Accurate estimates for the exponential decay of semigroups with non self-adjoint generators

Francis Nier,
IRMAR, Univ. Rennes 1
and INRIA project MICMAC.

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Outline

Some general estimates

The problem

$(e^{-tA})_{t\geq 0}$ contraction semigroup in a Hilbert space $\mathcal{H}$.

- A question which occur in many models is about the exponential decay
$$\|e^{-tA}\| \leq Ce^{-\lambda t} \quad C?\lambda?\,,$$

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$$\|e^{-tA}\| \leq Ce^{-\lambda t} \quad C?\lambda?\,,$$

- or possibly

$$\|e^{-tA} - \sum_{j=1}^{N} e^{-\lambda_j t}\Pi_j\| \leq Ce^{-\lambda t}\,, \lambda > \mathrm{Re}\,\lambda_j\,.$$

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Classical results

- Gearhardt-Prüss-Hwang-Greiner Theorem:
  - When $\|(z-A)^{-1}\|$ is uniformly bounded in $\{\mathrm{Re}\, z \leq \tau\}$ then there exists C_τ such that $\|e^{-tA}\| \leq C_\tau e^{-\tau t}$.
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$$\lim_{t\to+\infty} -\frac{\log \|e^{-tA}\|}{t} = \inf_{z\in Spec(A)} \mathrm{Re}\, z (=: \Sigma_A) .$$

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- When A is normal, the functional calculus gives

$$\|e^{-tA}\| \leq 1 e^{-\Sigma_A t}\,.$$

- But it is not true for general non self-adjoint generators. What about C_λ in $\|e^{-tA}\| < C_\lambda e^{-\lambda t}$ for $\lambda \leq \Sigma_A$? Is it important ?

Relevant quantities and the case of sectorial operators

- Consider the three quantities $\Sigma = \inf_{z \in SpecA} \mathrm{Re}\, z$,
  $\Xi = \inf_{\psi \in D(A)\,,\, \|\psi\|=1} \mathrm{Re}\, \langle \psi \,, A\psi \rangle$,
  $\Psi = \left(\sup_{\lambda \in i\mathbb{R}} \|(i\lambda - A)^{-1}\|\right)^{-1}$.

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- The first resolvent formula gives

$$\Xi \leq \Psi \leq \Sigma\,.$$

Relevant quantities and the case of sectorial operators

Theorem (Gallagher-Gallay-N.): Assume $|\arg\langle\psi, A\psi\rangle| \leq \frac{\pi}{2} - \alpha$ for all $\psi \in D(A)$ with $\alpha > 0$.

- **i)** If there exist $C \geq 1$ and $\mu > 0$ such that $\|e^{-tA}\| \leq C\, e^{-\mu t}$ for all $t \geq 0$, then

$$\Sigma \geq \mu\,, \quad \text{and} \quad \Psi \geq \frac{\mu}{1 + \log(C)}\,.$$

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- For any $\mu \in (0, \Sigma)$, the estimate $\|e^{-tA}\| \leq C(A, \mu)\, e^{-\mu t}$ holds for all $t \geq 0$, where

$$C(A, \mu) = \frac{\left(\mu N(A, \mu) + 2\pi\right)}{\pi \tan \alpha}\,,\; N(A, \mu) = \sup_{\lambda \in \mathbb{R}} \|(A - \mu - i\lambda)^{-1}\|\,.$$

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- For $\mu \in (0,\Psi)$, $N(A,\mu) \leq (\Psi-\mu)^{-1}$.

Relevant quantities and the case of sectorial operators

Consequences

- When $\mu = \frac{\Psi}{2}$ then $C(A, \mu) \leq \frac{1+2\pi}{\pi \tan \alpha}$ and

$$\|e^{-tA}\| \leq \frac{1+2\pi}{\pi \tan \alpha} e^{-\frac{\Psi t}{2}} \,. \tag{1.1}$$

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- When $\mu \leq \Sigma$, $\|e^{-tA}\| \leq C(A,\mu)e^{-\mu t}$ makes sense (i.e improves $\|e^{-tA}\| \leq 1$) for $t \geq \frac{\log C(A,\mu)}{\mu}$.
- If $\mu \in (\Psi, \Sigma)$ then $C(A,\mu) \geq e^{\frac{\mu}{\Psi}-1}$.
  The latter is exponentially large when $\mu = \frac{\Sigma}{2} \gg \Psi$.
  The time estimate does not make sense for $t \leq \frac{1}{\Psi} - \frac{2}{\Sigma}$.
  It becomes better than (1.1) only when $t \gg \frac{1}{\Psi}$.

