Additive noise does not destroy a pitchfork bifurcation

Mark Callaway, Doan Thai Son, Jeroen S.W. Lamb, Martin Rasmussen

Imperial College London

BIRS 2012, 6 November 2012
Contents

- What is a ”bifurcation” for a random dynamical system?
- Pitchfork bifurcation with additive noise has a unique attracting random fixed point [Crauel and Flandoli (1998)]
- Qualitative change in the attractivity.
- Lyapunov exponent and finite-time Lyapunov exponents.
- Dichotomy spectrum.
- Topological versus uniform topological equivalence.
Bifurcation

- Bifurcation ~ "Qualitative change in the dynamics."
- What signifies such a change and can we develop a mathematical theory (in analogy to deterministic setting)?
- History: [Arnold98] distinguishes Phenomenological (P) bifurcations, characterized by a change in the shape of the stationary measure, from Dynamic (D) bifurcations, characterized by a change in the Lyapunov exponent spectrum.
Case study.

\[ dx = (\alpha x - x^3)dt + \sigma dW_t \] with Wiener process \( W_t \)

- If \( \sigma = 0 \): classical pitchfork bifurcation with exchange of stability from \( x = 0 \) to branches \( x = \pm \sqrt{\alpha} \) when \( \alpha > 0 \).
- If \( \sigma \neq 0 \), a stationary distribution arises that changes shape when \( \alpha \) increases through 0. ([Arnold] “P-bifurcation”)
- [Crauel & Flandoli 1998] for all \( \alpha \)
  - Strictly negative Lyapunov exponent ([Arnold] no ”D-bifurcation”)
  - Unique attracting random fixed point: "Additive noise destroys a pitchfork bifurcation.”
SDE as Random Dynamical System

- $(\Omega, \mathcal{F}, \mathbb{P})$ probability space, with $\Omega = C_0(\mathbb{R}, \mathbb{R})$
- $\theta : \mathbb{R} \times \Omega \rightarrow \Omega$, $\theta_0 \omega = \omega$, $\theta_{t+s} \omega = \theta_t \theta_s \omega$.
- $\mathbb{P} \theta_t A = \mathbb{P} A$ (measure preserving)
- $\theta_t A = A \forall t \Rightarrow \mathbb{P} A \in \{0, 1\}$ (ergodicity)
- Skew product flow: $\Theta : \mathbb{R} \times \Omega \times \mathbb{R} \rightarrow \Omega \times \mathbb{R}$ with $\Theta_t(\omega, x) = (\theta_t \omega, \phi(t, \omega)x)$.
- Invariant probability measure $\mu$ on $(\Omega \times \mathbb{R}, \mathcal{F} \times \mathcal{B})$;
  - (i) $\mu(\Theta_t A) = \mu(A)$ and (ii) $\pi_{\Omega} \mu = \mathbb{P}$.
- Disintegration of $\mu$: $\exists \{\mu_\omega\}_{\omega \in \Omega}$ prob meas on $(\mathbb{R}, \mathcal{B})$ such that $\mu(A) = \int_{\Omega} \int_{\mathbb{R}} 1_A(\omega, x) d\mu_\omega(x) d\mathbb{P}(\omega)$. 

Callaway, Doan, Lamb, Rasmussen

Additive noise does not destroy a pitchfork bifurcation
Random pitchfork analysis.

\[ dx = (\alpha x - x^3)dt + \sigma dW_t \]

Arnold98, CF98: SDE has unique stationary measure
\[ \rho(B) = \int_{\mathbb{R}} T(x, B)d\rho(x) \quad \forall B \in \mathcal{B}(\mathbb{R}) \]
(where \( T(x, B) \) denotes the transition probability of the induced Markov semi-group) with density \( p_{\alpha,\sigma}(x) = N_{\alpha,\sigma} \exp\left(\frac{1}{\sigma^2}(\alpha x^2 - \frac{1}{4}x^4)\right) \) corresponding to global random attractor \( \{a_{\alpha}(\omega)\}_{\omega \in \Omega} \) with invariant measure \( \mu \) of the RDS with disintegration \( \mu_\omega = \lim_{t \to \infty} \phi(t, \theta_{-t}\omega)\rho = \delta_{a_{\alpha}(\omega)} \)
(random Dirac measure, i.e. random fixed point) and Lyapunov exponent \[ \lambda(\mu) = -\frac{2}{\sigma^2} \int_{\mathbb{R}} (\alpha x - x^3)^2 p_{\alpha,\sigma}(x)dx < 1. \]

\[ ^{1} \text{Lyapunov exponent: } \lambda = \lim_{t \to \pm \infty} \ln \|\Phi(t, \omega)x\|. \]

Callaway, Doan, Lamb, Rasmussen
Additive noise does not destroy a pitchfork bifurcation
Qualitative change in the attractivity.

- \( \{a_\alpha(\omega)\}_{\omega \in \Omega} \) is locally uniformly attractive if \( \exists \delta > 0 \) such that

\[
\lim_{t \to \infty} \sup_{x \in (-\delta, \delta)} \text{ess sup}_{\omega \in \Omega} |\phi(t, \omega)(a_\alpha(\omega) - x) - a_\alpha(\omega))| = 0
\]

- **Theorem:** (i) If \( \alpha < 0 \), the random attractor \( \{a_\alpha(\omega)\}_{\omega \in \Omega} \) is locally uniformly attractive (even globally uniformly exponential attractive),
(ii) if \( \alpha > 0 \), this is no longer the case.

