STABILITY OF SOLUTIONS TO EULER'S EQUATIONS

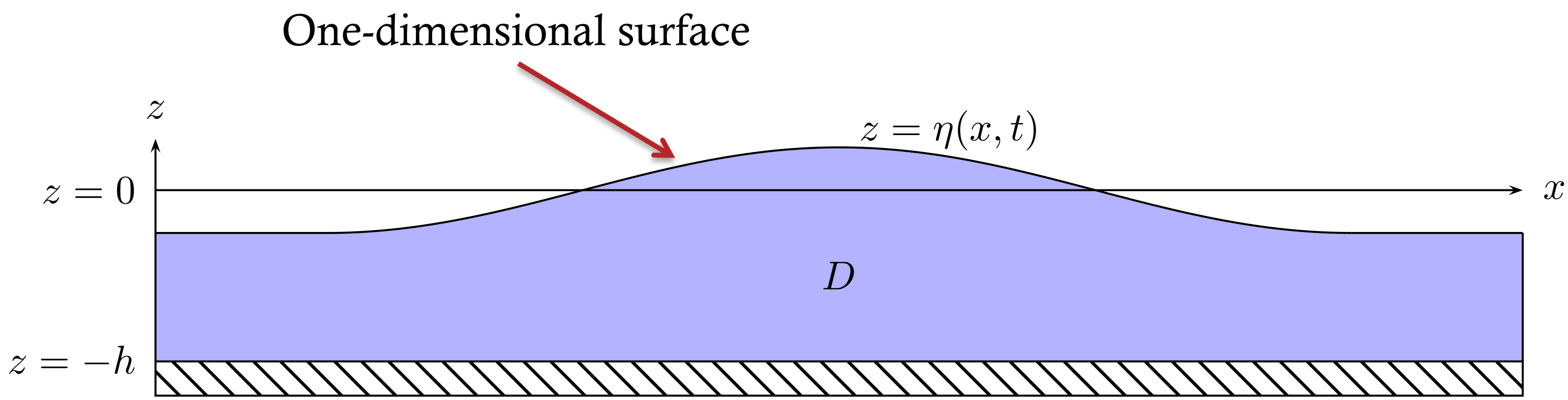
Katie Oliveras
Seattle University

Bernard Deconinck
University of Washington

Vishal Vasan
Penn State University

Patrick Sprenger
Seattle University

OVERVIEW

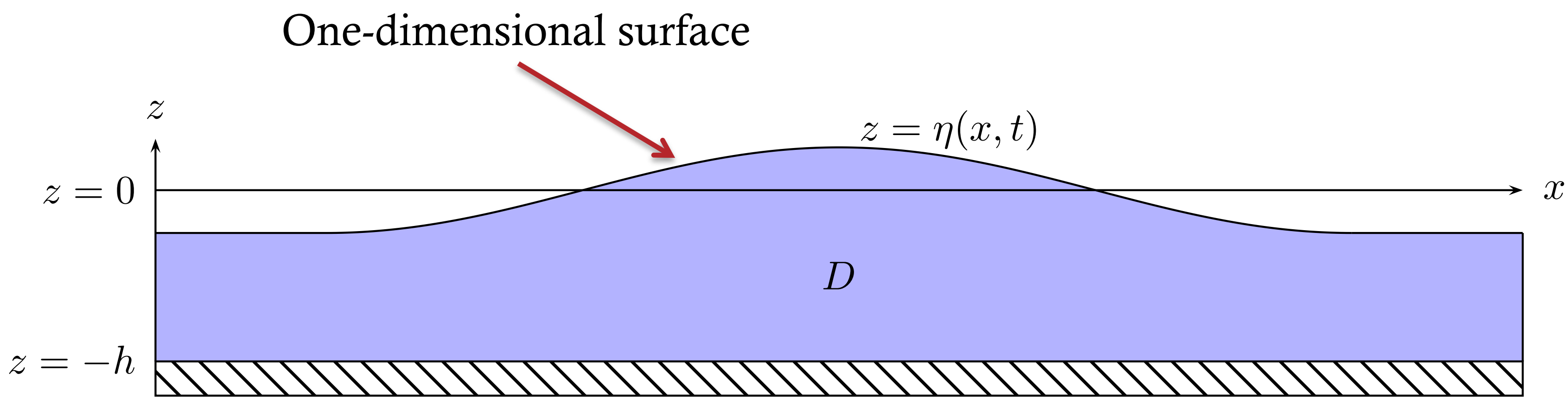


The problem of interest:

- Free Boundary Problem
- Flat bottom
- Inviscid, (irrotational)
- Periodic boundary conditions

Everything mentioned in this talk can be extended to a 2D surface.

OVERVIEW



$$
\begin{aligned}
\phi_{xx} + \phi_{zz} &= 0, & (x,z) &\in D,\\
\phi_z &= 0, & z &= -h\\
\eta_t + \eta_x \phi_x &= \phi_z, & z &= \eta(x,t),\\
\phi_t + \frac{1}{2}\phi_x^2 + \frac{1}{2}\phi_z^2 + g\eta &= 0, & z &= \eta(x,t).
\end{aligned}
$$

$$\phi_x(x,z,t) = \phi_x(x+L,z,t), \quad \eta(x,t) = \eta(x+L,t).$$

$$\vec{\mathbf{u}} = \nabla \phi$$

OVERVIEW

To consider the spectral stability of periodic solutions, we need to:

1. Determine stationary solutions (*traveling coordinate frame*).
2. Perturb the stationary solutions and linearize the equations of motion.
3. Solve the resulting eigenvalue problem.

But hasn't this already been done? Yes!

HISTORY - STABILITY

SEATTLE UNIVERSITY

- Lighthill (1965), Whitham (1967), Benjamin (1967), Bridges & Mielke (1995)
  - As kh increased to greater than 1.363 waves become unstable to long-wave perturbations. Benjamin-Feir or Modulational instability.

- Longuet-Higgins (1978)
  - The stability of periodic traveling waves in deep water changes as the dimensionless wave height, ak increases.

- McLean (1981), McLean et. al (1984), Francius & Kharif (2006), and others.
  - Transverse instability investigations.

- MacKay and Saffman (1984)
  - Collisions of opposite signature eigenvalues on the imaginary axis are necessary for instability.

- Nicholls (2008)
  - Exploits the analytic dependence of the spectra on the amplitude.

OBJECTIVE / GOAL

To understand the stability of traveling wave solutions for small amplitude solutions to Euler's equations through a new formulation.

OUTLINE OF THE TALK

SEATTLE UNIVERSITY

A NEW FORMULATION

TRAVELING WAVE SOLUTIONS

SPECTRAL STABILITY CALCULATIONS

TRANSVERSE STABILITY CALCULATIONS

Outline of the Talk

A New Formulation

Traveling Wave Solutions

Spectral Stability Calculations

Transverse Stability Calculations

EQUATIONS OF MOTION

SEATTLE UNIVERSITY

Why bother with an alternative formulation?

Results from others:

- Obtained a single equation for 1D traveling waves [Nekrasov, Bobenko, Toland, etc.].
- Hamiltonian formulation [Zakharov, Bridges & Laine-Pearson].
- Obtained results regarding the monotonicity of traveling waves [Strauss & Constantin].
- Results regarding existence/uniqueness of traveling wave solutions via the Dirichlet – Neuman operator [Nicholls, Craig].
- Many MANY more!

EQUATIONS OF MOTION

SEATTLE UNIVERSITY

Why bother with an alternative formulation?

Our Results (with the Ablowitz-Fokas-Musslimani [AFM] formulation):

- Obtained a single equation for traveling waves (1D surface, 2D problem). [Deconinck, O].
- Successfully investigated the spectral stability of 1D periodic travelling wave solutions w.r.t. all bounded 1D and 2D (transverse) perturbations. [Deconinck, O].
- Used pressure data to reconstruct the surface elevation of a wave [O, Vasan, Deconinck, Henderson].
- Created a single equation for the traveling waves (2D surface, 3D problem) [O, Vasan].
- The inverse problem: Bathymetry Detection [Vasan, Deconinck]

A NEW FORMULATION

Going back the the original equations of motion and transitioning to a traveling coordinate frame where $x \to x - ct$, we have

$$\begin{aligned} \phi_{xx} + \phi_{zz} &= 0, && (x,z) \in D, \\ \phi_z &= 0, && z = -h \\ \eta_t + \eta_x(\phi_x - c) &= \phi_z, && z = \eta, \\ \phi_t - c\phi_x + \frac{1}{2}\phi_x^2 + \frac{1}{2}\phi_z^2 + g\eta &= 0, && z = \eta. \end{aligned}$$

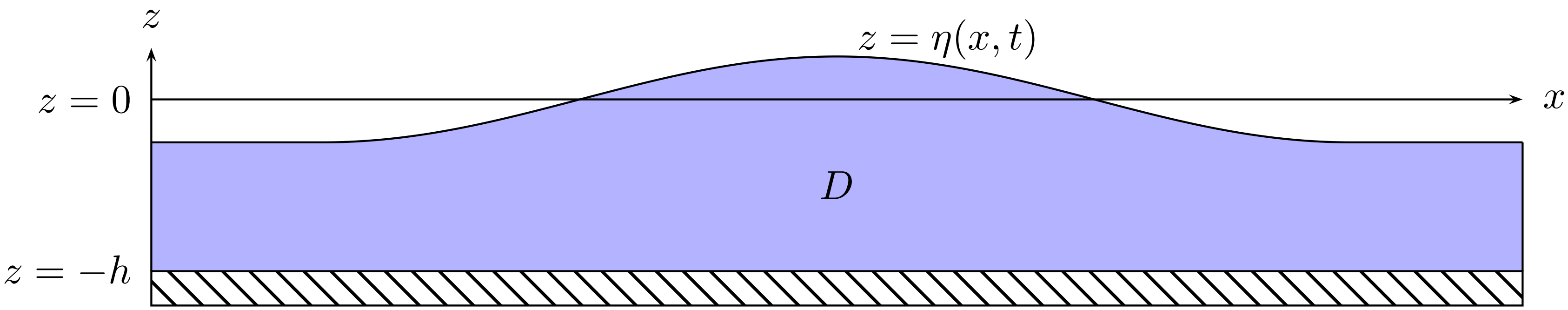
$$\phi_x(x,z,t) = \phi_x(x+L,z,t), \quad \eta(x,t) = \eta(x+L,t).$$

The goal is to consolidate this system of equations.

