol{p}) = (\boldsymbol{\xi}, \rho_0\partial_t\boldsymbol{\xi} + \rho_0\boldsymbol{v}_0 \cdot \nabla\boldsymbol{\xi})$

For $\boldsymbol{\xi} = \hat{\boldsymbol{\xi}}e^{-i\omega t}$, Action $\mu = \oint \boldsymbol{p} \cdot d\boldsymbol{q} = \int \overline{\hat{\boldsymbol{\xi}}} \cdot \rho_0(\omega + i\boldsymbol{v}_0 \cdot \nabla)\hat{\boldsymbol{\xi}} d^3x$ (**)

Both expressions are equivalent. But, (**) is more reduced and informative than (*).

Action-angle "variables" for continuous spectrum

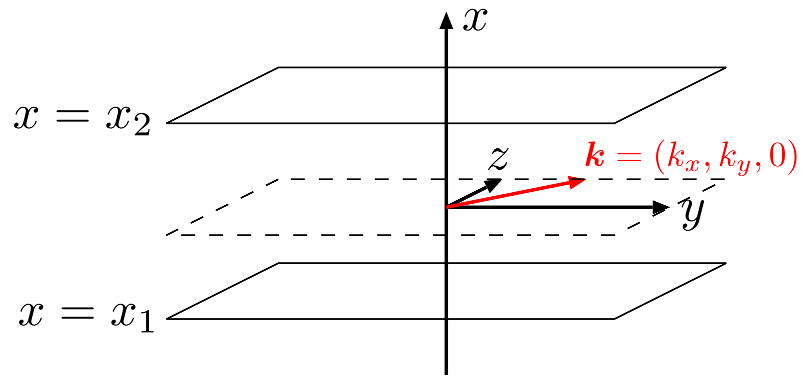
[Morrison (2000), Balmforth & Morrison (2002)]

Slab equilibria $x_1 \leq x \leq x_2$

$\Rightarrow$ further reduction $\boldsymbol{\xi} = (\xi_x, \xi_y, \xi_z) \mapsto \xi_x$

$\Rightarrow$ Sturm-Liouville Eigenvalue Problem

(Goedbloed 1971, Appert *et al.* 1974)



$$\frac{\partial}{\partial x}\left[P(\omega, x)\frac{\partial \hat{\xi}_x}{\partial x}\right] - Q(\omega, x)\hat{\xi}_x = 0,$$

$$\hat{\xi}_x|_{x=x_1} = \hat{\xi}_x|_{x=x_2} = 0,$$

Continuous spectrum $\{\omega \in \mathbb{R} |\ \exists x_s \in [x_1, x_2]$ s.t. $P(\omega, x_s) = 0\}$

Regular singular points

$\Updownarrow$

Continuum mode; "continuum of singular eigenfunctions" (Frobenius solutions)

Eample. Parallel shear flow $\boldsymbol{v}(x)$

$\Rightarrow$ Rayleigh equation: $P(\omega, x) = (\omega - \boldsymbol{k} \cdot \boldsymbol{v})^2$, $Q(\omega, x) = k^2(\omega - \boldsymbol{k} \cdot \boldsymbol{v})^2$

$\Rightarrow$ Balmforth & Morrison (2002) succeeded in transforming the continuum mode into action-angle variables via a generalized Hilbert transform.

What is more general strategy for various fluid systems?

Action-angle representation using the Laplace transform

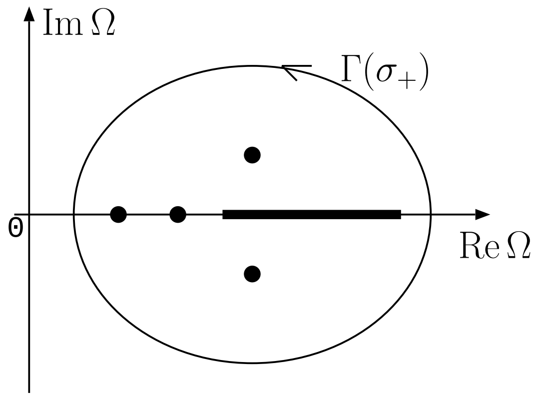
[Hirota & Fukumoto, J. Math. Phys. 49, 083101 (2008)]

Let $\xi_x(x,t) \mapsto \Xi(x,\Omega), \Omega \in \mathbb{C}$ be the Laplace transform. Define

$$D(\Omega) = \int_{x_1}^{x_2} \overline{\Xi(\overline{\Omega})} \left\{ \frac{\partial}{\partial x} \left[P(\Omega,x) \frac{\partial \Xi}{\partial x}(\Omega) \right] - Q(\Omega,x)\Xi(\Omega) \right\} dx$$

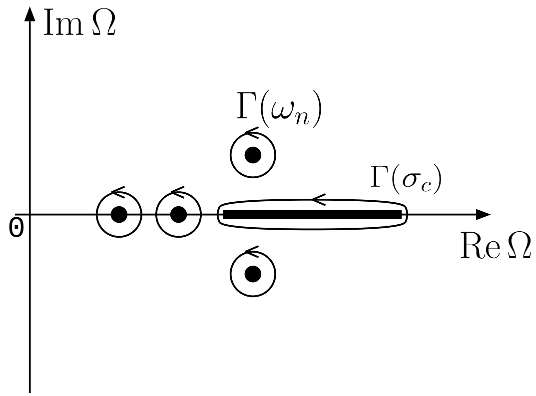
Action variables for eigenmode and continuum mode are given by

- Eigenvalues $\{\omega_n | n = 1,2,\dots\}$, $\quad \mu_n = \dfrac{1}{2\pi i} \oint_{\Gamma(\omega_n)} D(\Omega) d\Omega$, (residue)
- Continuous spectrum $\omega \in \sigma_c \subset \mathbb{R}$, $\quad \mu(\omega) = \dfrac{i}{2\pi} \left[D(\omega + i0) - D(\omega - i0) \right]$. (jump)



Deformation

$\Longrightarrow$



Example. Alfvén continuous spectrum in Ideal MHD

$$\Rightarrow P(\omega, x) = (\omega - \boldsymbol{k}\cdot\boldsymbol{v})^2 - \boldsymbol{k}\cdot\boldsymbol{B}^2$$

