The continuous spectrum in the Moore–Saffman–Tsai–Widnall instability

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Introduction

Vortex rings

Kazuhiro Nogi/Agence France-Presse – Getty Images.

Beluga whales in Hamada blowing bubble rings at an aquarium.

Can do experiments with a cigar and a cardboard box. Or using more sophisticated apparatus. . .

Or even more sophisticated apparatus. . . Stills from T. T. Lim's videos at (`http://serve.me.nus.edu.sg/limtt/`): colliding and leapfrogging rings.

Evident azimuthal instability. This has been looked at since the 1970s. Full equations are messy.

One approach (Widnall, Bliss & Tsai 1974) is to argue that for a thin vortex ring, locally the vortex looks like a vortex filament and locally the effect of the rest of the ring is a strain.

Hence examine the instability of a vortex in a strain field.

Other approaches are also possible, e.g. short-wavelength instability analysis (Hattori & Fukumoto).

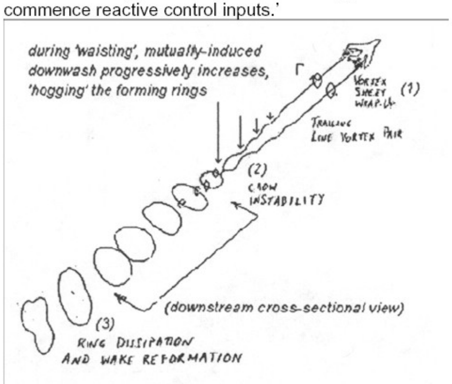
Wakes

Contrails left by aircraft are obvious in sky. Velocity perturbations induced by wakes can be dangerous for (smaller) following aircraft. Important in airport management and safety. Breakdown of wake due to instability. Instability of vortex filament (including non-zero core size) in Saffman (1989; Chapter 10).

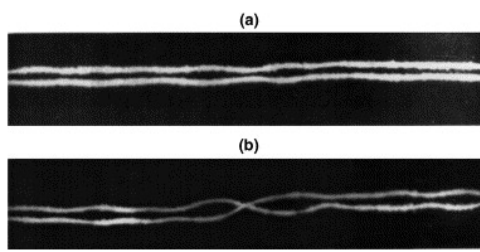
Crow instability Long wave cooperative instability acting on parallel vortex filaments.

The Crow Instability Process

'An instant flurry would likely be the GO trigger to commence reactive control inputs.'

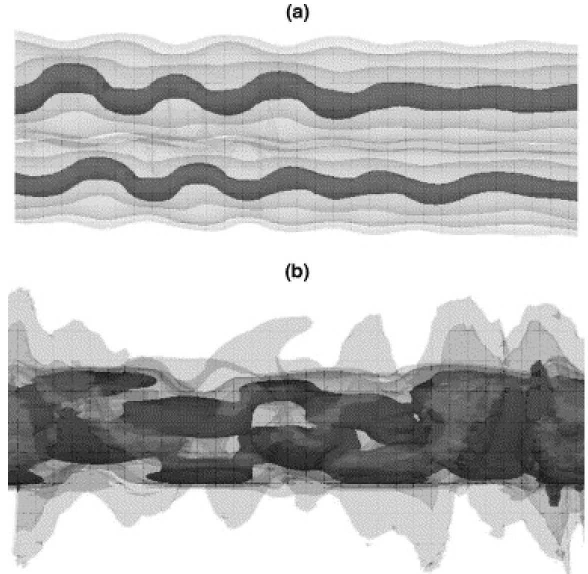


A trailing pair of wake vortices can merge into a series of vortex segments inclined more or less vertically. Hence, upon climb-out, a following aircraft at two-minute takeoff separation could encounter a preceding wake which is not a stable vortex pair, but which is in a state of breakdown or transition. *Source: Brown, in NTSB Docket No. SA-522, Exhibit 2-X, Aug. 2002*



Left: from analysis of Flight 587 crash. Center: image from Crow (1970).

Short wave cooperative instability Bending modes of a vortex column may be unstable in a straining field. Focus of this talk.



Iso-levels of the axial vorticity perturbation component in a direct numerical simulation of a co-rotating dipole of aspect ratio $a/b = 0.2$ for Re $= 5000$: (a) linear regime, (b) merger (Le Dizès and Laporte 2002, cited in Jacquin 2005).

Ultra short wave cooperative instability Elliptical instability. Generic mechanism for instability of flow with elliptical streamlines (Pierrehumbert 1986; Bayly 1986 and many others).

The Tsai–Widnall–Moore–Saffman instability

Fluid mechanics in one slide

Incompressible Euler equations:

$$\frac{\mathrm{D}\boldsymbol{u}}{\mathrm{D}t} = -\frac{1}{\rho}\boldsymbol{\nabla} p, \qquad \boldsymbol{\nabla}\cdot\boldsymbol{u} = 0.$$

These are basically Newton's law and mass conservation for a fluid particle. ρ is density, p is pressure, $\boldsymbol{u}$ is velocity. These are million dollar equations.

The vorticity is $\boldsymbol{\omega} = \boldsymbol{\nabla}\times\boldsymbol{u}$ and satisfies

$$\frac{\mathrm{D}\boldsymbol{\omega}}{\mathrm{D}t} = \boldsymbol{\omega}\cdot\boldsymbol{\nabla}\boldsymbol{u}.$$

The right-hand terms is vortex stretching.

A vortex jump is a step in vorticity. A vortex sheet is a delta function, corresponding to a discontinuity in velocity. Unstable to Kelvin–Helmholtz instability.

Linearized equations

Euler equations for incompressible, inviscid flow linearized about 2D radial basic state with streamfunction $\Psi(r)$ and vertical velocity $W(r)$. Take azimuthal dependence $\mathrm{e}^{\mathrm{i}m\theta}$ and vertical dependence $\mathrm{e}^{\mathrm{i}k_0 z}$:

$$\begin{aligned} u_t + \mathrm{i}m\Omega u + \mathrm{i}k_0 W u - 2\Omega v &= -p_r, \\ v_t + \mathrm{i}m\Omega v + \mathrm{i}k_0 W v + \frac{Z}{r^2}u &= -\frac{\mathrm{i}m}{r}p, \\ w_t + \mathrm{i}m\Omega w + \mathrm{i}k_0 W w &= \mathrm{i}k_0 p, \\ u' + \frac{u}{r} + \mathrm{i}mv + \mathrm{i}k_0 w &= 0. \end{aligned}$$

Here $\Omega(r) = r^{-1}\Psi'(r)$ is the angular velocity and $Z(r) = r^{-1}(r^2\Omega)'$ is the vorticity.