Ablowitz, Fokas & Musslimani, JFM 2005

AFM FORMULATION

Introduce a new surface variable $q(x,t) = \phi(x, \eta(x,t), t)$



The boundary conditions at the surface can be written in terms of surface variables as

$$q_t + \frac{1}{2}(q_x - c)^2 + g\eta - \frac{1}{2}\frac{(\eta_t + (q_x - c)\,\eta_x)^2}{1+\eta_x^2} = \frac{1}{2}c^2 - g\eta$$

AFM FORMULATION

Going back the the original equations of motion and transitioning to a traveling coordinate frame where $x \to x - ct$, we have

$$\begin{aligned} \phi_{xx} + \phi_{zz} &= 0, & (x,z) &\in D, \\ \phi_z &= 0, & z &= -h \end{aligned}$$

$$q_t + \frac{1}{2}\left(q_x - c\right)^2 + g\eta - \frac{1}{2}\frac{\left(\eta_t + \left(q_x - c\right)\eta_x\right)^2}{1 + \eta_x^2} = \frac{1}{2}c^2 - g\eta$$

$$\phi_x(x,z,t) = \phi_x(x+L,z,t), \quad \eta(x,t) = \eta(x+L,t).$$

The goal is to consolidate the system of equations.

AFM FORMULATION

SEATTLE UNIVERSITY

Consider two function that satisfy both Laplace's equation and the boundary condition at the bottom.

$$\psi_{xx} + \psi_{zz} = 0, \quad \psi_z\Big|_{z=-h} = 0, \qquad \phi_{xx} + \phi_{zz} = 0, \quad \phi_z\Big|_{z=-h} = 0,$$

It's easy to see that the following integral must also be zero

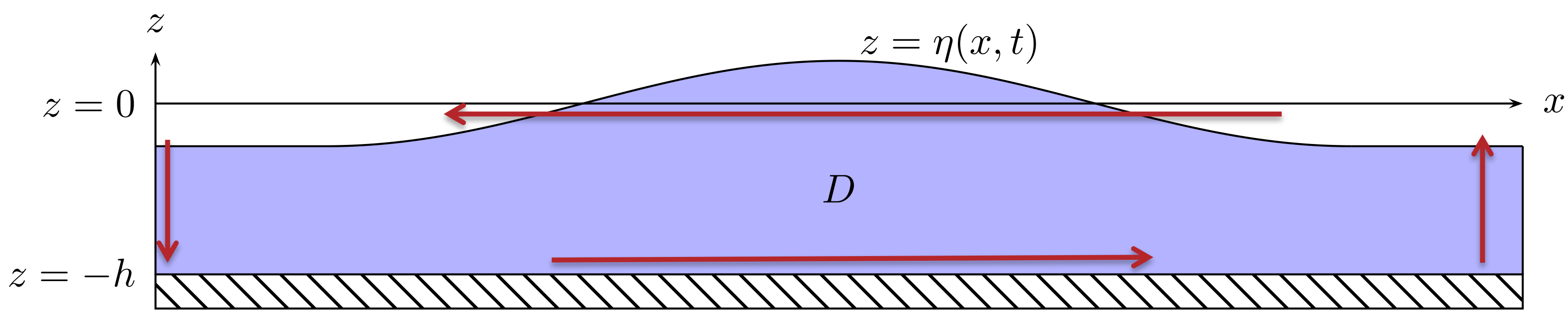
$$\int_D \left((\Delta\psi)\,\phi - (\Delta\phi)\,\psi \right) dV = 0$$

Using one of Green's identities, we can show:

$$\int_{\partial D} \left(\phi\,(\nabla\psi \cdot \vec{n}) - \psi\,(\nabla\phi \cdot \vec{n}) \right) dS = 0$$

AFM FORMULATION

Let $\psi = \sum_{k \in \Lambda'} e^{ikx} \hat{\xi}_k(t) \cosh\left(k(z+h)\right)$



$$\int_{\partial D} \left(\phi\left(\nabla\psi \cdot \vec{n}\right) - \psi\left(\nabla\phi \cdot \vec{n}\right)\right) dS = 0$$

$$\int_0^L e^{-ikx}\left(\eta_t \cosh\left(k(\eta+h)\right) + i\left(q_x - c\right)\sinh\left(k(\eta+h)\right)\right) dx = 0, \quad \forall k \in \Lambda'$$

AFM FORMULATION

Thus, the equations of motion are given in terms of surface variables by the following two equations:

- Local Equation

$$q_t + \frac{1}{2}\left(q_x - c\right)^2 + g\eta - \frac{1}{2}\frac{\left(\eta_t + \left(q_x - c\right)\eta_x\right)^2}{1 + \eta_x^2} = \frac{1}{2}c^2 - g\eta$$

- Nonlocal Equation

$$\int_0^L e^{-ikx}\left(\eta_t \cosh\left(k(\eta + h)\right) + i\left(q_x - c\right)\sinh\left(k(\eta + h)\right)\right)dx = 0, \quad \forall k \in \Lambda'$$

OUTLINE OF THE TALK

SEATTLE UNIVERSITY

A NEW FORMULATION

TRAVELING WAVE SOLUTIONS

SPECTRAL STABILITY CALCULATIONS

TRANSVERSE STABILITY CALCULATIONS

Looking for traveling wave solutions,

- Local Equation

$$q_t + \frac{1}{2}\left(q_x - c\right)^2 + g\eta - \frac{1}{2}\frac{\left(\eta_t + \left(q_x - c\right)\eta_x\right)^2}{1+\eta_x^2} = \frac{1}{2}c^2 - g\eta$$

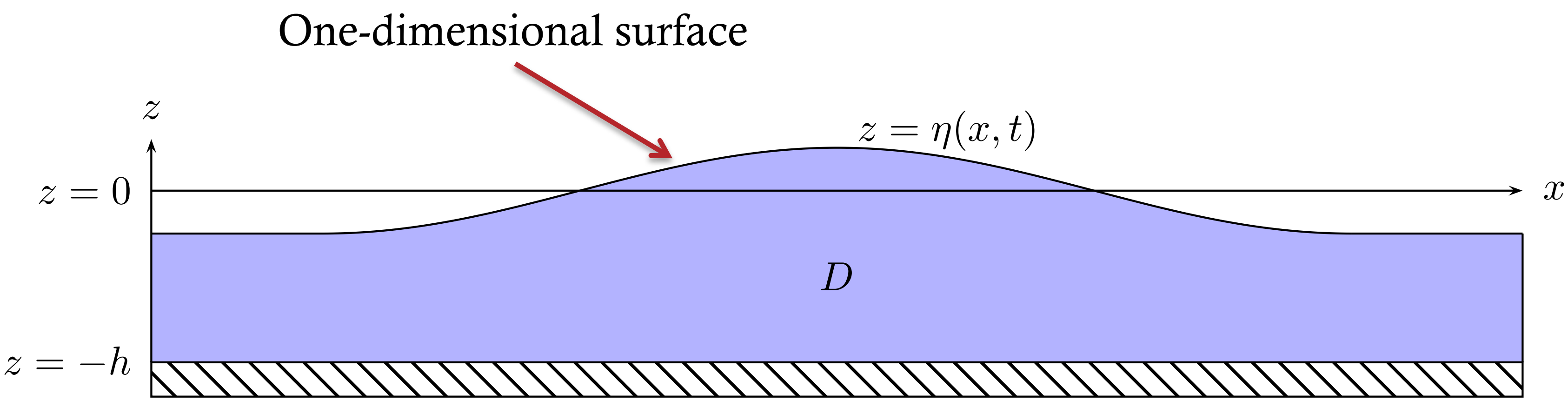
- Nonlocal Equation

Look for stationary solutions

$$\int_0^L e^{-ikx}\left(\eta_t \cosh\left(k(\eta+h)\right) + i\left(q_x - c\right)\sinh\left(k(\eta+h)\right)\right)dx = 0$$