Alfvén continuous spectrum: $\sigma_A^{\pm} = \{\boldsymbol{k}\cdot\boldsymbol{v}(x) \pm \omega_A(x) | x \in [x_1, x_2]\}$

$\omega_A(x) = |\boldsymbol{k}\cdot\boldsymbol{B}(x)|$: Alfvén frequency

Singular eigenfunction: (Frobenius series solution)

$$\hat{\boldsymbol{\xi}}(x,\omega) = \frac{\boldsymbol{B}}{|\boldsymbol{B}|} \times \boldsymbol{e}_x \left[\frac{\hat{C}_A(\omega)}{\pi} \mathrm{p.v.} \frac{1}{\omega - \boldsymbol{k}\cdot\boldsymbol{v} \mp \omega_A} + \hat{C}_A^{\dagger}(\omega)\delta(\omega - \boldsymbol{k}\cdot\boldsymbol{v} \mp \omega_A) \right] + \ldots,$$

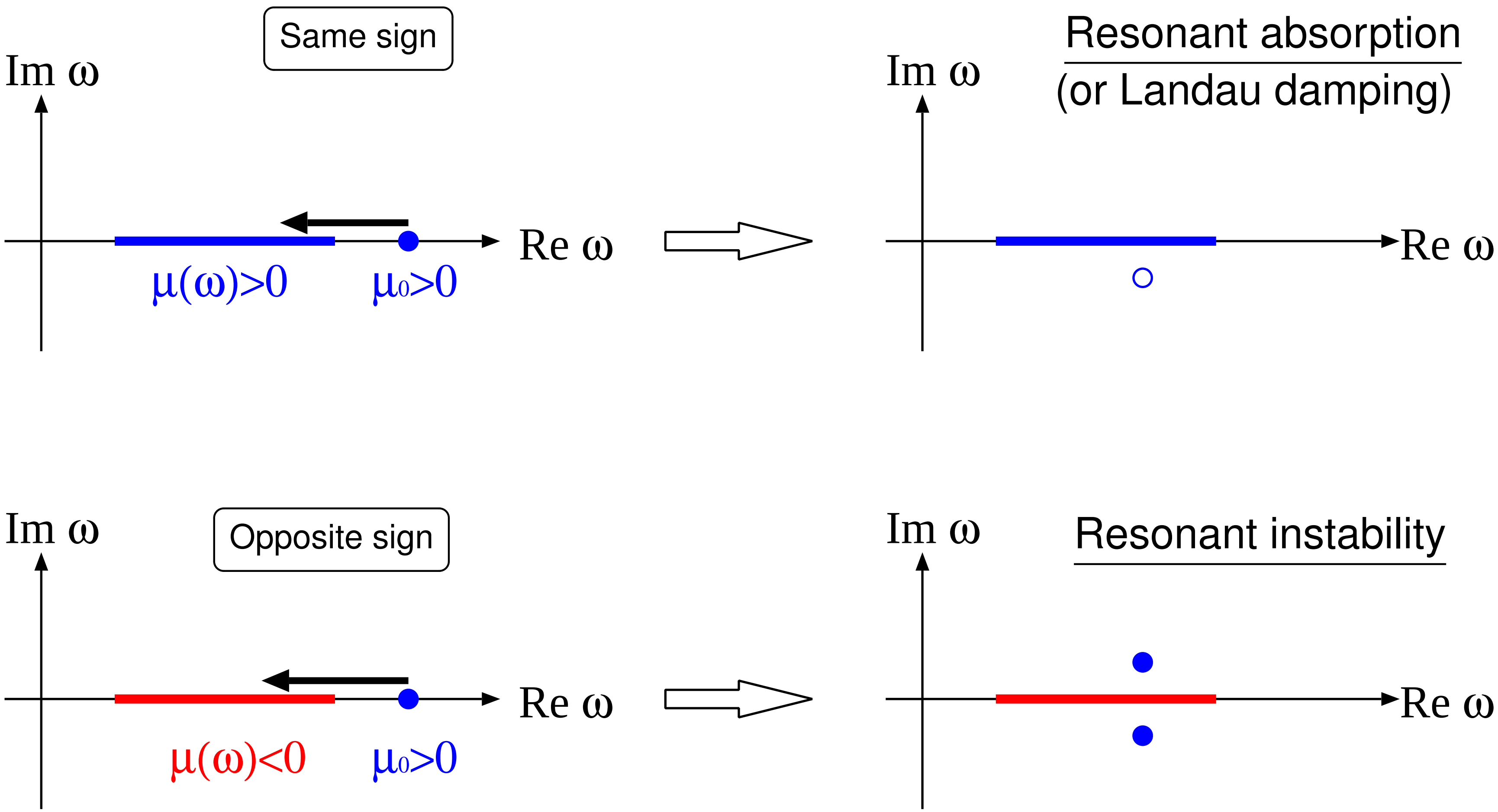
Action variable: [Hirota & Fukumoto, PoP **15**, 122101 (2008)]

$$\mu(\omega) = \left[|\hat{C}_A(\omega)|^2 + |\hat{C}_A^{\dagger}(\omega)|^2 \right] \int_{x_1}^{x_2} \omega_A [\delta(\omega - \boldsymbol{k}\cdot\boldsymbol{v} - \omega_A) - \delta(\omega - \boldsymbol{k}\cdot\boldsymbol{v} + \omega_A)] dx.$$

Krein signature, $\mathrm{sgn}(\mu(\omega))$, is evident from this expression!

Alfvén continuum mode has negative energy $\omega\mu(\omega) < 0$ if and only if $|\boldsymbol{k}\cdot\boldsymbol{v}| > |\boldsymbol{k}\cdot\boldsymbol{B}|$ somewhere on $[x_1, x_2]$.