Two-dimensional linearized equations can be written in streamfunction-vorticity form

$$\zeta_t + \mathrm{i}m\Omega\zeta - \frac{\mathrm{i}m}{r}Z'\psi = 0$$

with

$$\zeta = L_m\psi = \frac{\partial^2\psi}{\partial r^2} + \frac{1}{r}\frac{\partial\psi}{\partial r} - \frac{m^2}{r^2}\psi.$$

Strain

Infinitesimal two-dimensional strain at $O(\delta)$. Have to solve numerically in general.

Kelvin modes

The normal modes of a vortex are called Kelvin modes (sometimes expression used only for waves on edge of a vorticity jump).

Obtained from radial Rayleigh equation. Rayleigh's theorem in 2D geometry: if Z' does not change sign, the flow is stable (3D case is more complicated).

Kelvin modes can be found explicitly for the Rankine vortex and are discrete normal modes.

For smooth vorticity distributions, there are no discrete modes for $k_0 = 0$. Continuous spectrum exists. In general, obtain regular netrual modes and Landau damping (see talk by Stéphane Le Dizès). Will not discuss WKBJ analysis.

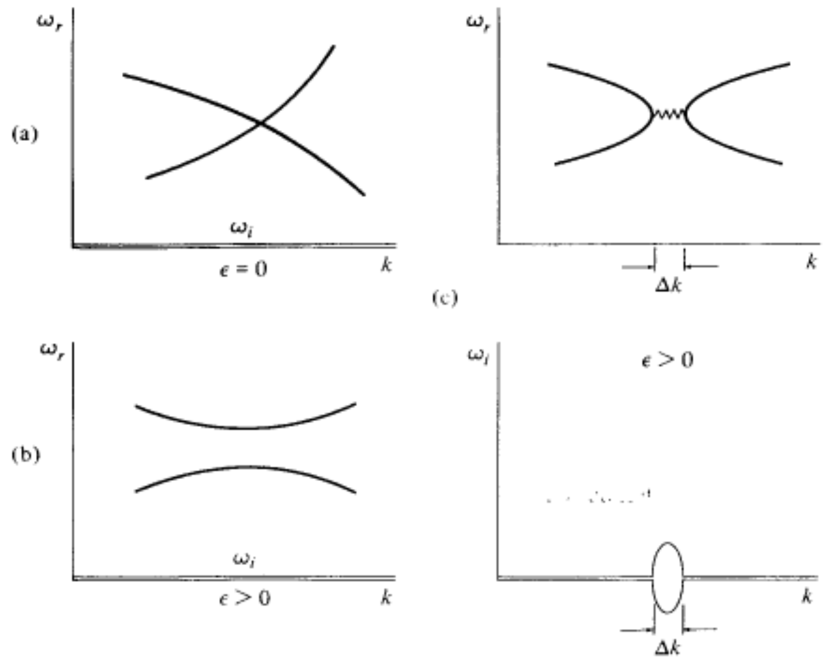
General analysis

This is in Moore & Saffman (1975). Solution expanded as a double series in strain perturbation. Required a discrete spectrum for the Kelvin modes.

Eventually a solvability condition is obtained in the case of degeneracy, i.e. equal frequencies. To find numbers, matrix elements must be computed, as was done by Tsai & Widnall (1976). In the absence of axial flow, instability can be shown without computing any matrix elements:

$$\omega_1^2 = \nu^2 Q^2 - R^2,$$

where ω_1 is the correction to the frequency, ν measures the departure from the resonant wavenumber, and P and Q are real numbers that depend on the basic state.



12.3-1 Instability associated with eigenvalue crossing. (a) Unperturbed. (b) Avoidance. (c) Instability.

Explicit results for the Rankine vortex (Fukumoto 2003)

Fukumoto (2003) was able to compute all the matrix elements exactly and correct mistakes in previous work.

How can one learn more about this problem? Rankine vortices are special, so one would like to examine smooth vorticity profiles, which do not have discrete normal modes in 2D. What happens? What happens in the initial-value problem? What about the infinitesimal strain assumption?

Interested in case when vortex is stable in the absence of strain, so not unstable modes, unlike Stèphane's talk.

Non-infinitesimal strain

Moore & Saffman (1975) explicitly discusses the case of non-infinitesimal strain: An exact solution of this problem for general F is not be hoped for, though an exact solution is known when ω is constant inside a finite core and zero elsewhere (Moore & Saffman 1971) and when the core is stagnant, or hollow (M. Hill, unpublished).

Robinson & Saffman (1984) studied the three-dimensional instability of a steady elliptical vortex in strain (a Moore–Saffman vortex). Note that Moore & Saffman had originally studied the two-dimensional instability. Miyazaki, Imai & Fukumoto (1994) looked at the 3D instability of Kirchhoff's elliptical vortex which rotates rigidly about a vertical axis. In both cases, the basic state vorticity is an elliptical vortex path, so elliptical coordinates are used.

What about other solutions with finite strain? How does one study the stability if they're not elliptic?

Hollow vortices

What is a hollow vortex? Fluid inside the vortex is stagnant in moving reference frame. If the vortex translates at a constant speed, pressure inside is constant. Boundary condition can be written as velocity of the fluid on the boundary is constant. If the vortex rotates (no known results), pressure field inside should be consistent with solid body rotation.

Few known hollow vortices:

- Pocklington (1894). Compressible extension by Pullin & Moore, Heister *et al.* (1990) and Leppington (2006).
- Hill (1975): hollow vortex in strain. But wait...
- Baker, Saffman & Sheffield (1976): array of hollow vortices. Compressible extension by Ardalon, Meiron & Pullin (1995).
- Crowdy and Roenby: hollow vortex surrounded by point vortices.
- Green and Crowdy: double vortex street.

Why? Always useful to find exact solutions to Euler equations. Also have advantage over vortex patches of involving a thermodynamic quantity, so useful as starting point for compressible vortices.

Here Basic state for TWMS is Hill's hollow vortex in strain.

Hill’s Hollow Vortex

Not to be confused with Hill’s Spherical Vortex. First reference in Baker, Saffman & Sheffield (1976); subsequently in papers by Baker and Pullin. Thesis never finished and work never published. I eventually found Mary Hill, who gave up science many years ago.