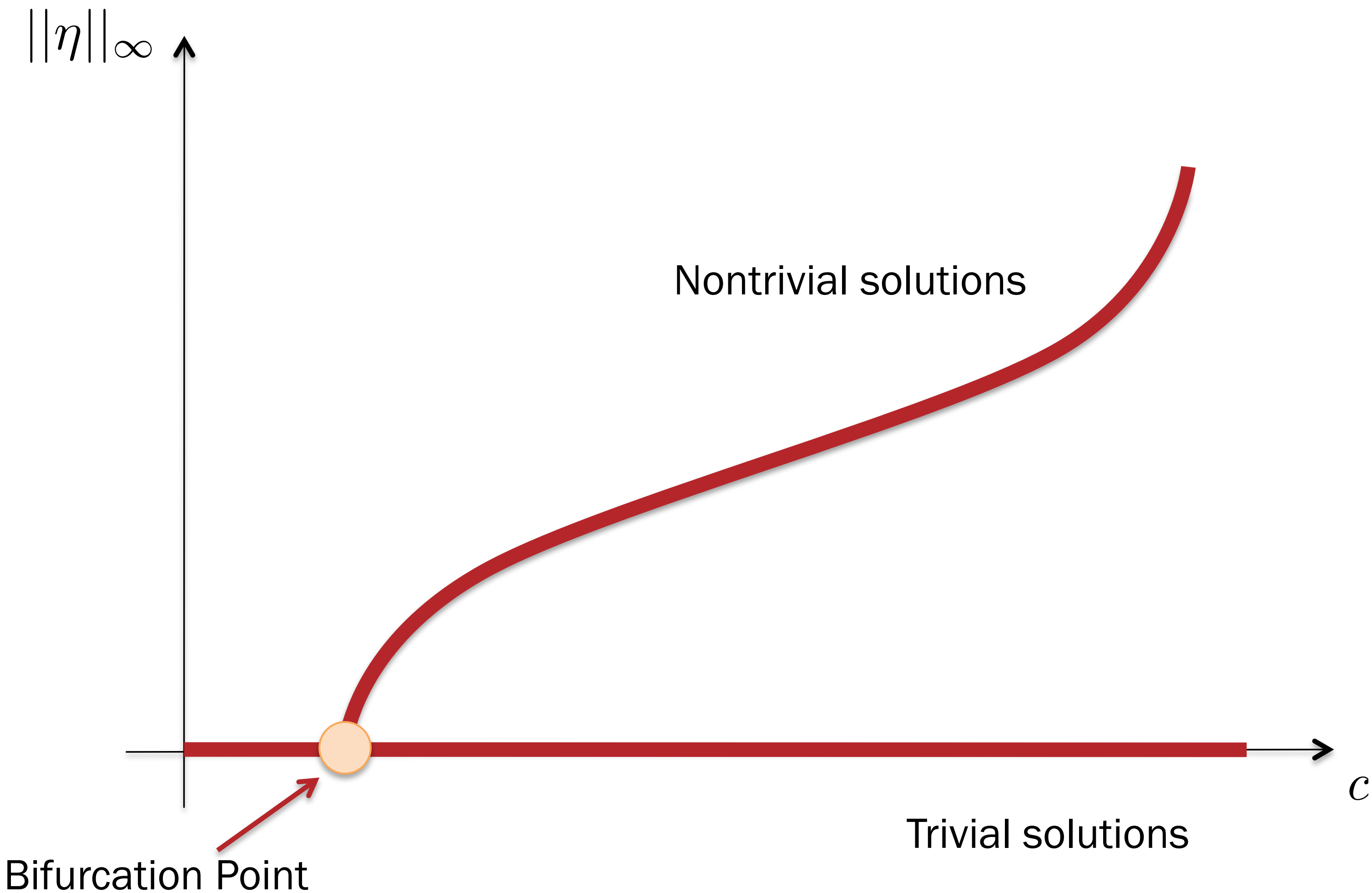
TRAVELING WAVE SOLUTIONS

SEATTLE UNIVERSITY



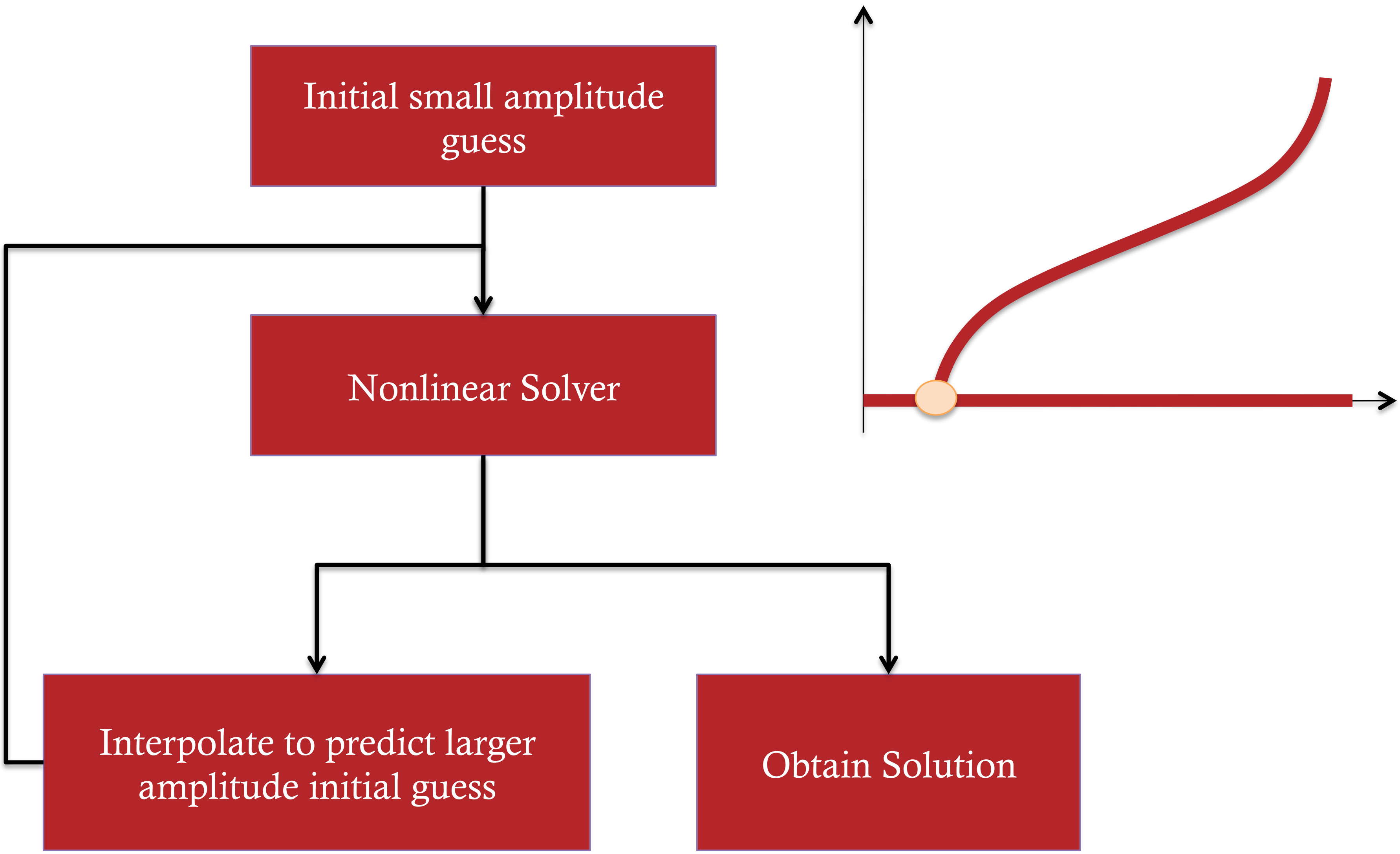
$$\int_0^L e^{-ikx} \sqrt{(c^2 - 2g\eta)\,(1+\eta_x^2)}\, \sinh\left(k(\eta + h)\right) dx = 0, \quad \forall k \in \Lambda$$

This single equation describes the surface for traveling wave solutions, and does not require knowledge of the velocity potential. [Deconinck, O]