Resonance between eigenmode and continuum mode



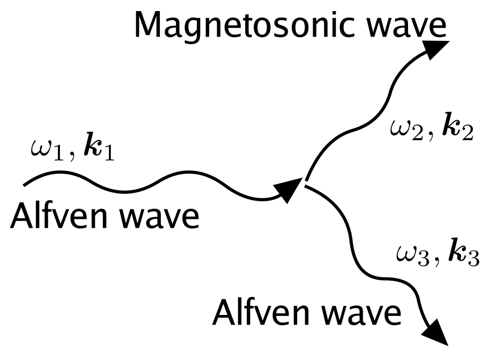
By using the averaged Lagrangian method (assuming $\mathrm{Re}\ \omega \gg 0$), adiabatic invariance of the total wave action $\mu_0 + \int \mu(\omega) d\omega$ holds. [Hirota & Tokuda, PoP **17**, 082109 (2010)]

3. Formulation of weakly nonlinear mode coupling

Difficulty of analysis under nonuniformity (or nonlocality)

Weakly nonlinear phenomena

- Three-wave resonance ⇒ Parametric decay (Sagdeev & Galeev 1969)
- Landau equation (1944) (four-wave resonance)
- Modulational instability (secondary instability)
- ...



$$\omega_1 = \omega_2 + \omega_3$$
$$\boldsymbol{k}_1 = \boldsymbol{k}_2 + \boldsymbol{k}_3$$

These require higer-order perturbation analysis and renormalization technique.
⇒ Naive expasion of fluid models often falls into tedious algebra.
⇒ Most analyses are limited to resonaces among plane waves or wave packets.

Whitham (1967) proposed the following approach to water waves.

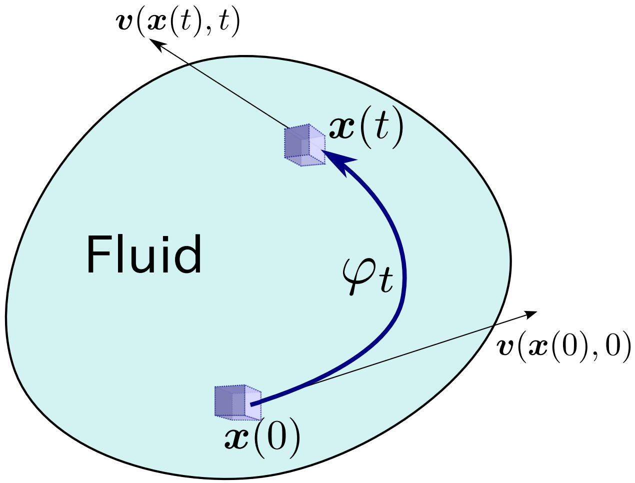
1. Small-amplitude expansion of Lagrangian
2. Averaging
3. Variational principle ⇒ Normal forms

☞ It would be beneficial to apply Whitham's method to various fluid models.

Newcomb's Lagrangian theory

The ideal MHD equations

$$\rho\frac{D\boldsymbol{v}}{Dt} = -\nabla p + (\nabla\times\boldsymbol{B})\times\boldsymbol{B}$$
$$\frac{\partial\boldsymbol{B}}{\partial t} = \nabla\times(\boldsymbol{v}\times\boldsymbol{B})$$
$$\frac{\partial\rho}{\partial t} + \nabla\cdot(\rho\boldsymbol{v}) = 0$$
$$\frac{\partial s}{\partial t} + \boldsymbol{v}\cdot\nabla s = 0$$



$\boldsymbol{B}, \rho, s$ are frozen into the flow map $\varphi_t : \boldsymbol{x}(0) \to \boldsymbol{x}(t)$

Lagrangian [Newcomb (1962)]

$$L[\varphi_t] = \int \left[\frac{\rho}{2}|\boldsymbol{v}|^2 - \frac{1}{2}|\boldsymbol{B}|^2 - \rho U(\rho,s)\right] d^3x, \qquad U(\rho,s) : \text{internal energy}$$

Nonlinear displacement: $\boldsymbol{x}(t) \mapsto \boldsymbol{x}(t) + \boldsymbol{\Xi}(\boldsymbol{x}(t),t)$
Small-amplitude expansion: $L = L^{(0)} + L^{(1)}(\boldsymbol{\Xi}) + L^{(2)}(\boldsymbol{\Xi},\boldsymbol{\Xi}) + L^{(3)}(\boldsymbol{\Xi},\boldsymbol{\Xi},\boldsymbol{\Xi}) + \ldots$

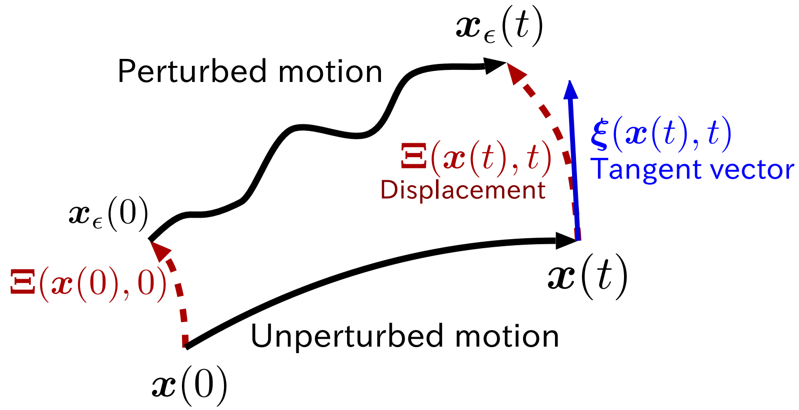
- Formulation of $L^{(2)}$ is established. [Frieman-Rotenberg (1960), Dewar (1970)]
- $L^{(3)}$ is derived by Pfirsch & Sudan (1993). But, no basic flow and an important symmetry is missing.