Natural examples

After linearization of non linear models

Jt work with T. Gallay and I. Gallagher (cont'd by W. Deng):
A simplified version of linearized fluid mech. equations is

$$H_\epsilon = -\frac{d^2}{dx^2} + x^2 + i\frac{f(x)}{\epsilon} \quad x \in \mathbb{R}\,,\ f(x) \sim \frac{1}{|x|^k}\,, k > 0\,.$$

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- $\Psi(\epsilon) \propto \epsilon^{-\nu_\psi}$ with $\nu_\psi = \frac{2}{k+4}$.

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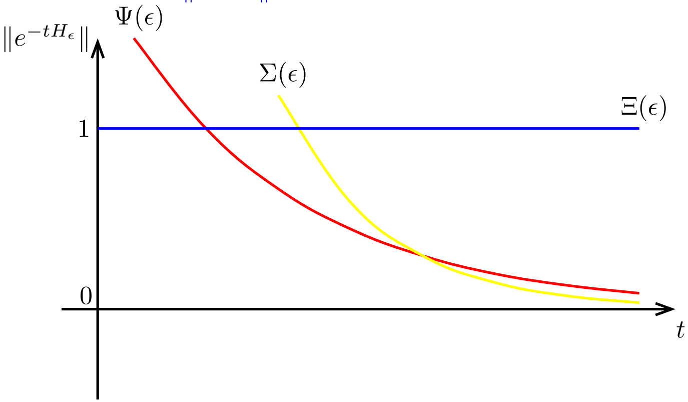
We proved

- $\Psi(\epsilon) \propto \epsilon^{-\nu_\psi}$ with $\nu_\psi = \frac{2}{k+4}$.
- When $f(x) = \frac{1}{(1+x^2)^{k/2}}$, $\Sigma(\epsilon) \geq \epsilon^{-\nu_\sigma}$ with $\nu_\sigma = \min\left\{\frac{1}{2}, \frac{2}{k+2}\right\}$ $\Sigma(\epsilon) \gg \Psi(\epsilon)$.

Gallay's numerical experiment

Time scales for $\|e^{-tH_\epsilon}\|$:



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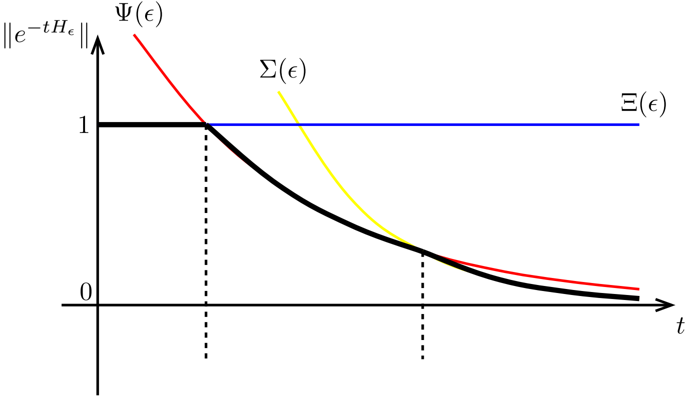

$t \leq \epsilon^{\frac{2}{k+4}}$ $\quad t \propto \epsilon^{\frac{2}{k+4}}|\log \epsilon|$ $\quad t \geq \epsilon^{\frac{2}{k+2}} \log[C(\epsilon)]$

$\epsilon^{\frac{2}{k+4}} \gg \epsilon^{\frac{2}{k+2}}$ but $C(\epsilon)$ exponentially large.