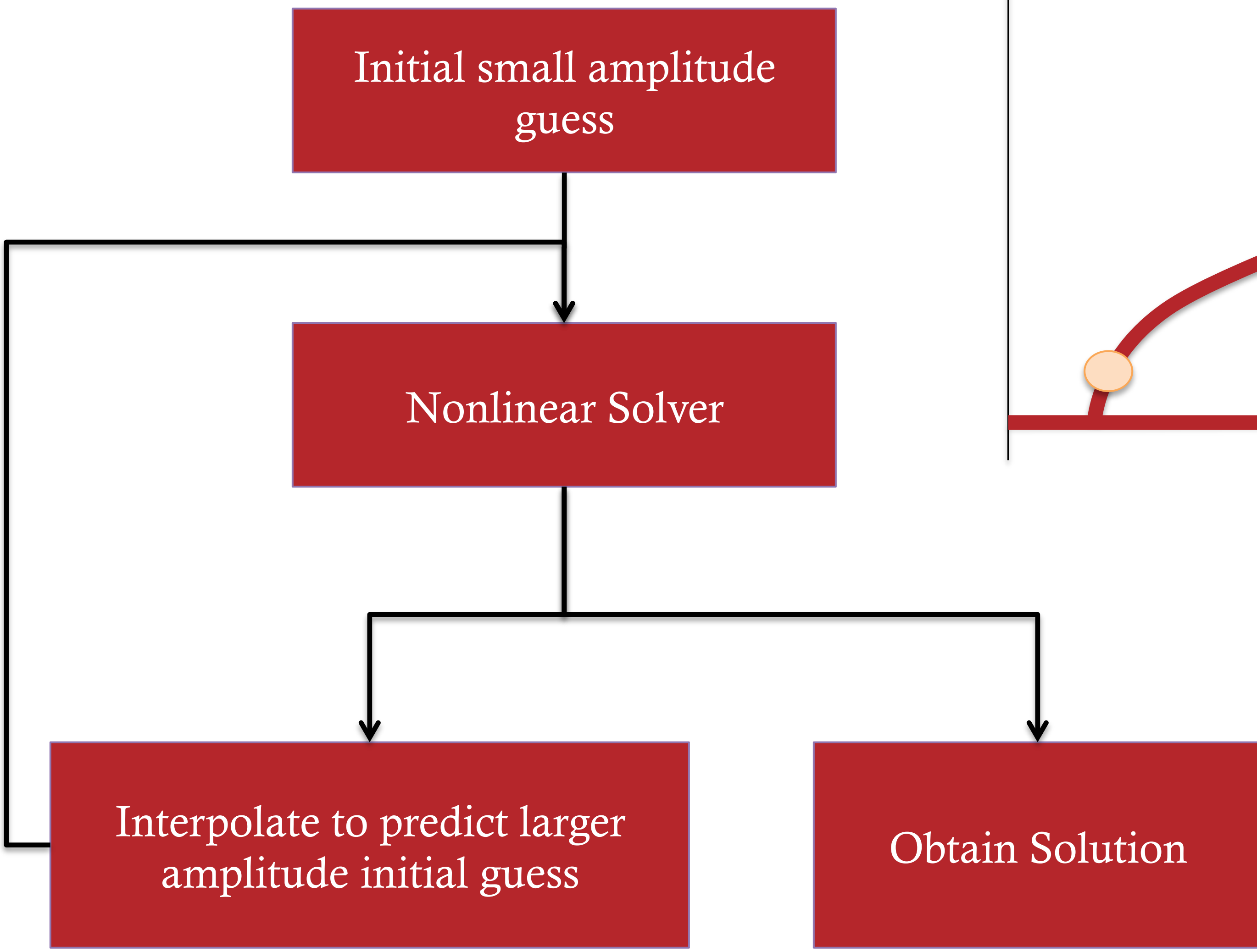

$||\eta||_\infty$
Nontrivial solutions
c
Trivial solutions
Bifurcation Point

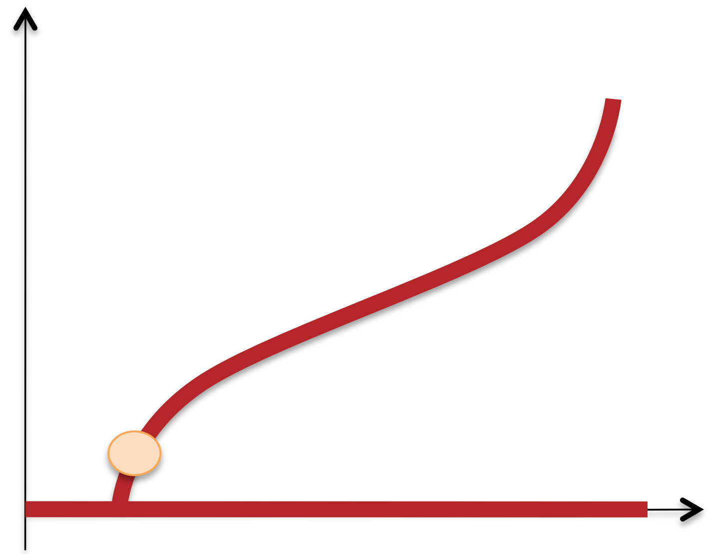
$L = 2\pi$

CONTINUATION



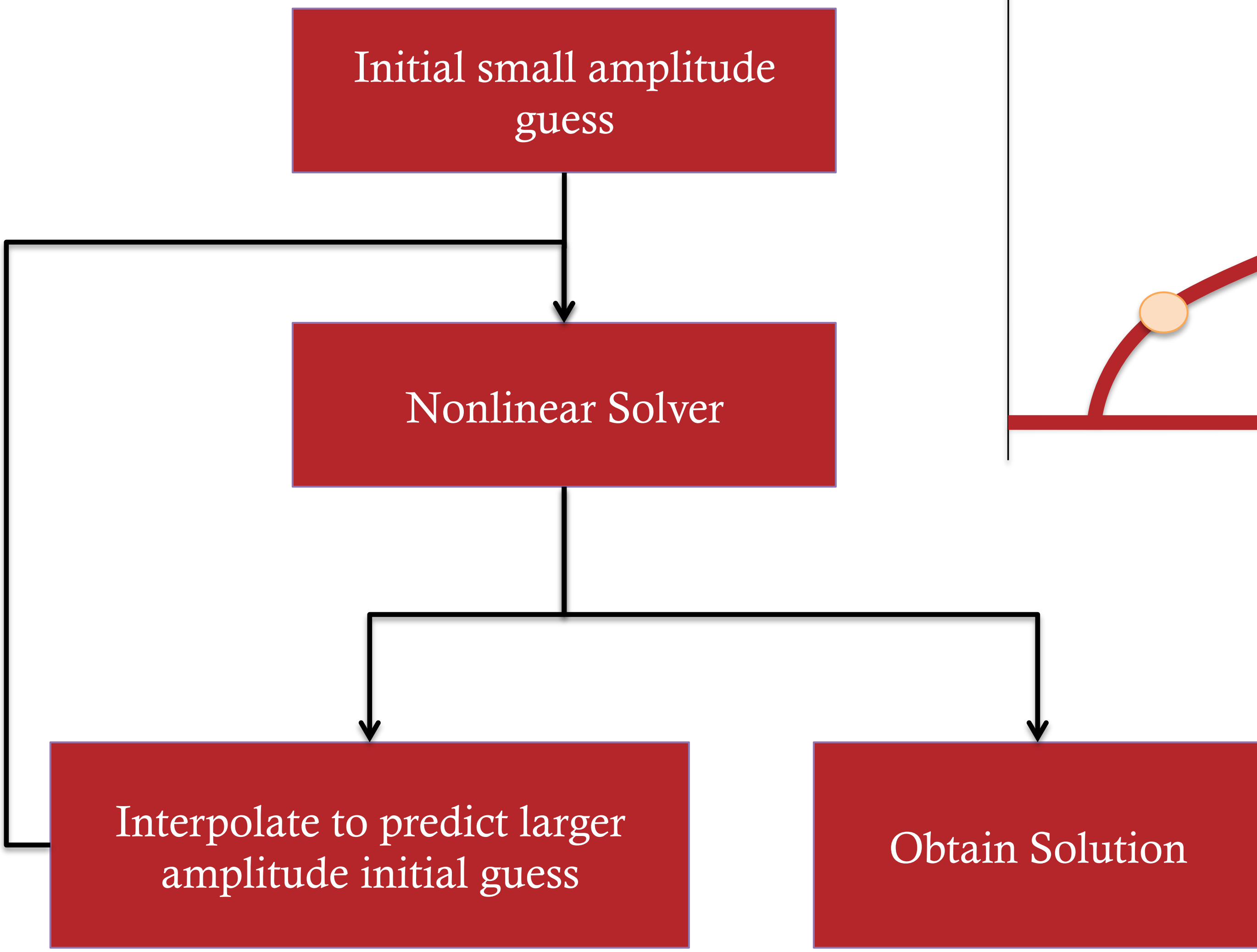
CONTINUATION

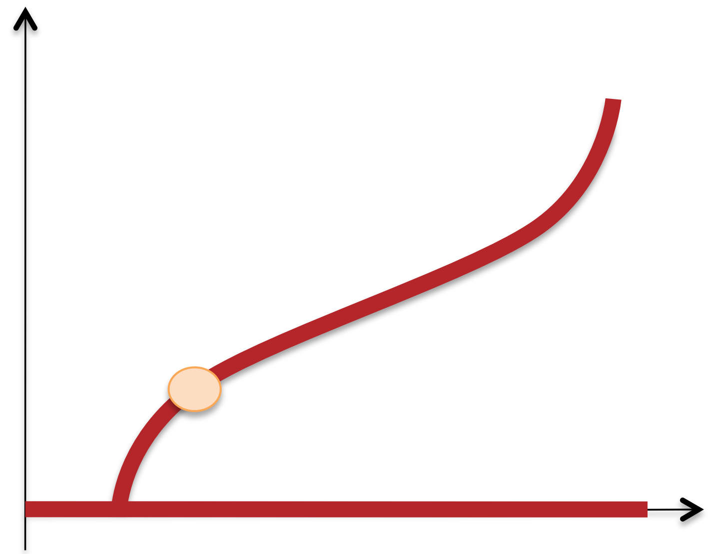




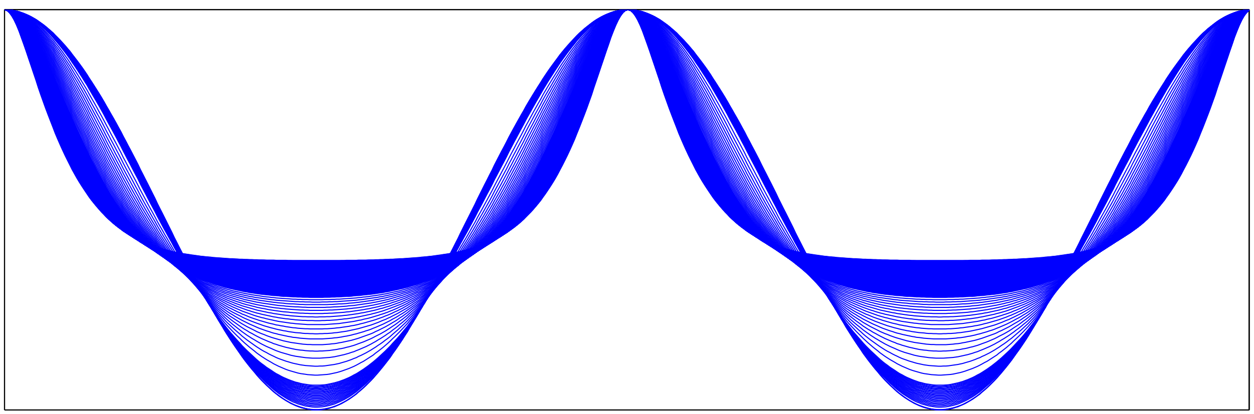
CONTINUATION

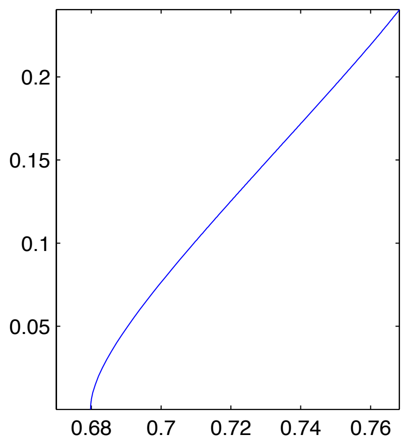
SEATTLE UNIVERSITY





Normalized solutions for $h = 0.5$, and $L = 2\pi$





OUTLINE OF THE TALK

SEATTLE UNIVERSITY

- A NEW FORMULATION
- TRAVELING WAVE SOLUTIONS
- **SPECTRAL STABILITY CALCULATIONS**
- TRANSVERSE STABILITY CALCULATIONS

SPECTRAL (IN)STABILITY

We consider perturbations of the form

$$\begin{aligned} \eta(x,t) &= \eta_0(x) + \epsilon \eta_1(x) e^{i\mu x} e^{\lambda t} + \dots \\ q(x,t) &= q_0(x) + \epsilon q_1(x) e^{i\mu x} e^{\lambda t} + \dots \end{aligned}$$

Time is only through the exponential term.

Spectrally unstable if there is any value of λ which has a positive real part.

Substituting the perturbed solution into the AFM formulation...

$$\mathcal{L}_\mu \mathbf{X} = \lambda \, \mathcal{M}_\mu \mathbf{X}$$

Range of the Floquet Parameter
We only need to consider the range $0 \leq \mu \leq 0.5$ instead of the full range. This allows us to reduce the size of the computational domain.

SPECTRAL (IN)STABILITY

SEATTLE UNIVERSITY

We would like to be efficient with our choice of Floquet parameter values μ.

We know the following:

1. A necessary condition for the loss of stability is the collision of two eigenvalues with opposite signatures (MacKay and Saffman)
2. The spectrum analytically depends on the amplitude of the traveling wave (Nicholls)
3. We can determine the spectrum analytically (in terms of μ) for the trivial solution with the appropriate wave speed corresponding to the location of the bifurcation of TWS. (Pen and Paper!)

IDEA: predict the location of instabilities for small amplitude waves and then track the location as we increase the amplitude of the TWS.

SPECTRAL (IN)STABILITY

Consider the trivial solution at the base of a bifurcation branch. For finite depth, the eigenvalues corresponding to the linear problem are given by

$$\lambda_m^{\pm} = -i\left(-ck_m \pm \sqrt{gk_m \tanh(hk_m)}\right), \quad k_m = m + \mu$$

An instability can arise if two eigenvalues with opposite signature collide:

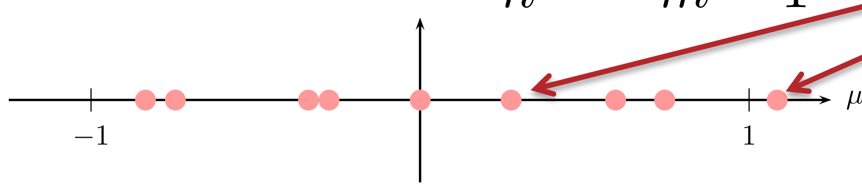
$$\lambda_m^{+} = \lambda_n^{-}, \quad m \neq n$$

We consider class I and class II instabilities such that

Dense grid in these neighborhoods.