Variational principle for nonlinear displacement field

[Hirota, J. Plasma Phys. **77**, 589 (2011)]

Difficulty: Nonlinear displacement Ξ is not a vector field, but a mapping!

$$
\begin{aligned}
\boldsymbol{x}_\epsilon &= \boldsymbol{x} + \boldsymbol{\Xi}(\boldsymbol{x}, t) \\
&= e^{\boldsymbol{\xi}\cdot\nabla}\boldsymbol{x} \\
&= \boldsymbol{x} + \boldsymbol{\xi} + \frac{1}{2}\boldsymbol{\xi}\cdot\nabla\boldsymbol{\xi} + \frac{1}{6}\boldsymbol{\xi}\cdot\nabla(\boldsymbol{\xi}\cdot\nabla\boldsymbol{\xi}) + \dots
\end{aligned}
$$



The corresponding variation of the Eulerian variables u is

$\Rightarrow$ Lie series: $u_\epsilon = u + \mathscr{L}_{\boldsymbol{\xi}}u + \frac{1}{2}\mathscr{L}_{\boldsymbol{\xi}}\mathscr{L}_{\boldsymbol{\xi}}u + \frac{1}{6}\mathscr{L}_{\boldsymbol{\xi}}\mathscr{L}_{\boldsymbol{\xi}}\mathscr{L}_{\boldsymbol{\xi}}u + \dots$

$$
u = \begin{pmatrix} \boldsymbol{v} \\ \boldsymbol{B} \\ \rho \\ s \end{pmatrix}
\begin{matrix} \leftarrow \text{vector} \\ \leftarrow \text{2-form} \\ \leftarrow \text{3-form} \\ \leftarrow \text{0-form} \end{matrix}
\quad \Rightarrow \quad \text{Lie derivative: } \mathscr{L}_{\boldsymbol{\xi}}u = \begin{pmatrix} \partial_t\boldsymbol{\xi} + (\boldsymbol{v}\cdot\nabla)\boldsymbol{\xi} - (\boldsymbol{\xi}\cdot\nabla)\boldsymbol{v} \\ \nabla\times(\boldsymbol{\xi}\times\boldsymbol{B}) \\ -\nabla\cdot(\rho\boldsymbol{\xi}) \\ -\boldsymbol{\xi}\cdot\nabla s \end{pmatrix}
$$

Rearrangement of Lie series

<u>Theorem:</u> In terms of $\boldsymbol{\Xi} = \boldsymbol{\xi} + \frac{1}{2}\boldsymbol{\xi}\cdot\nabla\boldsymbol{\xi} + \frac{1}{6}\boldsymbol{\xi}\cdot\nabla(\boldsymbol{\xi}\cdot\nabla\boldsymbol{\xi}) + \ldots$,

$$e^{\mathscr{L}_{\boldsymbol{\xi}}} = 1 + \mathscr{L}_{\boldsymbol{\xi}} + \frac{1}{2}\mathscr{L}_{\boldsymbol{\xi}}\mathscr{L}_{\boldsymbol{\xi}} + \frac{1}{6}\mathscr{L}_{\boldsymbol{\xi}}\mathscr{L}_{\boldsymbol{\xi}}\mathscr{L}_{\boldsymbol{\xi}} + \ldots$$
$$= 1 + \mathscr{L}_{\boldsymbol{\Xi}} + \frac{1}{2}\mathscr{L}^2_{\boldsymbol{\Xi},\boldsymbol{\Xi}} + \frac{1}{6}\mathscr{L}^3_{\boldsymbol{\Xi},\boldsymbol{\Xi},\boldsymbol{\Xi}} + \ldots.$$

where

$$\mathscr{L}^2_{\boldsymbol{\eta},\boldsymbol{\xi}} \overset{\text{def}}{=} \mathscr{L}_{\boldsymbol{\eta}}\mathscr{L}_{\boldsymbol{\xi}} - \mathscr{L}_{\boldsymbol{\eta}\cdot\nabla\boldsymbol{\xi}},$$
$$\mathscr{L}^3_{\boldsymbol{\zeta},\boldsymbol{\eta},\boldsymbol{\xi}} \overset{\text{def}}{=} \mathscr{L}_{\boldsymbol{\zeta}}\mathscr{L}^2_{\boldsymbol{\eta},\boldsymbol{\xi}} - \mathscr{L}^2_{\boldsymbol{\zeta}\cdot\nabla\boldsymbol{\eta},\boldsymbol{\xi}} - \mathscr{L}^2_{\boldsymbol{\eta},\boldsymbol{\zeta}\cdot\nabla\boldsymbol{\xi}},$$
$$\mathscr{L}^n_{\boldsymbol{\xi}_1,\boldsymbol{\xi}_2,\ldots,\boldsymbol{\xi}_n} \overset{\text{def}}{=} \mathscr{L}_{\boldsymbol{\xi}_1}\mathscr{L}^{n-1}_{\boldsymbol{\xi}_2,\ldots,\boldsymbol{\xi}_n} - \sum_{j=2}^{n}\mathscr{L}^{n-1}_{\boldsymbol{\xi}_2,\ldots,\boldsymbol{\xi}_1\cdot\nabla\boldsymbol{\xi}_j\ldots,\boldsymbol{\xi}_n},$$

are symmetric with respect to any permutation of subscript vector fields.
(Proof) Use the Jacobi identity; $\mathscr{L}_{\boldsymbol{\xi}}\mathscr{L}_{\boldsymbol{\eta}} - \mathscr{L}_{\boldsymbol{\eta}}\mathscr{L}_{\boldsymbol{\xi}} = \mathscr{L}_{\boldsymbol{\xi}\cdot\nabla\boldsymbol{\eta}-\boldsymbol{\eta}\cdot\nabla\boldsymbol{\xi}}$ for all $\boldsymbol{\xi}$ and $\boldsymbol{\eta}$.

<u>Example.</u> If $\mathscr{L}_{\boldsymbol{\xi}} = \boldsymbol{\xi}\cdot\nabla$ in Cartesian coordinates,

$$e^{\mathscr{L}_{\boldsymbol{\xi}}}s = s + \Xi_i\frac{\partial s}{\partial x_i} + \frac{1}{2}\Xi_i\Xi_j\frac{\partial^2 s}{\partial x_i\partial x_j} + \frac{1}{6}\Xi_i\Xi_j\Xi_k\frac{\partial^3 s}{\partial x_i\partial x_j\partial x_k} + \ldots$$

Perturbation expansion of the Lagrangian around an equilibrium state u results in