Djview – Saffman P.G. Vortex dynamics (CUP, 1992)(T)(322s).djvu

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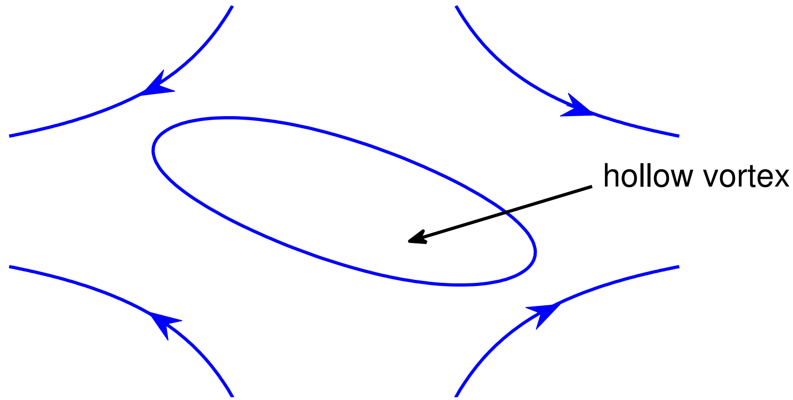
Problem statement

Turns out to be a very interesting problem. What is the shape of a hollow vortex in a strain field with potential γz^n at infinity?

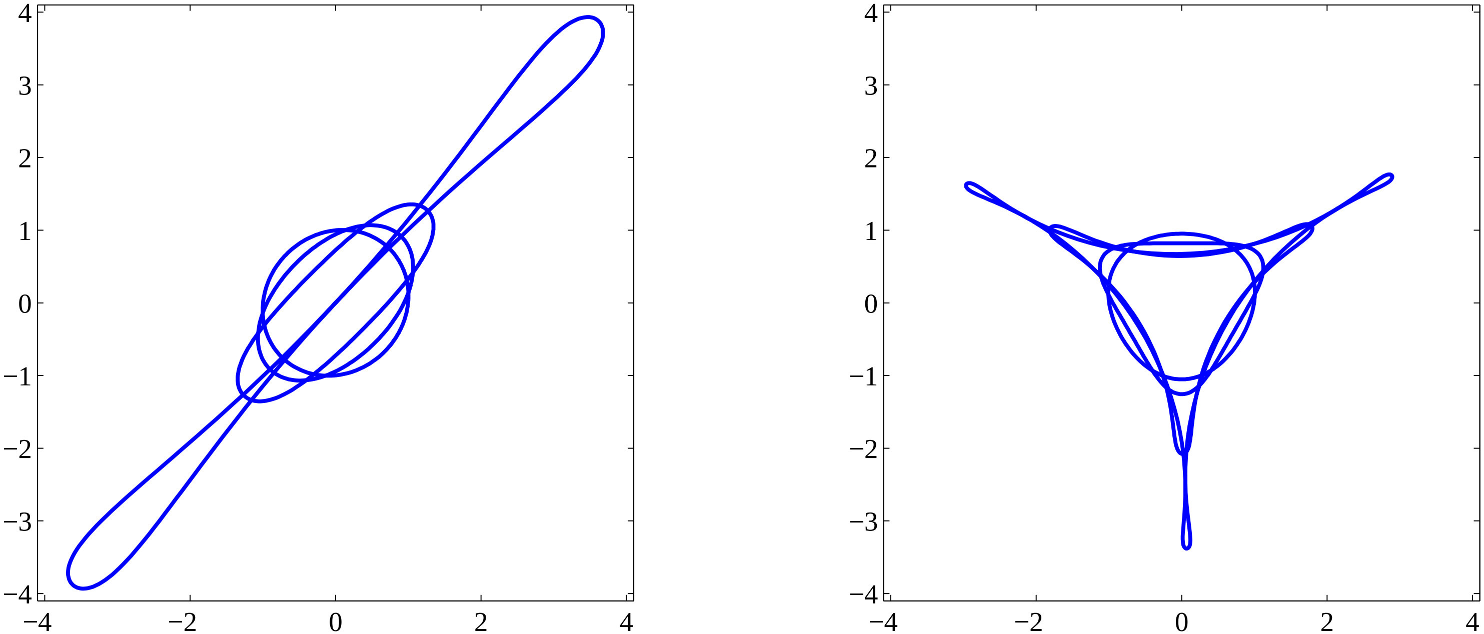


Free boundary value problem. Solve by constructing conformal map (classical technique, cf. Birkhoff & Zarantanello). Map inside of unit circle $|\zeta| = 1$ to outside of vortex, with $z \sim a\zeta^{-1}$ for large $|z|$.

Find

$$z(\zeta) = a\left[\frac{1}{\zeta} - \frac{2\mathrm{i}\beta}{(n-1)}\zeta^{n-1} + \frac{\beta^2\zeta^{2n-1}}{(2n-1)}\right],$$

where μ is related to the strain and must be smaller than μ_c for univalent solutions.



Hollow vortex shapes for $n = 2$ with $\mu = 0.05$, 0.245, 0.5 and $\mu = \mu_c^{(2)}$ (left) and $n = 3$ with $\mu = 0.1$, 0.4, 0.8 and $\mu = \mu_c^{(3)}$ (right). Each vortex has area π.

Stability

Results for $\gamma = 0$ vortex can be found analytically. In non-dimensional form

$$\sigma_m^{\pm} = \mathrm{i}(m \pm \sqrt{|m|}), \quad m \neq 0,$$

Audience participation question: the hollow vortex has a vortex sheet on the boundary, so why no Kelvin–Helmholtz instability?

There are modes sharing eigenvalues:

$$\sigma_1^- = \sigma_{-1}^+ = 0, \qquad \sigma_1^+ = \sigma_4^- = 2\mathrm{i}, \qquad \sigma_{-1}^- = \sigma_{-4}^+ = -2\mathrm{i}.$$

This suggests the possibility of resonance between modes with common eigenfrequencies.

For $n = 2$, the configuration is always unstable to a mode with growth rate $\omega = 2\gamma$ for small γ: this corresponds to the instability associated with a point vortex situated at the stagnation point of a linear straining flow.

In contrast, $n = 3$ and higher modes are linearly stable. Instability would be a finite-area effect.