Class I: $n = -m$

Class II: $n = -m - 1$

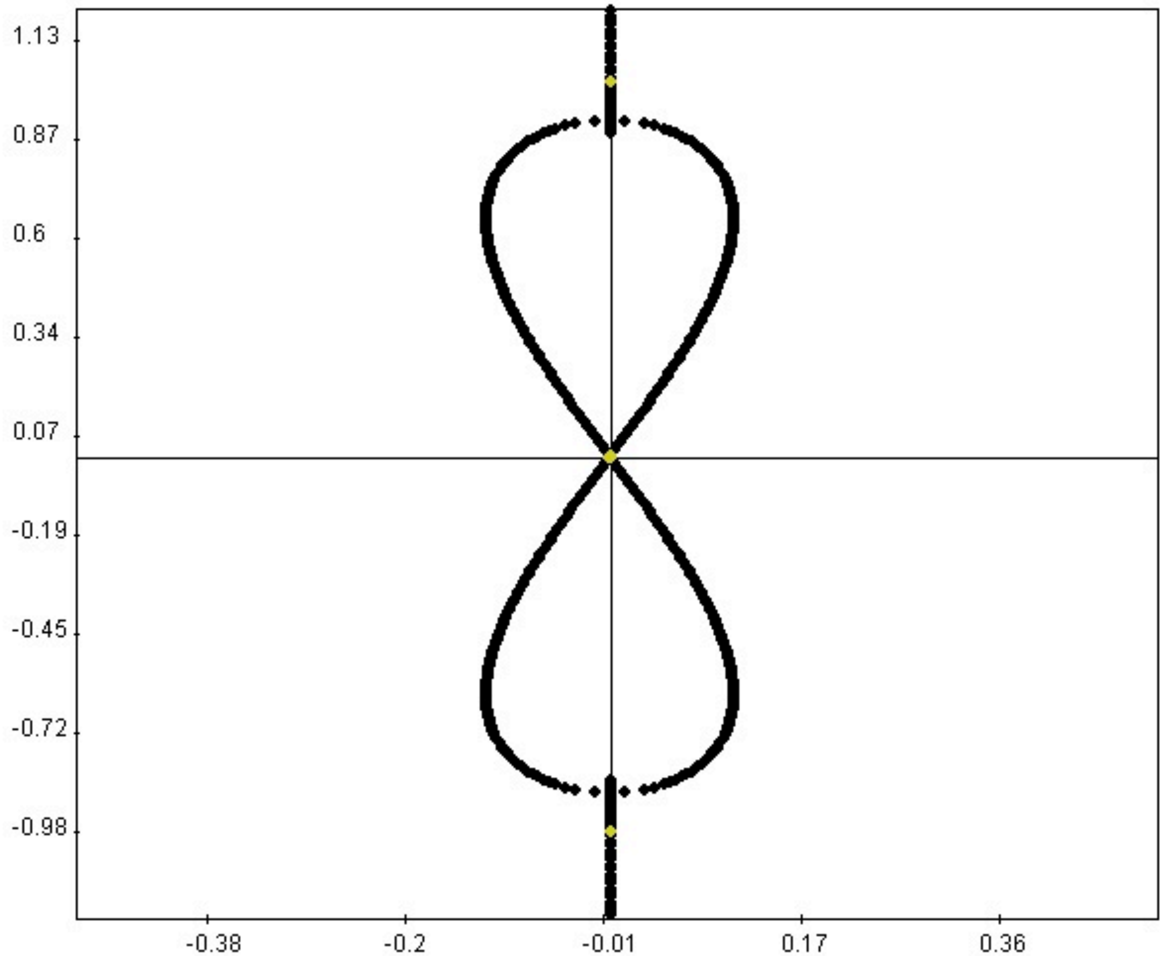


These are the same techniques used by McLean (1981), Ioulalaen, et. al (1999), Francius and Kharif (2006), and many others when investigating the stability with respect transverse perturbations.