Lagrangian for nonlinear displacement (Hirota, J. Plasma Phys. 2011)

$$L[\boldsymbol{\Xi}] = \int \frac{\rho}{2}\left|\frac{D\boldsymbol{\Xi}}{Dt}\right|^2 d^3x - \frac{W^{(2)}(\boldsymbol{\Xi},\boldsymbol{\Xi})}{2} - \frac{W^{(3)}(\boldsymbol{\Xi},\boldsymbol{\Xi},\boldsymbol{\Xi})}{3!} - \frac{W^{(4)}(\boldsymbol{\Xi},\boldsymbol{\Xi},\boldsymbol{\Xi},\boldsymbol{\Xi})}{4!} - \dots$$

where $D/Dt = \partial_t + \boldsymbol{v}\cdot\nabla$.

nth-order potential energy: $W^{(n)}(\boldsymbol{\Xi},\dots,\boldsymbol{\Xi}) = -\int \boldsymbol{\Xi}\cdot\mathcal{F}^{(n-1)}(\boldsymbol{\Xi},\dots,\boldsymbol{\Xi})d^3x$

Equation of motion

$$\Rightarrow \quad \rho\frac{D^2\boldsymbol{\Xi}}{Dt^2} = \mathcal{F}\boldsymbol{\Xi} + \frac{1}{2}\mathcal{F}^{(2)}(\boldsymbol{\Xi},\boldsymbol{\Xi}) + \frac{1}{3!}\mathcal{F}^{(3)}(\boldsymbol{\Xi},\boldsymbol{\Xi},\boldsymbol{\Xi}) + O(\epsilon^4),$$

Nonlinear extension of the Frieman-Rotenberg equation!

Case 1. Nonlinear three-mode coupling

Resonant three eigenmodes: $\boldsymbol{\Xi} = \sum_{j=a,b,c} A_j(\epsilon t)\hat{\boldsymbol{\xi}}_j e^{-i\omega_j t} + \text{c.c.}, \quad (\omega_a = \omega_b + \omega_c)$

<u>Amplitude equations</u>

$$\mu_a \frac{dA_a}{dt} = -iW^{(3)}_{a,b,c} A_b A_c, \quad \mu_b \frac{dA_b^*}{dt} = iW^{(3)}_{a,b,c} A_a^* A_c, \quad \mu_c \frac{dA_c^*}{dt} = iW^{(3)}_{a,b,c} A_a^* A_b$$

- Wave action: $N_j = \mu_j |A_j|^2$ where $\mu_j = 2\int \left[\hat{\boldsymbol{\xi}}_j^* \cdot \rho(\omega_j + i\boldsymbol{v}\cdot\nabla)\hat{\boldsymbol{\xi}}_j\right] d^3x$
- Coupling coefficient: $W^{(3)}_{a,b,c} = W^{(3)}(\hat{\boldsymbol{\xi}}_a^*, \hat{\boldsymbol{\xi}}_b, \hat{\boldsymbol{\xi}}_c)$ $\cdots$ strength of coupling

<u>Remark</u>:
The energy conservation, $\omega_a N_a + \omega_b N_b + \omega_c N_c =$ const., holds due to the cubic symmetry of $W^{(3)}$.

Case 2. Nonlinear hydrodynamic stability

Landau's idea (1944)

"Nonlinear self-interaction of the dominant mode generates second harmonics and distorts the mean fields."

- Seek the solution in the form of

$$\boldsymbol{\Xi} = \boldsymbol{\Xi}^{(1)} + \tfrac{1}{2}\boldsymbol{\Xi}^{(2)} \quad \text{with} \quad \boxed{\boldsymbol{\Xi}^{(1)} = A(\epsilon t)(\hat{\boldsymbol{\xi}}_1 e^{-i\omega t} + \text{c.c.})}$$

$$(\rho\tfrac{D^2}{Dt^2} - \mathcal{F})\boldsymbol{\Xi}^{(2)} = \mathcal{F}^{(2)}(\boldsymbol{\Xi}^{(1)}, \boldsymbol{\Xi}^{(1)}) \quad \Rightarrow \quad \boxed{\boldsymbol{\Xi}^{(2)} = 2|A|^2\hat{\boldsymbol{\xi}}_0^{(2)} + A^2(\hat{\boldsymbol{\xi}}_2^{(2)} e^{-2i\omega t} + \text{c.c.})}$$

- By substituting this $\boldsymbol{\Xi}$ into the Lagrangian,

$$L[\boldsymbol{\Xi}] = I\left|\frac{dA}{dt}\right|^2 - W_2|A|^2 - W_4\frac{|A|^4}{4} \quad \Rightarrow \quad \boxed{I\frac{d^2A}{dt^2} = -W_2 A - \frac{W_4}{2}A|A|^2}$$

where $I = \int \rho|\hat{\boldsymbol{\xi}}_1|^2 d^3x$ and $W_2 = W^{(2)}(\hat{\boldsymbol{\xi}}_1, \hat{\boldsymbol{\xi}}_1^*)$,

$$W_4 = W^{(3)}(\hat{\boldsymbol{\xi}}_1, \hat{\boldsymbol{\xi}}_1^*, \hat{\boldsymbol{\xi}}_0^{(2)}) + \text{Re}W^{(3)}(\hat{\boldsymbol{\xi}}_1, \hat{\boldsymbol{\xi}}_1, \hat{\boldsymbol{\xi}}_2^{(2)*}) + W^{(4)}(\hat{\boldsymbol{\xi}}_1, \hat{\boldsymbol{\xi}}_1, \hat{\boldsymbol{\xi}}_1^*, \hat{\boldsymbol{\xi}}_1^*)$$

4. Lagrangian approach to explosive instability

Strong nonlinearity of explosive instability

Fates of linear instabilities

- Saturation at small amplitude

  $\Rightarrow$ Weakly nonlinear problem; perturbation analysis is applicable.

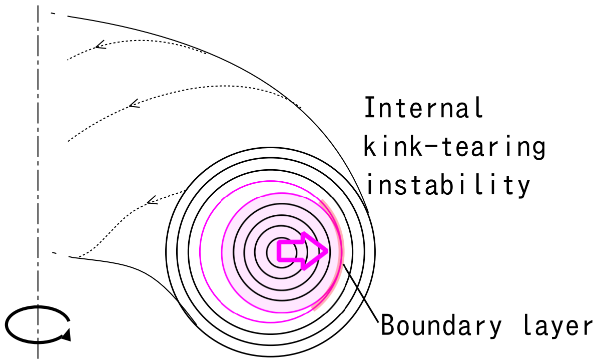
- Explosive growth (abrupt collapse)

  $\Rightarrow$ Strongly nonlinear problem; perturbation expansion fails to converge.