Formulation

BSS derive linearized equations to describe the stability of their basic state working in the potential plane: $W = \phi + \mathrm{i}\psi$ is the independent variable. The perturbation velocity potential Φ is a harmonic function in $\psi < 0$ decaying as $\psi \to -\infty$. In these coordinates, the dynamic and kinematic boundary conditions are

$$\frac{1}{q_0^2}\frac{\partial \delta}{\partial t} + \frac{\partial \delta}{\partial \phi} = \frac{\partial \Phi}{\partial \psi}, \qquad \frac{1}{q_0^2}\frac{\partial \Phi}{\partial t} + \frac{\partial \Phi}{\partial \phi} + \left(\frac{\partial}{\partial \psi}\frac{1}{2}\frac{q^2}{q_0^2}\right)_{\psi=0} \delta = 0.$$

Now work in the ζ-plane. Equations on boundary become

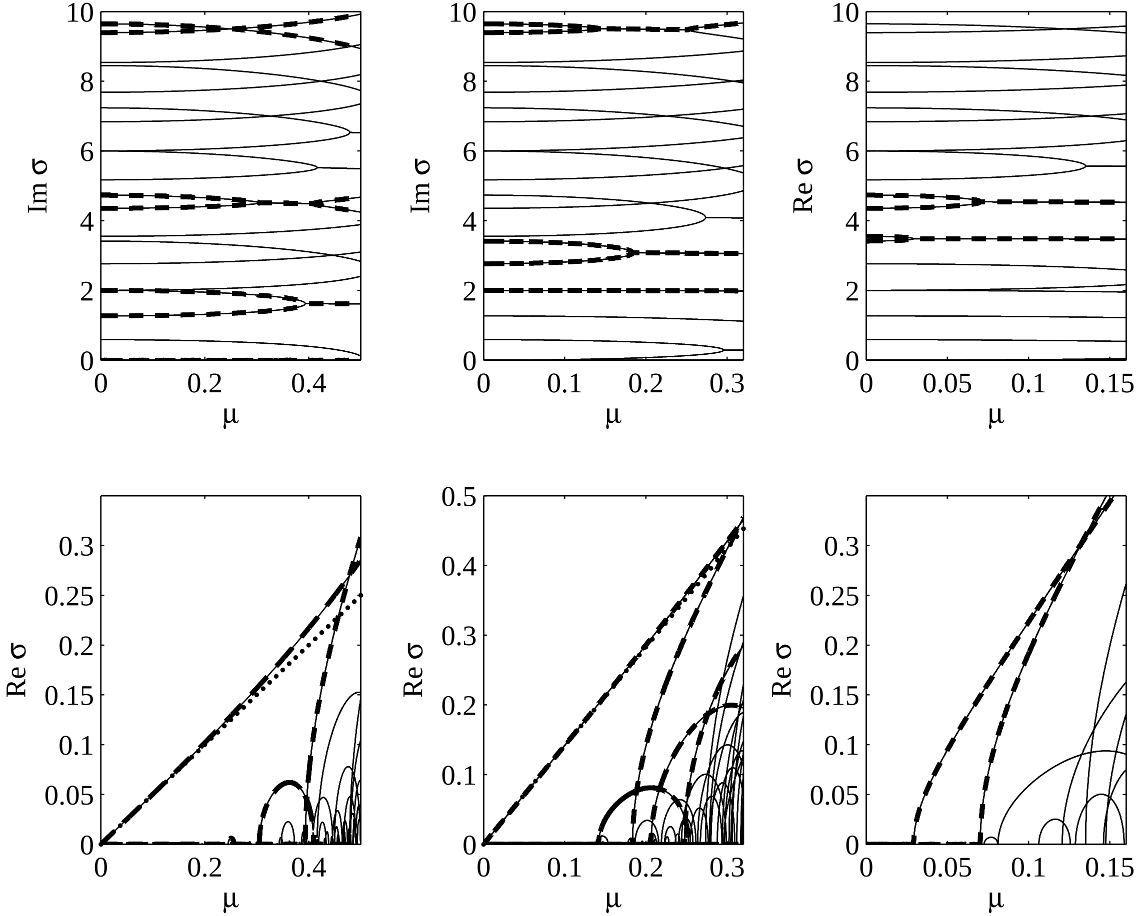
$$\frac{1}{q_0^2}\frac{\partial \delta}{\partial t} + \frac{1}{\phi_\theta}\frac{\partial \delta}{\partial \theta} = \frac{1}{\psi_\rho}\frac{\partial \Phi}{\partial \rho}, \qquad \frac{1}{q_0^2}\frac{\partial \Phi}{\partial t} + \frac{1}{\phi_\theta}\frac{\partial \Phi}{\partial \theta} + \left(\frac{1}{\psi_\rho}\frac{\partial}{\partial \rho}\frac{1}{2}\frac{q^2}{q_0^2}\right)_{\rho=\rho_0} \delta = 0,$$

Finally

$$\sigma\Phi + Q\frac{\partial \Phi}{\partial \theta} = G\delta, \qquad \sigma\delta + Q\frac{\partial \delta}{\partial \theta} = -Q\frac{\partial \Phi}{\partial \rho},$$

where $\sigma = 2\pi\lambda a^2/q_0\Gamma$ is the non-dimensional growth rate, and where Q and G are known functions.

Results



Imaginary and real parts of σ for the vortex in strain with $n = 2, 3, 4$.

3D instability

Only vertical dependence is in wavenumber of perturbation. Take vertical dependence $\mathrm{e}^{\mathrm{i}k_0 z}$ and solve

$$\nabla_h^2 \phi - k_0^2 \phi = 0.$$

Can still use the the mapping from z to ζ and

$$\nabla_h^2 = 4\frac{\partial^2}{\partial z \partial \bar{z}} = 4\left|\frac{\mathrm{d}\zeta}{\mathrm{d}z}\right|^2 \frac{\partial^2}{\partial \zeta \partial \bar{\zeta}},$$