WHAT WE ALREADY KNOW…

SEATTLE UNIVERSITY

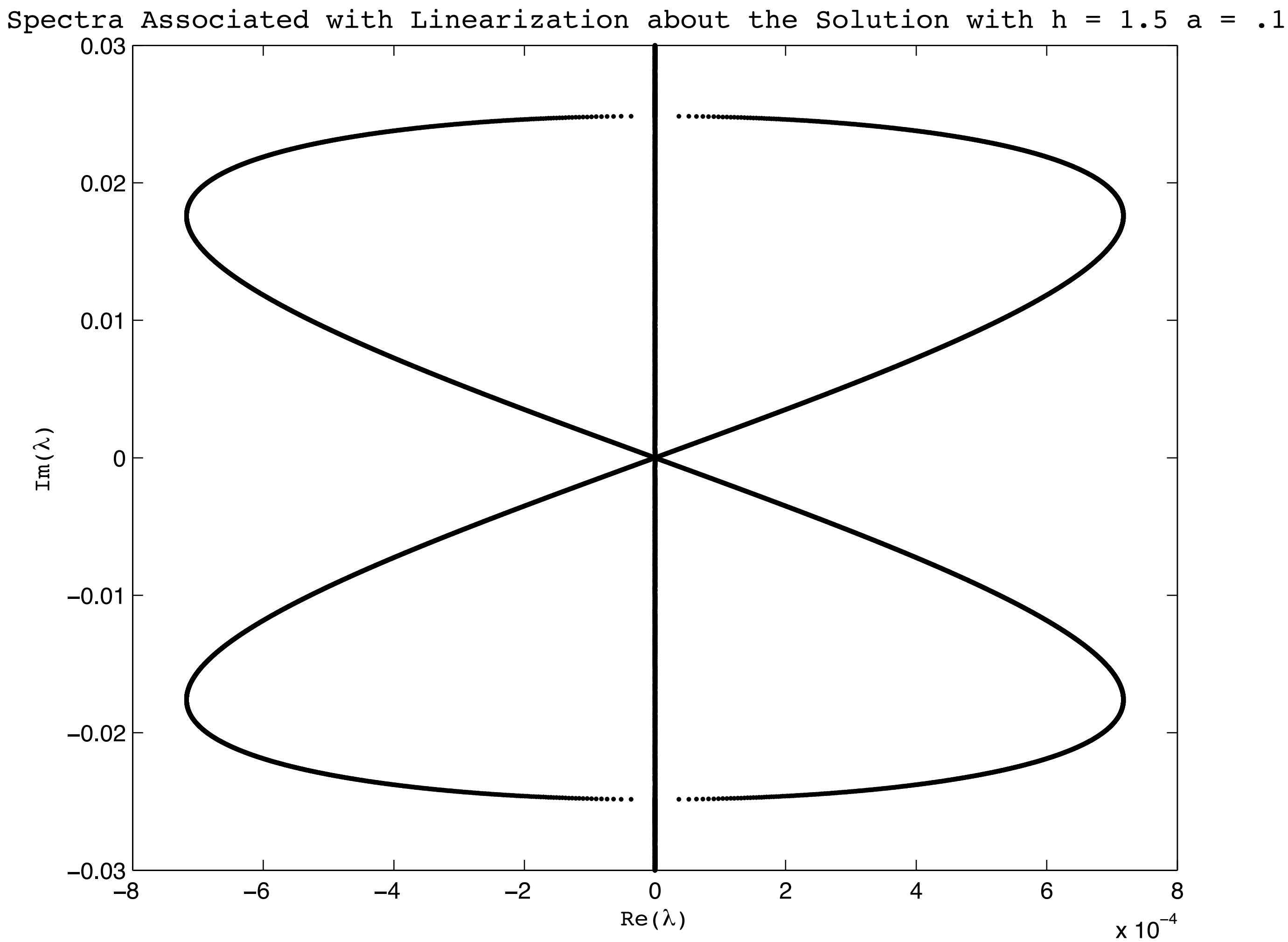
Eigenvalues λ in the complex plane



Nonlinear Schrodinger Equation

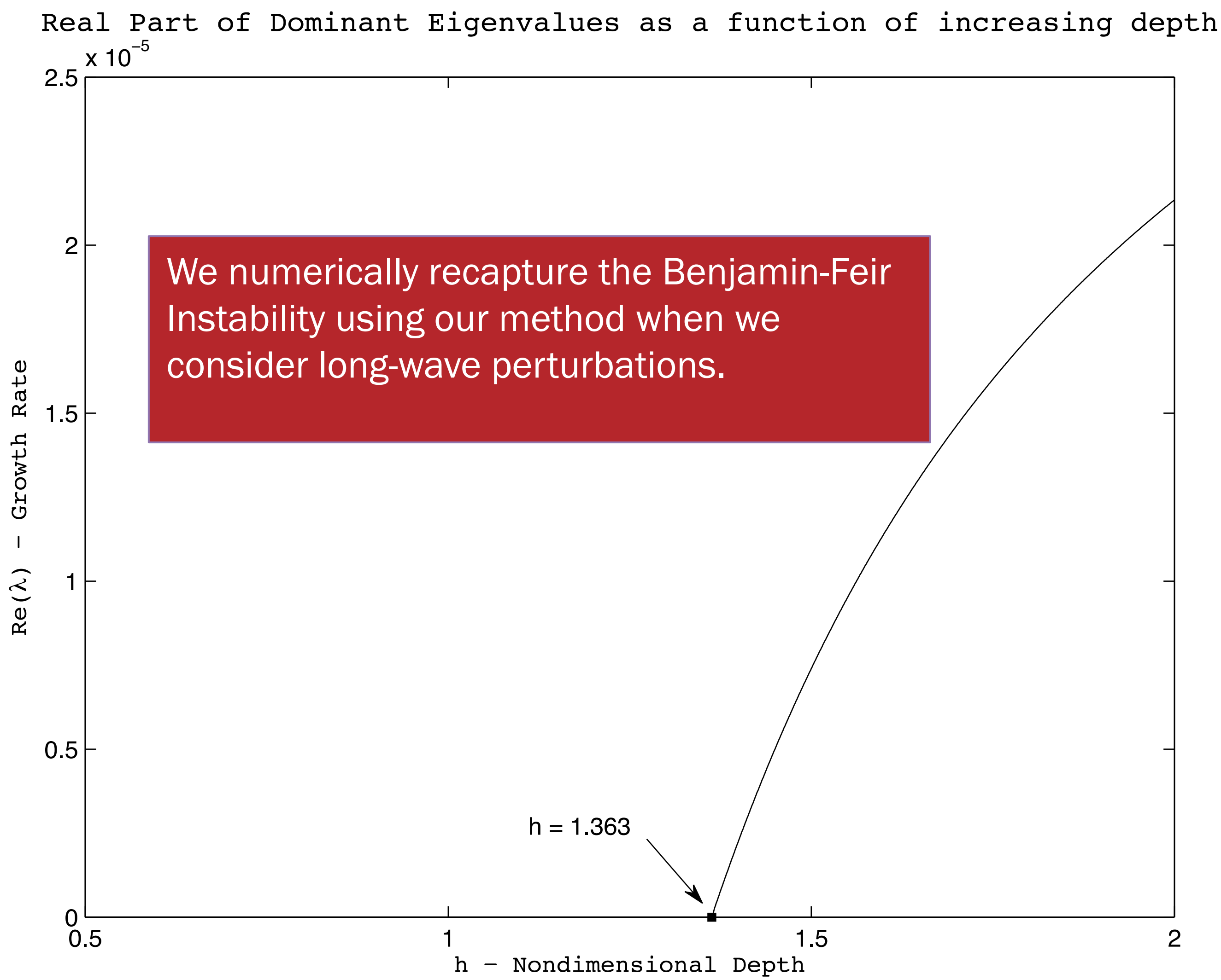
Known to exhibit long-wave instability

FINITE DEPTH EIGENVALUES



kh > 1.363

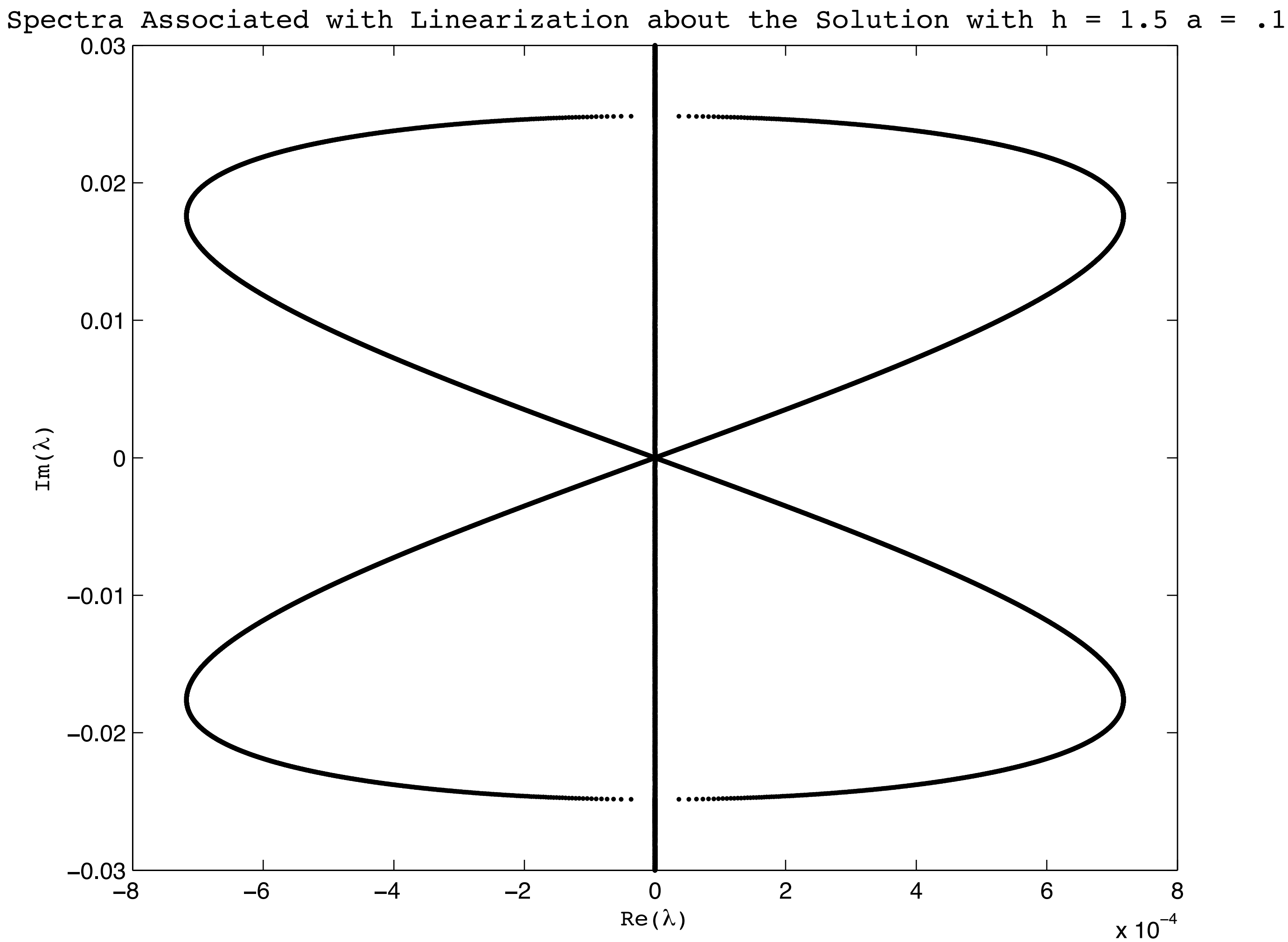
BF INSTABILITY