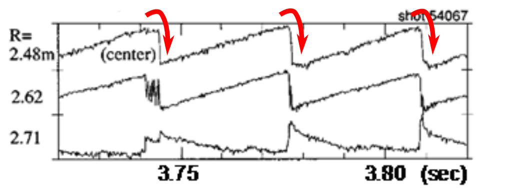
  which is often the case with boundary layer problem (singular perturbation)

Example. Collisionless magnetic reconnection

Magnetic confinement of plasma

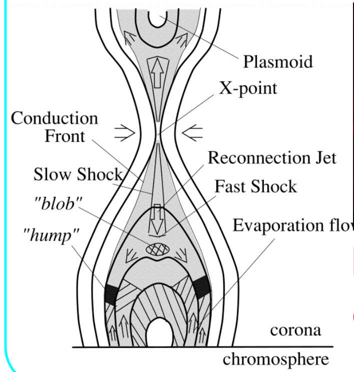


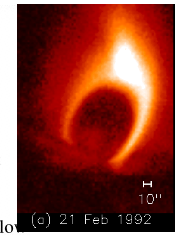
Sawtooth oscillation
[Goeler et al. 1974]



(fast collapses)

Solar flare
(from Yokoyama & Shibata ApJ 2001)





Explosive
outflow

- Boundary layer width (d) $\ll$ System size (L)
- Linearly unstable eigenfunction has a steep gradient within the thin layer; $\partial/\partial x \sim 1/d$
- Perturbation expantion will not converge when amplitude (ϵ) $\rightarrow$ layer width (d)

A model of collisionless magnetic reconnection

For $\boldsymbol{v} = \nabla\phi(x,y,t) \times \boldsymbol{e}_z$ and $\boldsymbol{B} = \nabla\psi(x,y,t) \times \boldsymbol{e}_z + B_0\boldsymbol{e}_z$,

Vorticity equation: $$\frac{\partial \nabla^2\phi}{\partial t} - [\phi, \nabla^2\phi] - [\nabla^2\psi, \psi] = 0, \quad (1)$$

(Collisionless) Ohm's law: $$\frac{\partial(\psi - d_e^2\nabla^2\psi)}{\partial t} - [\phi, \psi - d_e^2\nabla^2\psi] = 0, \quad (2)$$

where $[f,g] = \frac{\partial f}{\partial x}\frac{\partial g}{\partial y} - \frac{\partial g}{\partial x}\frac{\partial f}{\partial y}$, $d_e (\ll L)$: electron skin depth

The frozen-in flux is not ψ, but $\psi_e = \psi - d_e^2\nabla^2\psi$.

By introducing the flow map $(x,y)(t) = \varphi_t(x_0,y_0)$,

Lagrangian: $$L[\varphi_t] = \frac{1}{2}\int \left(|\nabla\phi|^2 - |\nabla\psi|^2 - d_e^2|\nabla^2\psi|^2\right) d^2x = K - W$$

This play the role of potential energy

where $\frac{\partial \varphi_t}{\partial t}(x_0,y_0) = \nabla\phi(\varphi_t(x_0,y_0),t) \times \boldsymbol{e}_z$ and $\psi_e(\varphi_t(x_0,y_0),t) = \psi_e(x_0,y_0,0)$

☞ If the potential energy decreases ($\delta W < 0$) for some displacement map, then such a displacement tends to grow with the release of free energy.

1D slab equilibrium

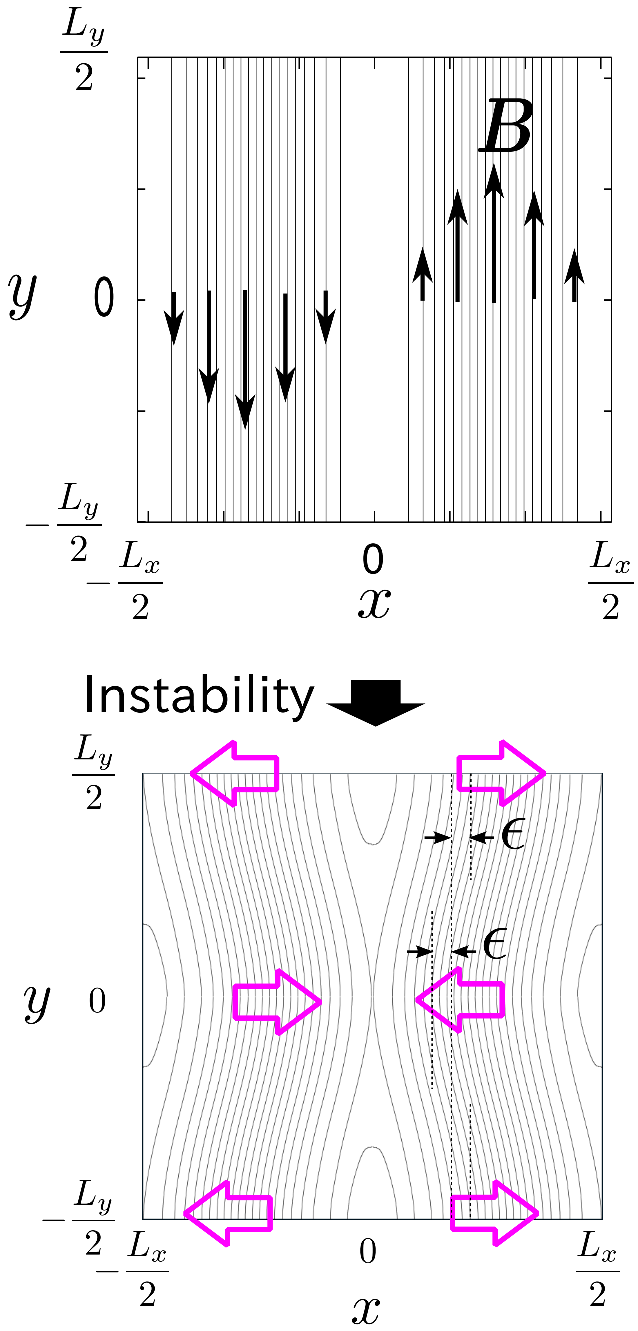
On a doubly-periodic box $D = [-\frac{L_x}{2}, \frac{L_x}{2}] \times [-\frac{L_y}{2}, \frac{L_y}{2}]$

$$\phi \equiv 0 \text{ (no flow)}, \quad \psi(x) = \psi_0 \cos \frac{2\pi x}{L_x}$$