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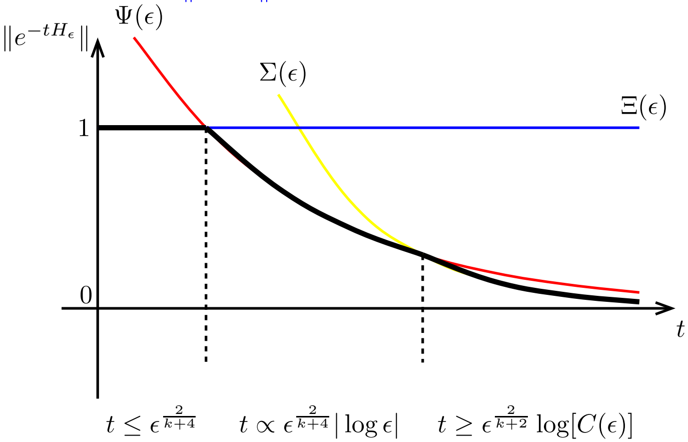

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T. Gallay was stopping the numerical simulation when $\|e^{-tH_\epsilon}u\| \ll \|u\|$ but before reaching the exponential regime $\propto e^{-t\Sigma(\varepsilon)}$.

Kramers-Fokker-Planck operators

Jt work with F. Hérau (cont'd by Hérau, Hitrik, Sjöstrand, Helffer, Hairer...)

$$K = v.\partial_x - \frac{1}{m}\partial_x V(x).\partial_v + \frac{\gamma_0}{m\beta}\left(-\Delta_v + \frac{m^2\beta^2 v^2}{4} - \frac{m\beta}{2}\right)\,,\ x, v \in \mathbb{R}^d\,.$$

- Assumptions: $C^{-1}\langle x\rangle^m - C \leq V(x) \leq C\langle x\rangle^m + C\,,\ m > 1\,,\ +$ derivatives...

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- Equilibrium $M(x, v) = e^{-\frac{\beta}{2}\left(\frac{mv^2}{2} + V(x)\right)}$,

$$\|e^{-tK}u - c_u M\| \leq Q(m, \beta, \gamma_0, \omega, t)e^{-\omega t}\,,$$

where $\omega = \omega(\gamma_0, \beta, m)$ is related to the spectral gap of some Witten Laplacian and Q is an algebraic expression.

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  where $\omega = \omega(\gamma_0, \beta, m)$ is related to the spectral gap of some Witten Laplacian and Q is an algebraic expression.
- K is not sectorial but hypoelliptic estimates allow resolvent estimates and contour deformations in the complex plane. For the refined analysis of the low-lying spectrum, the "PT"-symmetry is important.

Artificial examples

Adiabatic evolution for 1D shape resonances

Jt work with A. Faraj and A. Mantile (cont'd by A. Mantile)
$H^h = -h^2\Delta + V(x) \quad V(x)$ well in an island, $x \in \mathbb{R}$.

- Resonances are unveiled by a complex deformation parametrized by $\theta \in i\mathbb{R}$ which makes $\sigma(iH^h(\theta)) \subset \{\Re z \geq 0\}$ and $\sigma_{ess}(iH^h(\theta)) \setminus \{0\} \subset \{\Re z > 0\}$.

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- Time-adiabatic evolution of resonances $i\varepsilon\partial_t u = [-h^2\Delta + V(x,t)]u$ or $i\varepsilon\partial_t u_\theta = H^h(\theta,t)u_\theta$, when $u_\theta(0)$ is a resonant state and $\varepsilon = e^{-\frac{C'}{h}}$.

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- Adiabatic evolution justified under well controlled estimates for $\|U_\theta(t,s)\|$ with $i\varepsilon\partial_t U_\theta(t,s) = H^h(\theta,t)$ and $U_\theta(s,s) = \mathrm{Id}$. $H^h(\theta)$ non-selfadjoint makes it almost impossible.

Adiabatic evolution of 1D shape resonances

Jt work with A. Faraj and A. Mantile (cont'd by A. Mantile)

We decided to introduce an additional deformation of H^h by modifying $-\Delta$:

Exterior dilation.

$$\begin{aligned} H^h(\theta) &= U_\theta H^h U_{-\theta} \\ &= -h^2 e^{-2\theta\times 1_{\mathbb{R}\setminus[a,b]}}\Delta + V \end{aligned}$$

$$D(H^h(\theta)) = \left\{ u \in H^2(\mathbb{R}\setminus\{a,b\})\,,\ \begin{array}{l} e^{-\frac{\theta}{2}}u(b^+) = u(b^-)\,, \\ e^{-\frac{3\theta}{2}}u'(b^+) = u'(b^-)\,, \\ e^{-\frac{\theta}{2}}u(a^-) = u(a^+)\,, \\ e^{-\frac{3\theta}{2}}u'(a^-) = u'(a^+)\,, \end{array} \right\}$$

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We decided to introduce an additional deformation of H^h by modifying $-\Delta$:
Exterior dilation + modification.