See Bridges & Mielke for detailed/complete proof

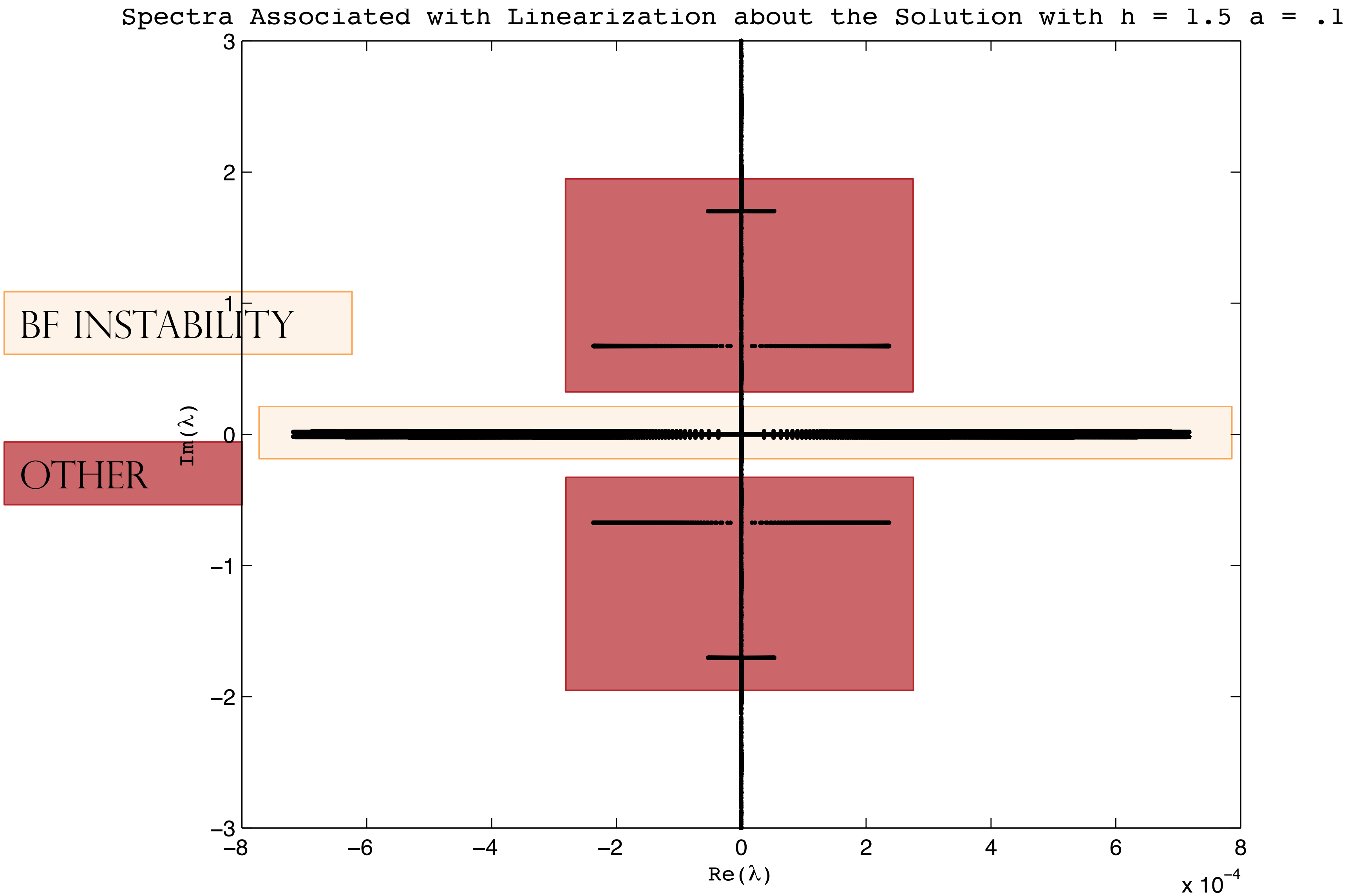
FINITE DEPTH EIGENVALUES

SEATTLE UNIVERSITY

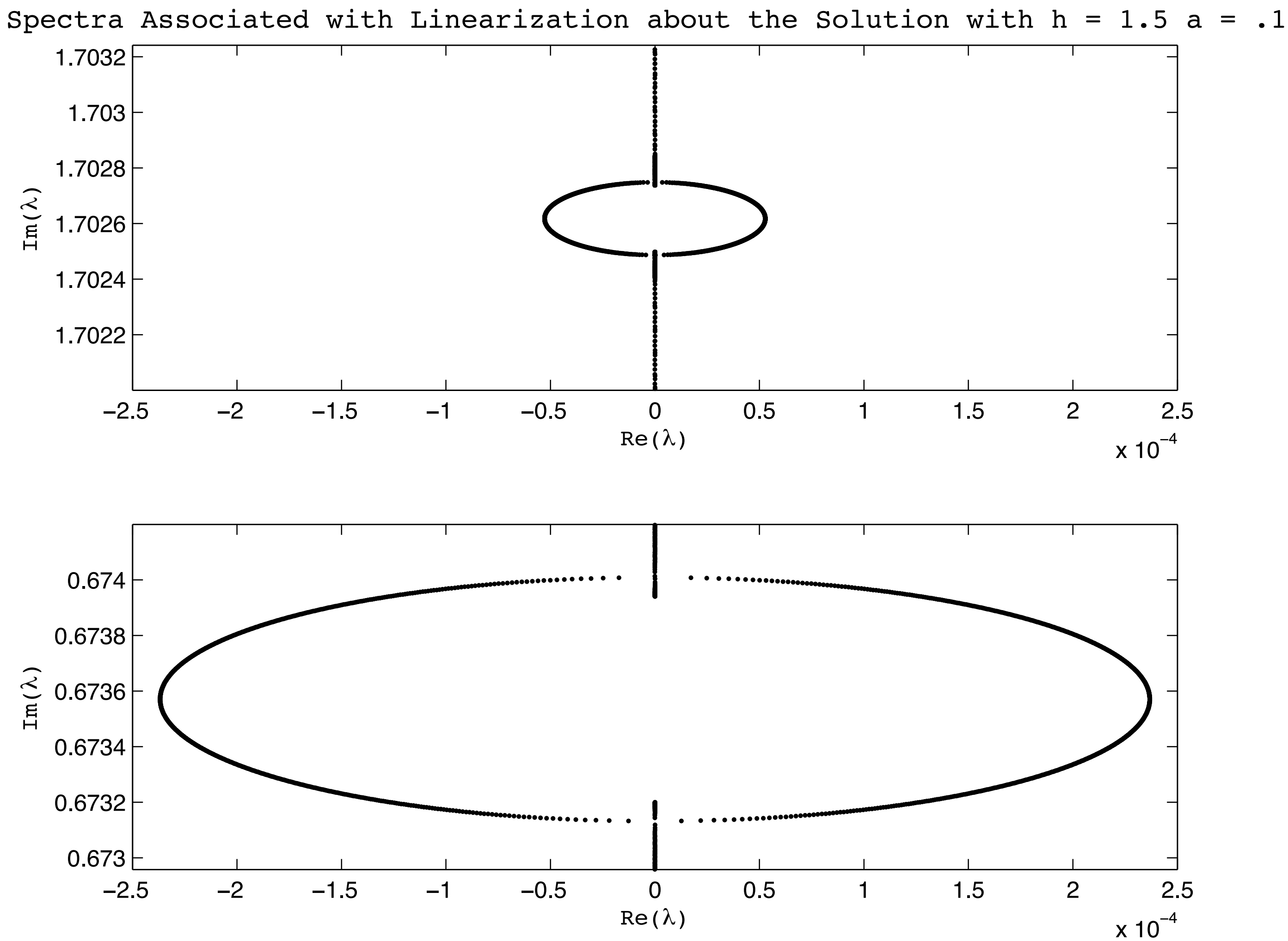


FINITE DEPTH EIGENVALUES

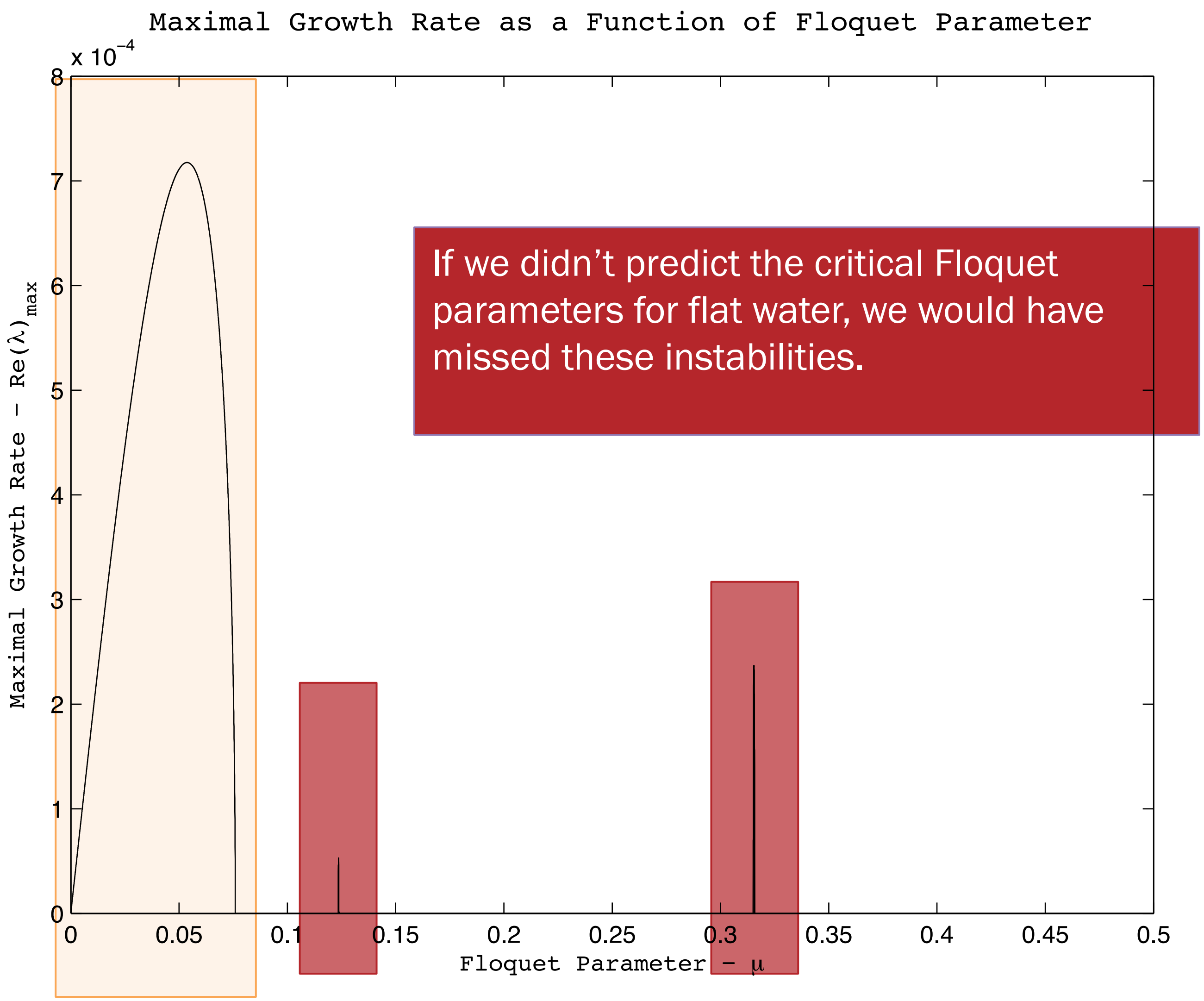
SEATTLE UNIVERSITY



FINITE DEPTH EIGENVALUES



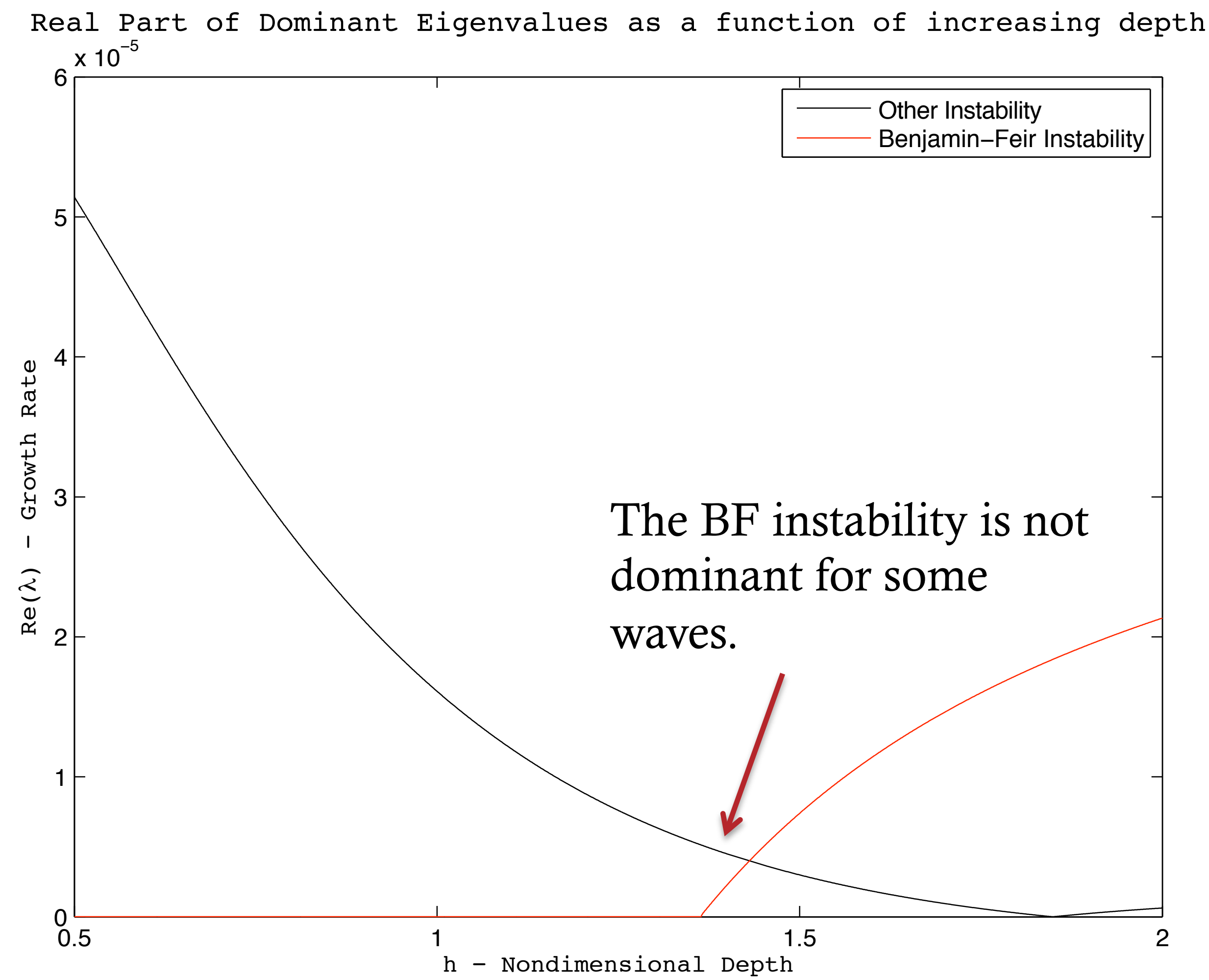
FINITE DEPTH EIGENVALUES