to obtain

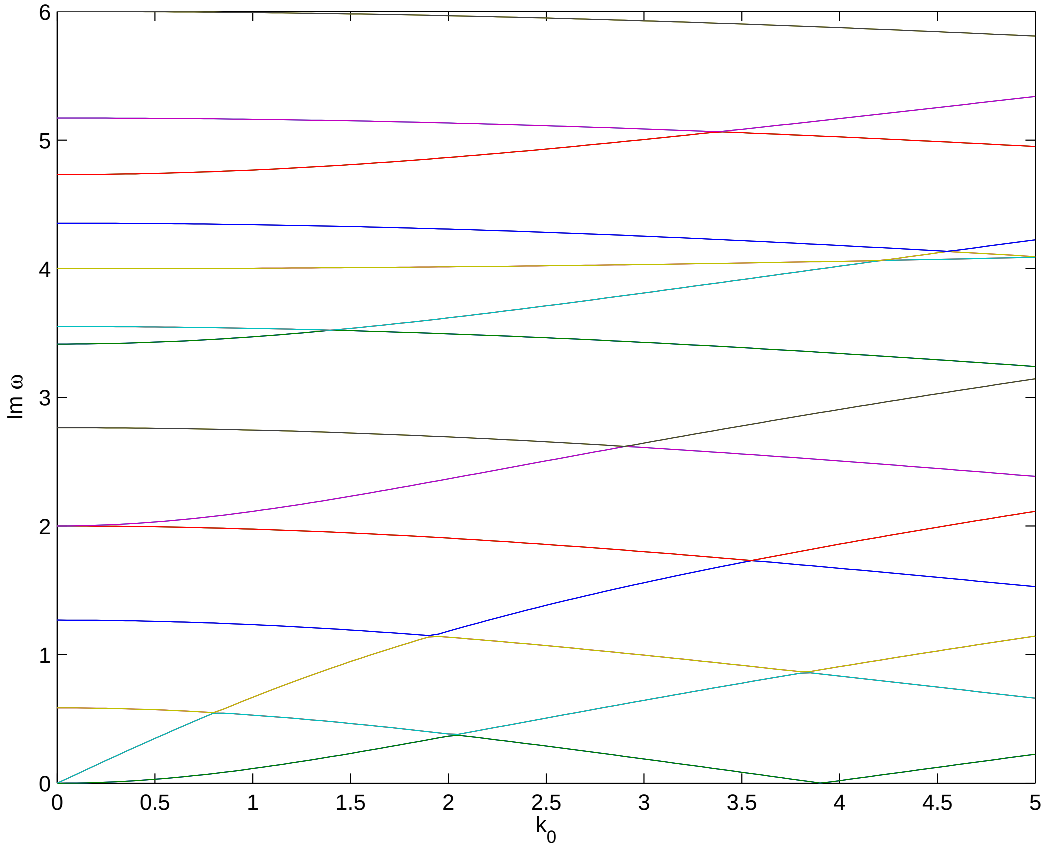
$$\nabla_\zeta^2 \phi - k_0^2 |z'|^2 \phi = 0,$$

Unlike the 2D case, the different azimuthal modes are coupled. Need to solve numerically.

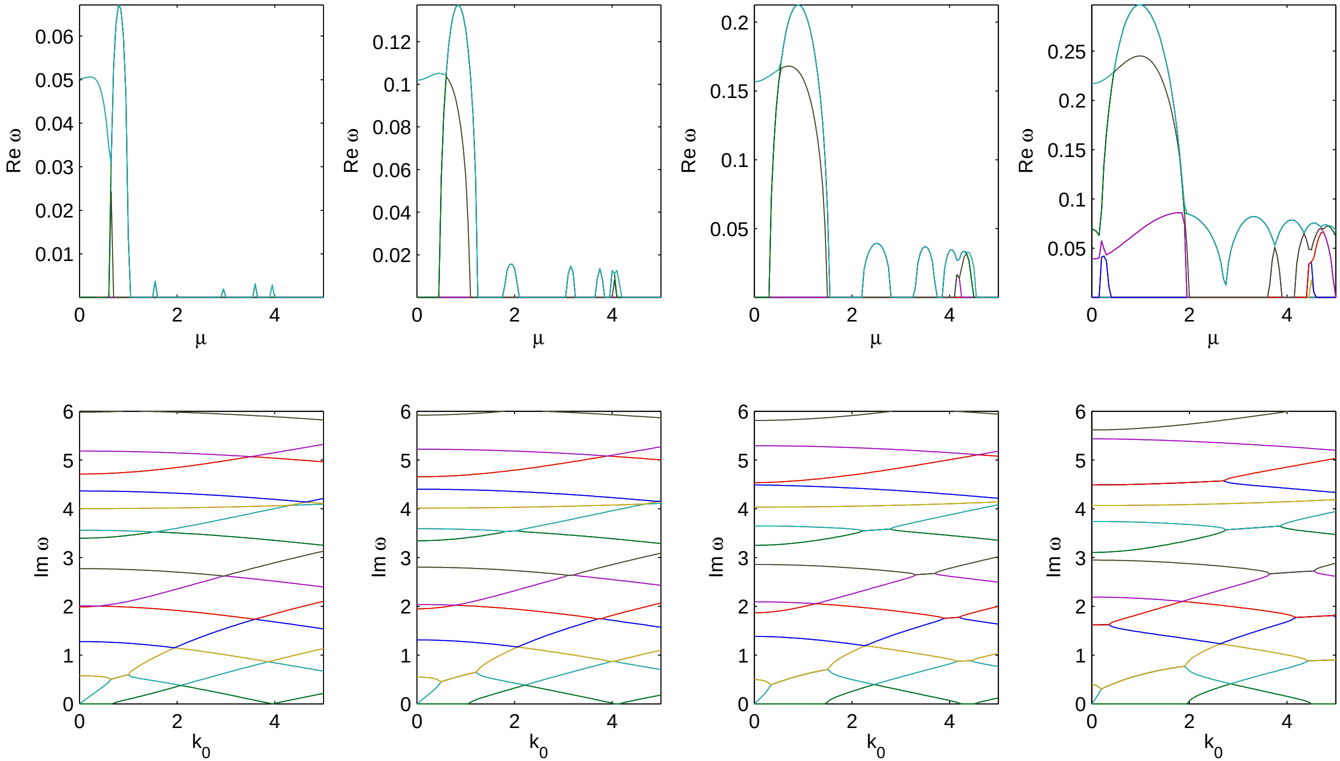
Can add a vertical velocity W. W cannot depend on the vertical coordinate and the vertical momentum equation gives $W = W(\psi)$. The boundary conditions become

$$\frac{1}{q_0^2}\frac{\partial \delta}{\partial t} + \left(\frac{\partial}{\partial \phi} + \frac{\mathrm{i}k_0 W_0}{q_0^2}\right)\delta = \frac{\partial \Phi}{\partial \psi}, \qquad \frac{\partial \Phi}{\partial t} + \left(\frac{\partial}{\partial \phi} + \frac{\mathrm{i}k_0 W_0}{q_0^2}\right)\Phi + \left(\frac{\partial}{\partial \psi}\frac{1}{2}\frac{q^2 + W^2}{q_0^2}\right)_{\psi=0}\delta = 0,$$

where W_0 is the (constant) value of W on the boundary of the vortex (a streamline of the basic state). If there is no axial flow, the boundary conditions are unchanged.



Frequencies for vortex column, i.e. $\mu = 0$. Can find these frequencies from an explicit dispersion relation. Different from usual figure since not Rankine vortex.