$$\begin{aligned} H^h_{\theta_0}(\theta) &= U_\theta H^h_{\theta_0} U_{-\theta} \\ &= -h^2 e^{-2\theta \times 1_{\mathbb{R}\setminus[a,b]}} \Delta_{\theta_0} + V \end{aligned}$$

$$D(H^h_{\theta_0}(\theta)) = \left\{ u \in H^2(\mathbb{R}\setminus\{a,b\}) \,,\ \begin{array}{l} e^{-\frac{\theta_0+\theta}{2}} u(b^+) = u(b^-)\,, \\ e^{-\frac{3\theta_0+3\theta}{2}} u'(b^+) = u'(b^-)\,, \\ e^{-\frac{\theta_0+\theta}{2}} u(a^-) = u(a^+)\,, \\ e^{-\frac{3\theta_0+3\theta}{2}} u'(a^-) = u'(a^+)\,, \end{array} \right\}$$

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We proved

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- When $\theta_0 = \theta = i\tau$ with $\tau \in \mathbb{R}$ the equation $i\varepsilon\partial_t U_{\theta_0,\theta}(t,s) = H^h_{\theta_0}(\theta, t)U_{\theta_0,\theta}(t,s)$ defines a dyn. syst. of contractions
$$\|U_{i\tau,i\tau}(t,s)\| \leq 1 \quad , \ \forall t \geq s\,;$$
$\rightarrow$ good adiabatic evolution of (modified) resonant states.

Optimizing the sampling of equilibrium distributions

Jt work with T. Lelièvre and G. Pavliotis

When one wants to sample the equilibrium distribution $\psi_\infty = \frac{e^{-V}}{\int_{\mathbb{R}^N} e^{-V(x)}\, dx}$ a natural way is to use the reversible dynamics

$$dX_t = -\nabla V(X_t) + \sqrt{2} dW_t \,.$$

Reversible means a self-adjoint semigroup generator for the Fokker-Planck equation.

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When one wants to sample the equilibrium distribution $\psi_\infty = \frac{e^{-V}}{\int_{\mathbb{R}^N} e^{-V(x)}\ dx}$ an alternative approach consists in using the non reversible dynamics

$$dX_t = (-\nabla V(X_t) + b(X_t))dt + \sqrt{2}dW_t \quad \text{with} \quad \nabla.(be^{-V}) \equiv 0\,.$$

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$$dX_t^J = -(I+J)SX_t^J dt + \sqrt{2}dW_t \quad J^t = -J$$

in the linear case with $\psi_\infty(x) = \frac{\det(S)^{1/2}}{(2\pi)^{N/2}} e^{-\frac{x^t Sx}{2}}$.

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By setting $\mathcal{L}_J = -((I+J)Sx).\nabla + \Delta$ the problem is about the optimization of $(\lambda_S(J)$ and $C_S(J))$ w.r.t J for

$$\|e^{t\mathcal{L}_J}u - (\int_{\mathbb{R}^N} u\psi_\infty)\psi_\infty\|_{L^2(\psi_\infty dx)} \leq C_S(J)e^{-\lambda_S(J)t}\|u\|_{L^2(\psi_\infty dx)}$$

while $\mathcal{L}_J$ is no more self-adjoint for $J \neq 0$.

Optimizing the sampling of equilibrium distributions

Jt work with T. Lelièvre and G. Pavliotis
Results

- The optimum value $\lambda_S = \frac{\mathrm{Tr}\,[S]}{N}$ can be reached by constructing $\tilde{J} = S^{1/2} J S^{1/2} = -\tilde{J}^t$ and $Q = Q^t > 0$ s.t.

$$\tilde{J}Q - Q\tilde{J} = -QS - SQ + \frac{2\,\mathrm{Tr}\,[S]}{N} Q\,.$$

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- The constant $C_S(J)$ is bounded by $C_N \kappa(S)^{7/2}$ where $\kappa(S) = \|S\| \|S^{-1}\|$ and $C_N = \mathcal{O}(N^3)$.
  This estimate uses some bosonic QFT inequality:

$$\sum_{1 \leq i,j,k,\ell \leq N} A_{i,j,k,\ell}\, a_j^* a_i^* a_k a_\ell \geq 0$$

when $A \in L(\mathbb{C}^{N^2})$ satisfies $A = A^t \geq 0$.