- Assume sufficiently small wavenumber $k = 2\pi/L_y$ in the y-direction such that

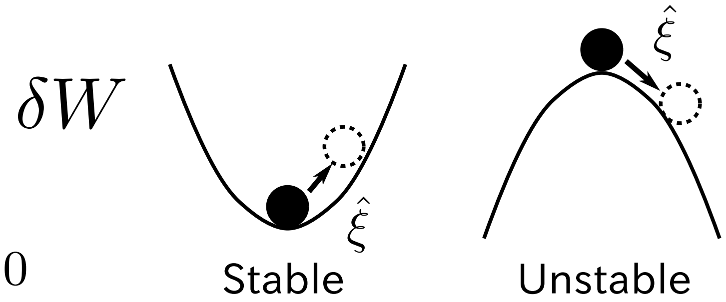
$$L_x^3/8L_y^2 \ll d_e \ll L_x.$$

- Define ϵ as maximum displacement in x direction ($\approx$ half width of magnetic island).



Energy principle for linear stability ($\epsilon \ll d_e$)

Eigenvalue probelm (4th order ODE) $\Leftrightarrow$ $-\gamma^2 \delta I = \delta W$

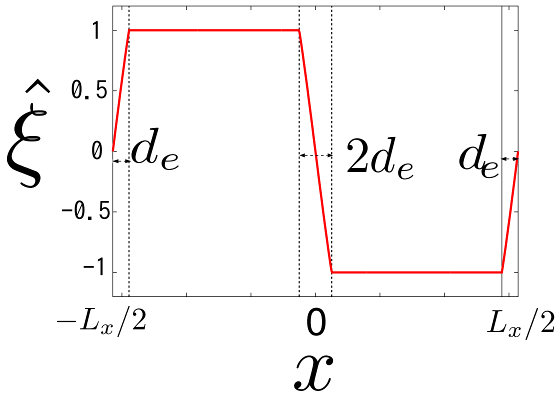


where
$$\delta I = \int dx \frac{1}{k^2}\left(|\hat{\xi}'|^2 + k^2|\hat{\xi}|^2\right) > 0$$
$$\delta W = \int dx \left[-(\psi_e'\hat{\xi}^*)\frac{\nabla^2}{1-d_e^2\nabla^2}(\psi_e'\hat{\xi}) + \psi_e'\psi'''|\hat{\xi}|^2 \right]$$

Energy principle (or Rayleigh-Ritz method)

The most unstable eigenvalue $\gamma > 0$ is found by minimizing $\dfrac{\delta W}{\delta I}$ with respect to $\hat{\xi}$.

By substituting the following test function $\hat{\xi}$,



$$-\gamma^2 = \frac{\delta W}{\delta I} \simeq -\frac{1+27e^{-2}}{6\tau_0^2} = -0.776/\tau_0^2$$

where $\tau_0^{-1} = d_e k B_{y0}'$

$\Rightarrow$ Linear growth rate: $\gamma = \sqrt{0.776}/\tau_0 = 0.881/\tau_0$

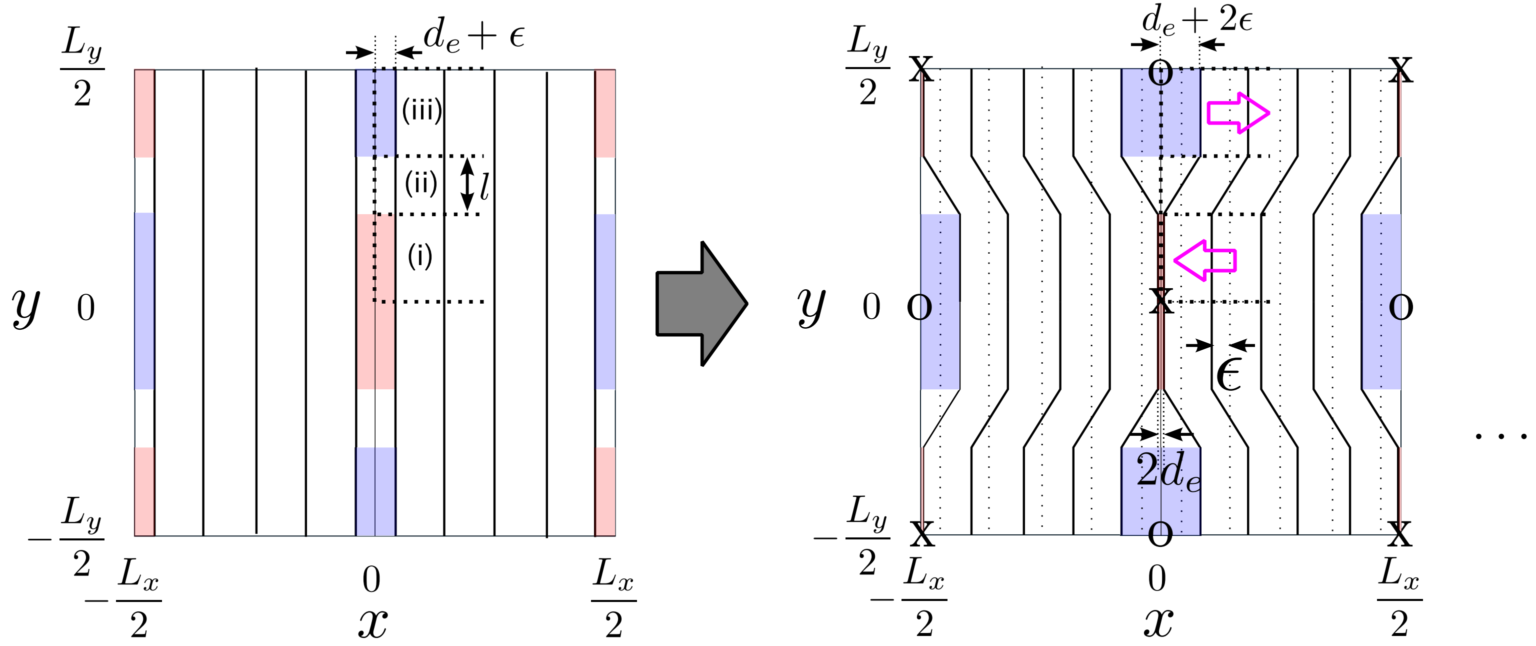
Nonlinear stability analysis ($\epsilon > d_e$)