Frequencies for vortex column with $\mu = 0.1$, 0.2, 0.3, 0.4.

The initial-value problem

Use the Laplace transform. The strain correction is steady. The terms forced by the coupling between strain and Kelvin modes satisfy a forced linear equation in which the forcing can be obtained as a Laplace transform.

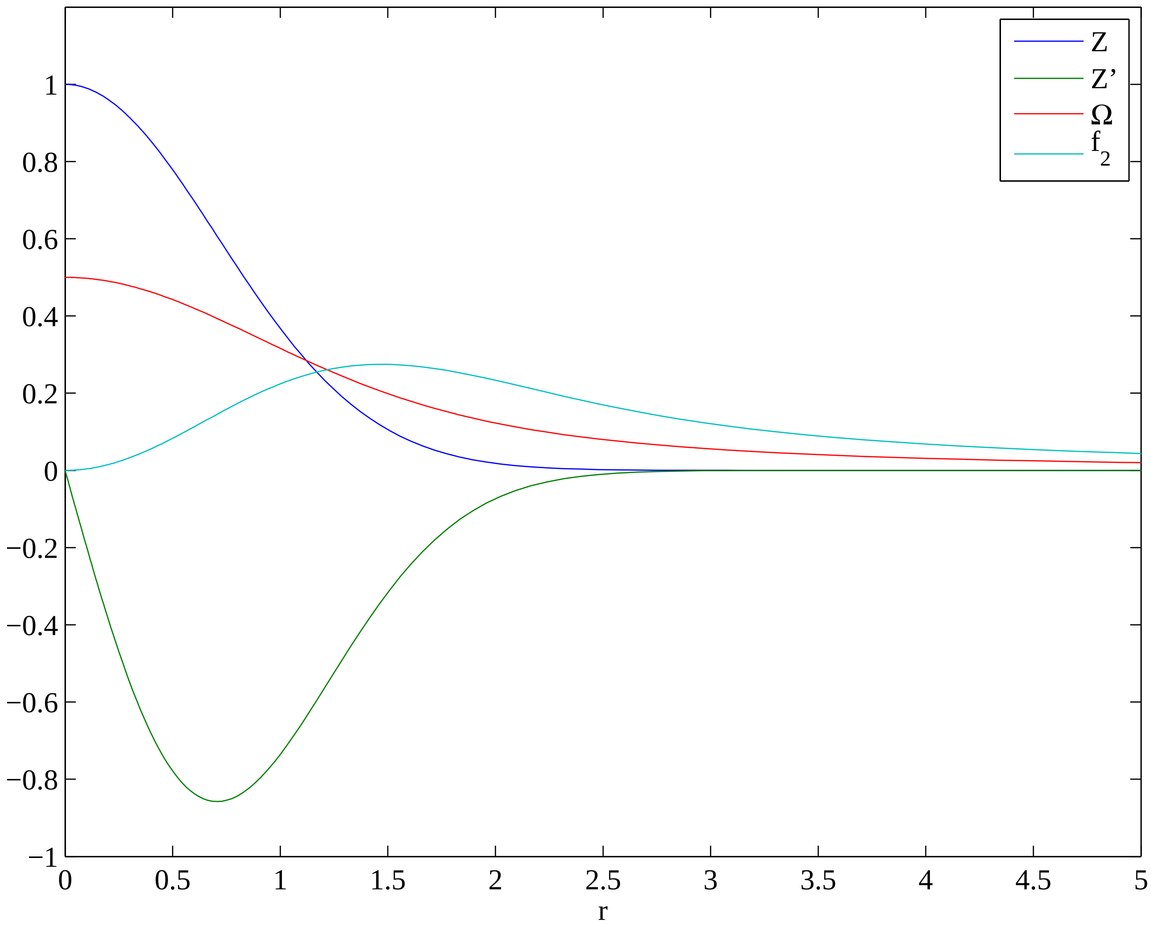
Critical issue becomes the initial condition for the Kelvin mode. The initial condition for the forced response can be taken to be zero.

Strain correction

Look for solution $\psi = (r^2 + f_s)\mathrm{e}^{2\mathrm{i}\theta} + c.c.$ (note r^2 is irrotational). Then f_s satisfies

$$\Omega L_2 f_s - \frac{Z'}{r} f_s = r Z'.$$

Solve numerically.



Correction to far-field strain for Gaussian vortex.

The 2D problem

Kelvin modes

Solution is $\psi_K = \sum_m \psi_{K,m} \mathrm{e}^{\mathrm{i}m\theta} + c.c.$, so $\psi_{K,-m} = \psi^*_{K,m}$. Modes satisfy

$$\frac{\partial}{\partial t} L_m \psi_{K,m} + \mathrm{i}m\Omega L_m \psi_{K,m} - \frac{\mathrm{i}m}{r} Z' \psi_{K,m} = 0.$$

Normal mode solutions correspond to setting $\partial_t = \mathrm{i}\omega$

Solve initial-value problem numerically using Laplace transforms:

$$(s + \mathrm{i}m\Omega) L_m \bar{\psi}_{K,m} - \frac{\mathrm{i}m}{r} Z' \bar{\psi}_{K,m} = L_m \zeta^{(0)}_{K,m},$$

Will need to think about the initial condition.

Resonant terms

Need to solve

$$\frac{\partial}{\partial t}\zeta + \Omega\zeta_\theta - \frac{Z'}{r}\psi_\theta = -\frac{1}{r}\left[\frac{\partial\psi_s}{\partial r}\frac{\partial\zeta_K}{\partial\theta} + \frac{\partial\psi_K}{\partial r}\frac{\partial\zeta_s}{\partial\theta} - \frac{\partial\psi_s}{\partial\theta}\frac{\partial\zeta_K}{\partial r} - \frac{\partial\psi_K}{\partial\theta}\frac{\partial\zeta_s}{\partial r}\right].$$

The strain couples modes m and $m+2$. Limit ourselves to the case $(-1,1)$ case for now and write $\psi_{K,1} = g_K$. Then

$$\frac{\partial}{\partial t}L_1\psi + \Omega L_1\psi - \frac{Z'}{r}\psi = -\frac{1}{4r}\left[-\mathrm{i}f_s' L_1 g_K^* + 2\mathrm{i}g_K'^* L_2 f_s - 2\mathrm{i}f_s(L_1 g_K^*)' + \mathrm{i}\psi_K(L_2 f_s^*)' x b\right].$$