- In fact \( |\phi(t, \omega)(a_\alpha(\omega) - x) - a_\alpha(\omega)| \leq K(\omega) \exp(-\lambda t)x \), where \( K(\omega) < \hat{K} < \infty \) iff \( \alpha < 0 \).
Finite-time Lyapunov exponents.

- $\lambda_\alpha(T, \omega) := \frac{1}{T} \ln \left| \frac{\partial \phi_\alpha}{\partial x} (T, \omega, a_\alpha(\omega)) \right|$. (random variable!)

- Lyapunov exponent is $\lambda_\alpha := \lim_{T \to \infty} \lambda_\alpha(T, \omega)$.

- **Theorem:** (i) If $\alpha < 0$, the random attractor is **finite-time attractive**: $\lambda_\alpha(\omega) \leq \alpha < 0$. (ii) If $\alpha < 0$, the random attractor is **not** finite-time attractive and $\mathbb{P}\{\omega \in \Omega : \lambda_\alpha(T, \omega) > 0\} > 0$.

- **Corollary:** The (negative) Lyapunov exponent can only be observed ”almost surely” in finite time, if $\alpha < 0$. 

Additive noise does not destroy a pitchfork bifurcation
Lyapunov spectrum

- Linear RDS in $\mathbb{R}^N$:
  \[
  \phi(t, \omega)(ax_1 + bx_2) = a\phi(t, \omega)x_1 + b\phi(t, \omega)x_2.
  \]
  Denoted as $\Phi : \mathbb{R} \times \Omega \to \mathbb{R}^{N \times N}$.

- Osceledets: (under mild assumptions) $\exists k$ Lyapunov exponents $\lambda_1 < \lambda_2 < \ldots < \lambda_k$ and $\mathbb{R}^N = W_1(\omega) \oplus \ldots W_k(\omega)$ so that
  \[
  \lambda_i := \lim_{t \to \pm \infty} \frac{1}{t} \ln \|\Phi(t, \omega)\| \text{ for } 0 \neq x \in W_i(\omega).
  \]

- But we have just seen that ”bifurcation” is not necessarily associated with a change of stability in the Lyapunov spectrum.

- We claim that a better concept for this purpose is the Dichotomy spectrum.
Dichotomy spectrum

- Definition: \((\theta, \Phi)\) has an exponential dichotomy wrt growth rate \(\gamma \in \mathbb{R}\) if there exists a splitting \(\mathbb{R}^N = S(\omega) \oplus U(\omega)\), measurable and invariant \((\Phi(t,\omega)S(\omega) = S(\theta_t\omega), \text{ etc})\), satisfying for some \(K, \varepsilon > 0\)
  \[||\Phi(t,\omega)x|| \leq Ke^{(\gamma-\varepsilon)t}||x||, \text{ for all } t \geq 0, x \in S(\omega).\]
  \[||\Phi(t,\omega)x|| \geq K^{-1}e^{(\gamma+\varepsilon)t}||x||, \text{ for all } t \geq 0, x \in U(\omega).\]
- Dichotomy spectrum \(\Sigma := \mathbb{R} \setminus \bigcup \text{growth rates } \gamma \{\gamma\}\).
- **Spectral Theorem:** \(\Sigma = I_1 \cup \ldots \cup I_k\) with \(I_i = \{W_i(\omega)\}_{\omega \in \Omega}\) and corresponding decomposition \(\mathbb{R}^N = W_1(\omega) \oplus \ldots \oplus W_k(\omega)\).
- In the pitchfork example, \(\Sigma = (-\infty, \alpha]\), so that the random pitchfork bifurcation corresponds to a loss of hyperbolicity of the Dichotomy spectrum.
Topological versus uniform topological equivalence.

- RDSs $\phi_1(t, \omega)$ and $\phi_2(t, \omega)$ are **topologically conjugate** iff $\exists$ homeomorphism $h : \Omega \times \mathbb{R} \rightarrow \mathbb{R}$ so that for all $\omega \in \Omega$, $\phi_2(t, \omega)h(\omega, x) = h(\theta t\omega, \phi_1(t, \omega)x)$ for all $t, x$.

- **Theorem:** For the pitchfork example all $\phi_\alpha$ are topologically equivalent.

- **Theorem:** A topological conjugacy $h$ from $\phi_\alpha$ to $\phi_{\alpha'}$ with $\text{sgn}(\alpha) = -\text{sgn}(\alpha')$ cannot be uniformly continuous. 
  *Proof:* uniformly continuous conjugacies preserve local uniform attractivity.
Main result and some questions:

- Additive noise does **not** destroy a random pitchfork bifurcation. (cf [CF98])

- Is a change in the signature of the Dichotomy Spectrum a good indicator for bifurcation of RDS?

- Is **uniform topological equivalence** a suitable equivalence relation to define the notion of bifurcation in RDS?
References


CDLR12 Mark Callaway, Doan Thai Son, Jeroen S.W. Lamb and Martin Rasmussen. The dichotomy spectrum for random dynamical systems and pitchfork bifurcations with additive noise. Preprint (2012).


http://www.ma.imperial.ac.uk/DynamIC
mailto:jeroen.lamb@imperial.ac.uk