We devise a displacement map $\varphi_\epsilon : (x_0, y_0) \mapsto (x, y)$ that tends to decrease the potential energy W as much as possible.

$$x = \begin{cases} g_\epsilon(x_0), & 0 < y_0 < \frac{L_y}{4} - \frac{l}{2}, & \text{(i)} \\ x_0 + \frac{2}{l}\left(y_0 - \frac{L_y}{4}\right)[x_0 - g_\epsilon(x_0)], & \frac{L_y}{4} - \frac{l}{2} < y_0 < \frac{L_y}{4} + \frac{l}{2}, & \text{(ii)} \\ 2x_0 - g_\epsilon(x_0), & \frac{L_y}{4} + \frac{l}{2} < y_0 < \frac{L_y}{2}, & \text{(iii)} \end{cases}$$

and

$$g_\epsilon(x_0) = \begin{cases} e^{-\hat{\epsilon}} x_0, & 0 < x_0 < d_e, \\ d_e e^{\frac{x_0 - \epsilon}{d_e} - 1}, & d_e < x_0 < d_e + \epsilon, \\ x_0 - \epsilon, & d_e + \epsilon < x_0. \end{cases}$$



$\cdots$ (Contours of ψ_e are deformed into a "Y-shape".)

Acceleration of collisionless reconnection

$$\mathrm{L}[\varphi_{\epsilon(t)}] \simeq L_y B_{y0}'^2 d_e^3 \hat{I} \left[\left(\frac{d\hat{\epsilon}}{d\hat{t}} \right)^2 - U(\hat{\epsilon}) \right] \quad \begin{pmatrix} \hat{\epsilon} = \epsilon / d_e, \\ \hat{t} = t/\tau_0 \end{pmatrix}$$

- In linear phase ($\hat{\epsilon} \ll 1$),

$U(\hat{\epsilon}) \simeq -0.776\hat{\epsilon}^2 \quad \Rightarrow$ Exponential growth

$\hat{\epsilon} \propto \exp(\sqrt{0.776}\hat{t})$

- In nonlinear phase ($\hat{\epsilon} \gg 1$),

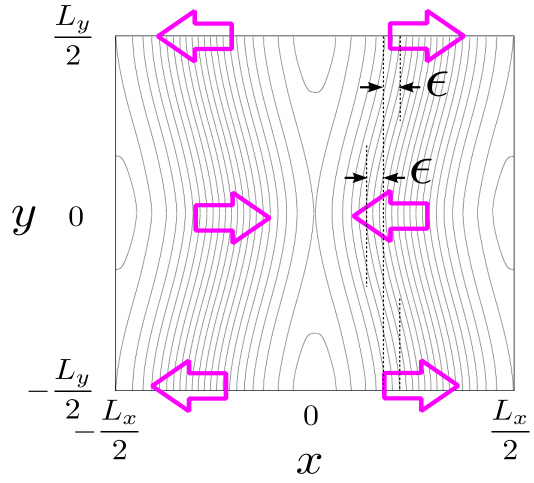
$U(\hat{\epsilon}) \simeq -0.439\hat{\epsilon}^3 \quad \Rightarrow$ Explosive growth

$\hat{\epsilon} \to \infty$ in $\Delta\hat{t} = 2 \sim 3$

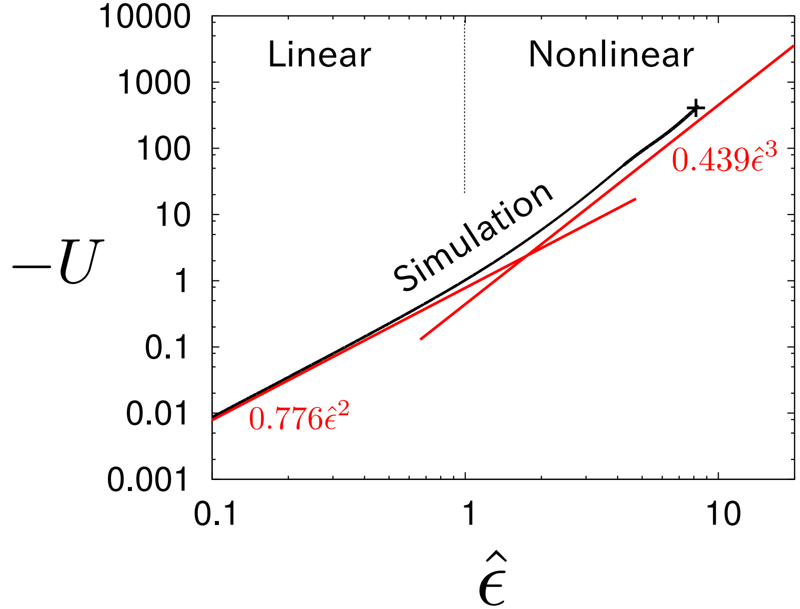
☞ Nonlinear force $F(\hat{\epsilon}) = -U'(\hat{\epsilon}) \sim \hat{\epsilon}^2$ obtained here is different from $F(\hat{\epsilon}) \sim \hat{\epsilon}^4$ in Ottaviani & Porcelli [PRL **71**, 3802 (1993)].

☞ Direct numerical simulation shows an agreement with our scaling (right figure).

Simulation with $\frac{d_e}{L_x} = 0.01, \ \frac{L_y}{L_x} = 4\pi$



Change of potential U



[Hirota, Morrison, Ishii, Yagi and Aiba, arXiv:1210.0630]