Exact solution for the $(-1, 1)$ resonance

The radial Rayleigh equation can be solved exactly for $m = \pm 1$. Start with

$$\zeta_t \pm \mathrm{i}\Omega\zeta \mp \mathrm{i}r^{-1}Z'\psi = g(r,t),$$

with

$$\zeta = \psi'' + \frac{1}{r}\psi' - \frac{1}{r^2}\psi.$$

Now take the Laplace transform in time:

$$(s \pm \mathrm{i}\Omega)\bar{\zeta} \mp \mathrm{i}r^{-1}Z'\bar{\psi} = \zeta_0(r) + \bar{g}(r,s).$$

Follow Llewellyn Smith (1995) and write $\psi = r(s \pm \mathrm{i}\Omega)f$. This leads to

$$[r^3(s \pm \mathrm{i}\Omega)^2 f']' = r^2[\zeta_0(r) + g(r,s)].$$

The solution to this equation that decays at infinity is

$$\psi = -r(s \pm \mathrm{i}\Omega)\int_r^\infty \frac{m(v)}{(s \pm \mathrm{i}\Omega)^2}\,\mathrm{d}v$$

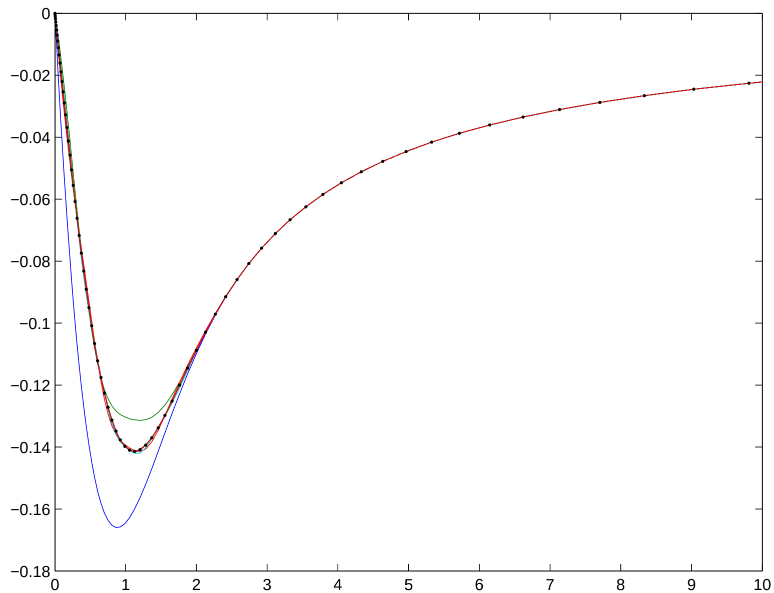
with

$$m(v) = \frac{1}{v^3}\int_0^v u^2[\zeta_0(u) + g(u,s)]\,\mathrm{d}u = \frac{\partial}{\partial v}\left(\frac{\psi_0(v)}{v}\right) + \frac{1}{v^3}\int_0^v u^2 g(u,s)\,\mathrm{d}u.$$

Go back to LS95 and check that long-time result for evolution of Kelvin mode is correct. Result is

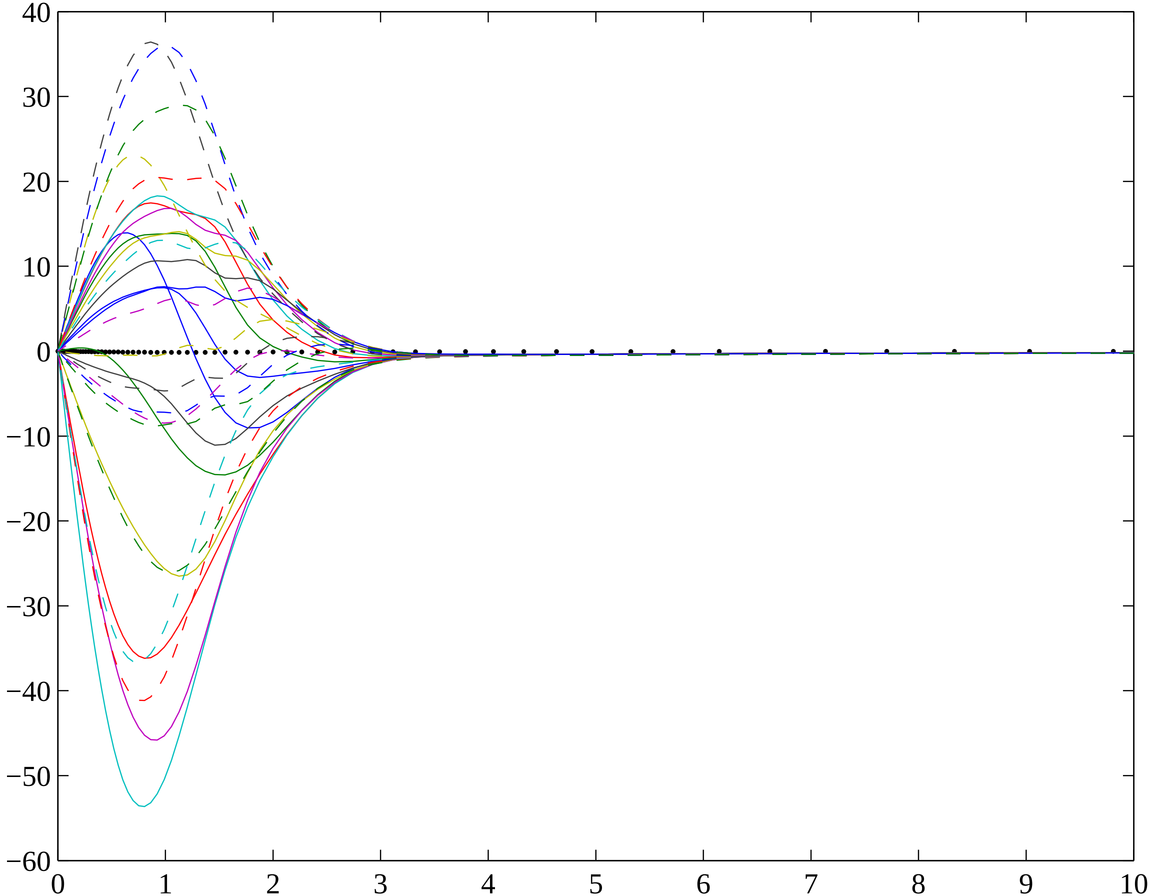
$$\psi \sim -\frac{\pi M}{\Gamma} r\Omega$$

for long time, where $M = \int_0^\infty u^2 \zeta_0(u)\,\mathrm{d}u$. For Gaussian vorticity $Z = \mathrm{e}^{-r^2}$, $\Gamma = \pi$. Take $\zeta_0 = \mathrm{e}^{-r^2}$ (the actual form is irrelevant) so that $M = \pi^{1/2}/4$.



Time evolution of $m = 1$ Kelvin mode for $t = 0$ to 200. The long-term asymptotic result is the dotted black curve.

Numerical solution



Time evolution of resonant mode induced by $m = 1$ Kelvin mode as above for $t = 0$, 5, 10, 15 (solid: real part, dashed: imaginary part).

The 3D problem

The next step. . .

Conclusion

Hollow vortices

- Have obtained solution for hollow vortex in strain field. Presumably this is what Mary Hill found.
- Examined two-dimensional stability of hollow vortex in strain.
- Same technique works for three-dimensional stability and is a general method for examining the stability of free-streamline solutions to the Euler equations.
- Used the same method to examine the stability of Pocklington's hollow vortex. Stable if sufficiently close to vortex pair.
- Next steps: Sadovskii vortices with delta and Heaviside functions of vorticity. Relevant to bluff-body wakes from Prandtl–Batchelor theorem. Surface tension: expect to stabilize vortex. Stability method should still apply (with some changes in boundary conditions) but need to compute basic states again.

The initial-value problem

- Strain correction is simple to obtain.
- Kelvin mode evolution for 2D case has been computed. Agrees with exact $m = \pm 1$ solution.
- Exact 2D solution is available for $(-1, 1)$ resonance. Asymptotics should be possible.
- Have started with numerics. Need to be compared to asymptotics.
- 3D case remains to be investigated.

Density effects

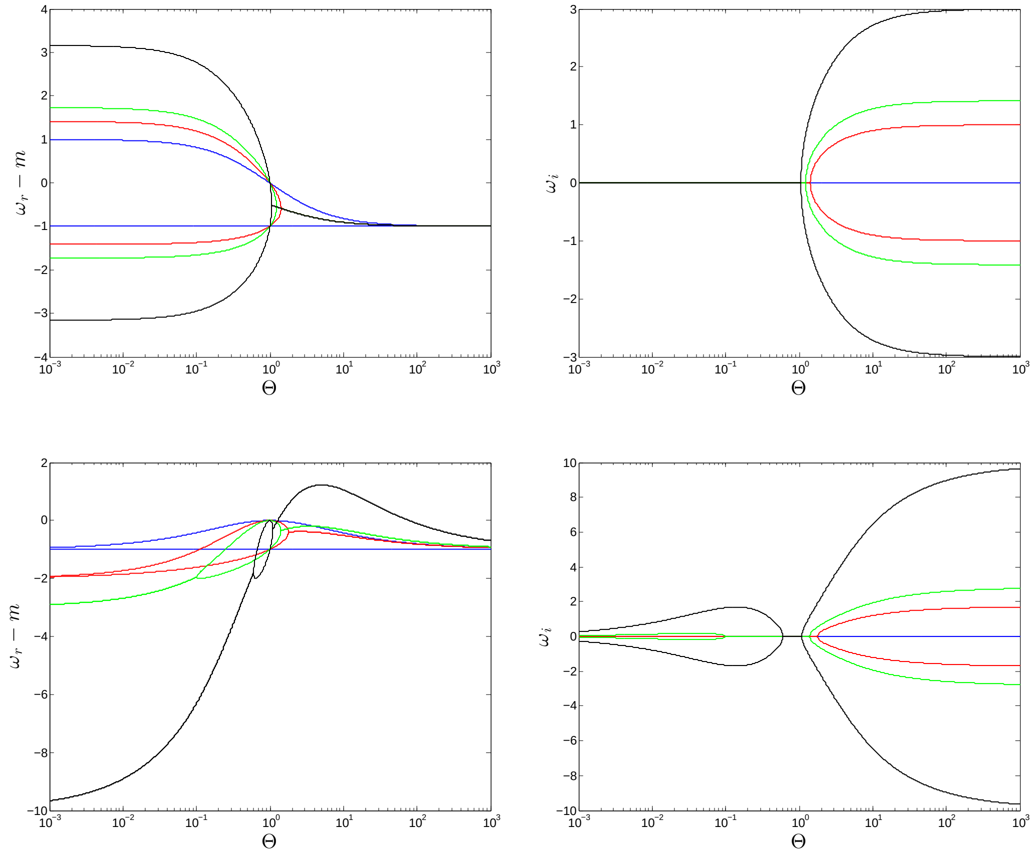
Dolphins!

Rings seem very stable. Led to wonder about density effects in vortex ring stability and hence in TWMS. Current work with Laurent Lacaze (IMFT).

Set density $\rho = \rho_i$ and $\rho = \rho_o$ for $r < 1$ and $r > 1$ respectively. Two profiles (i) Rankine and (ii) modified Rankine depending on continuity condition (modified Rankine probably not physical).

Analysis as in usual MS case. Θ is the density ratio.

Related to previous work on vortex stability with density differences by Jacquin and coworkers.



Dispersion curves as a function of Θ for the Rankine vortex (top row) and Modified Rankine vortex (bottom row). Frequency and growth rate are shown: $m = 1$ (blue), $m = 2$ (red), $m = 3$ (green) and $m = 10$ (black).

Numerical solution

Since Φ is harmonic, the functions Φ and δ can be written in the fluid region as

$$\Phi = \sum_{n=-\infty}^{\infty} \Phi_n \mathrm{e}^{\mathrm{i}n\theta} \rho^{|n|}, \qquad \delta = \sum_{n=-\infty}^{\infty} \delta_n \mathrm{e}^{\mathrm{i}n\theta}.$$

Obtain matrix equations

$$\begin{aligned} -\mathrm{i} \sum_{m=-\infty}^{\infty} Q_{n-m} m \Phi_m + \sum_{m=-\infty}^{\infty} G_{n-m} \delta_m &= \sigma \Phi_n, \\ -\sum_{m=-\infty}^{\infty} Q_{n-m} |m| \Phi_m - \mathrm{i} \sum_{m=-\infty}^{\infty} Q_{n-m} m \delta_m &= \sigma \delta_n \end{aligned}$$

(G_n and Q_n obtained using FFT).

Truncate and solve for the vector $\mathbf{r} = [\Phi_{-N}, \cdots, \Phi_0, \cdots, \Phi_N, \delta_{-N}, \cdots, \delta_0, \cdots, \delta_N]^T$: generalized eigenvalue problem.