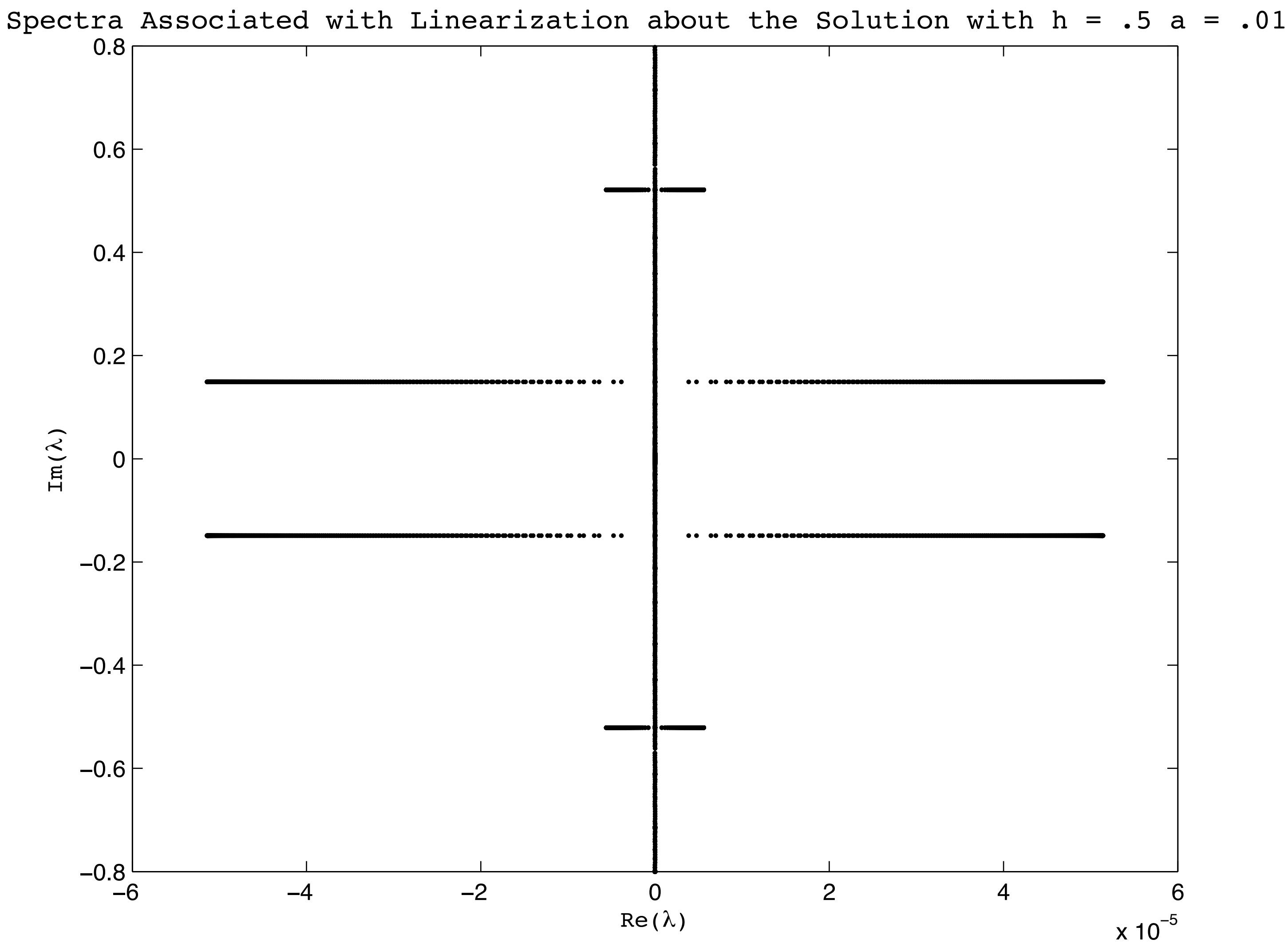
If we didn't predict the critical Floquet parameters for flat water, we would have missed these instabilities.

COMPARISON

SEATTLE UNIVERSITY

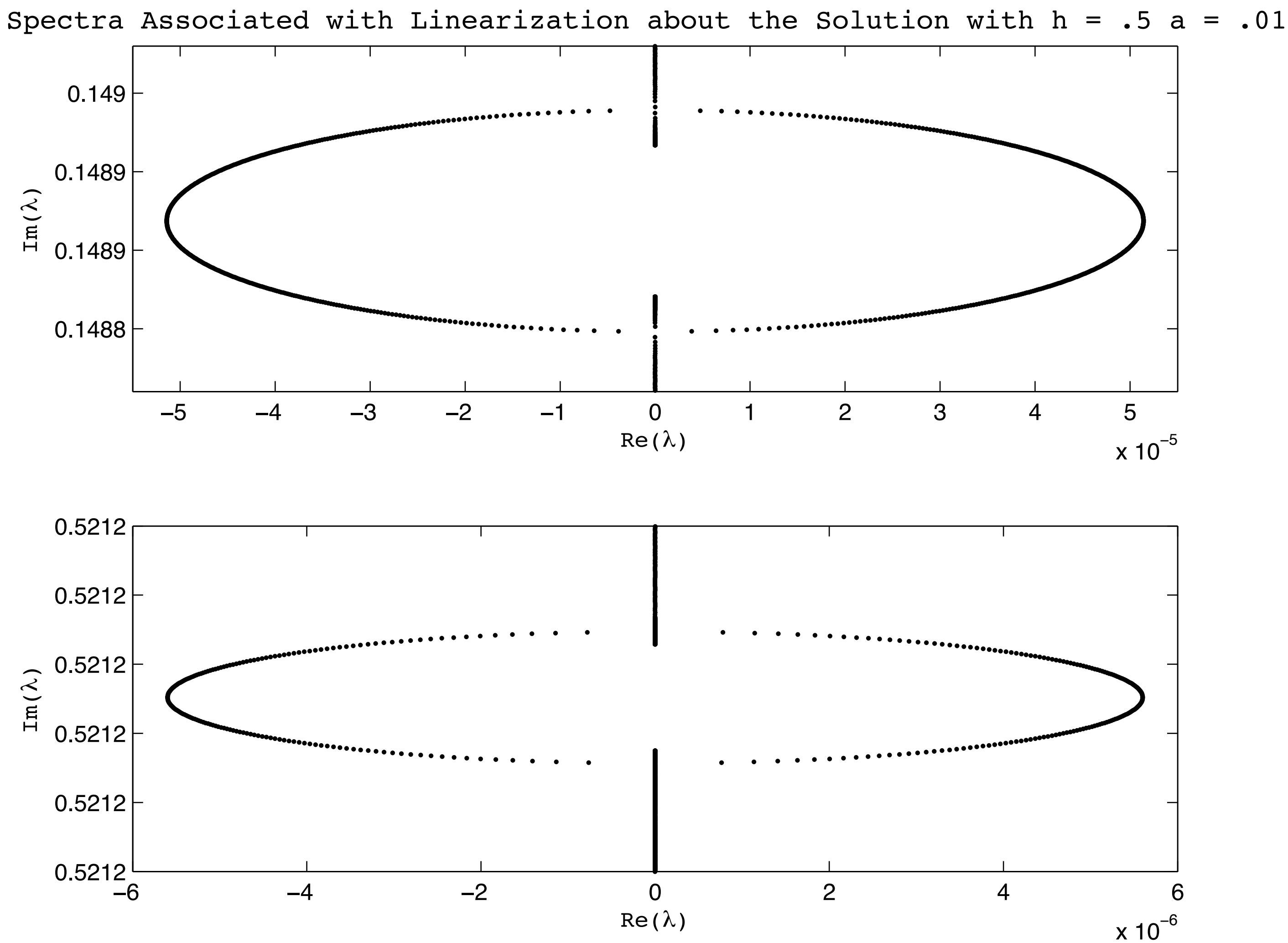


FINITE DEPTH EIGENVALUES

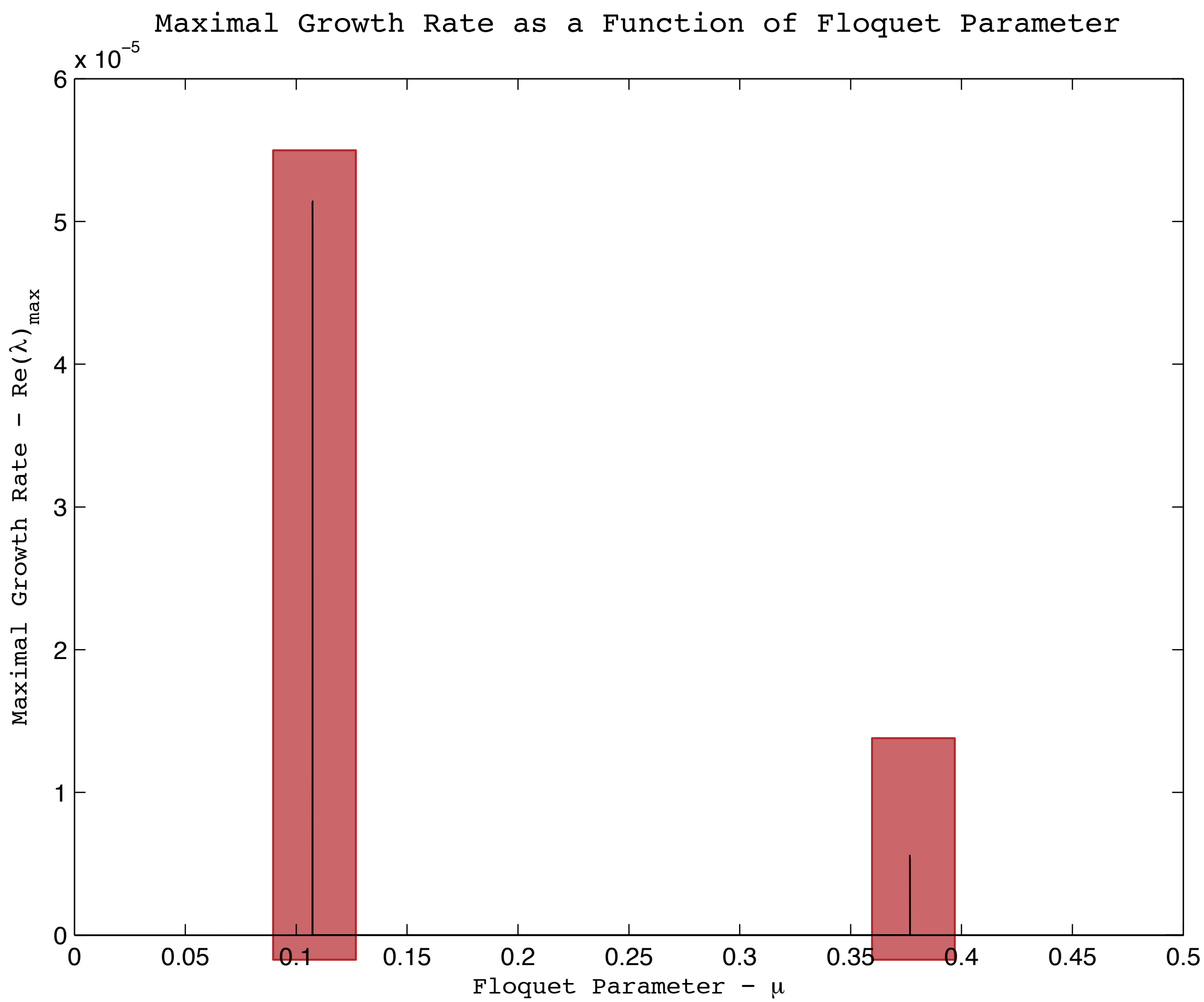


$\approx$ 6% of the limiting wave height

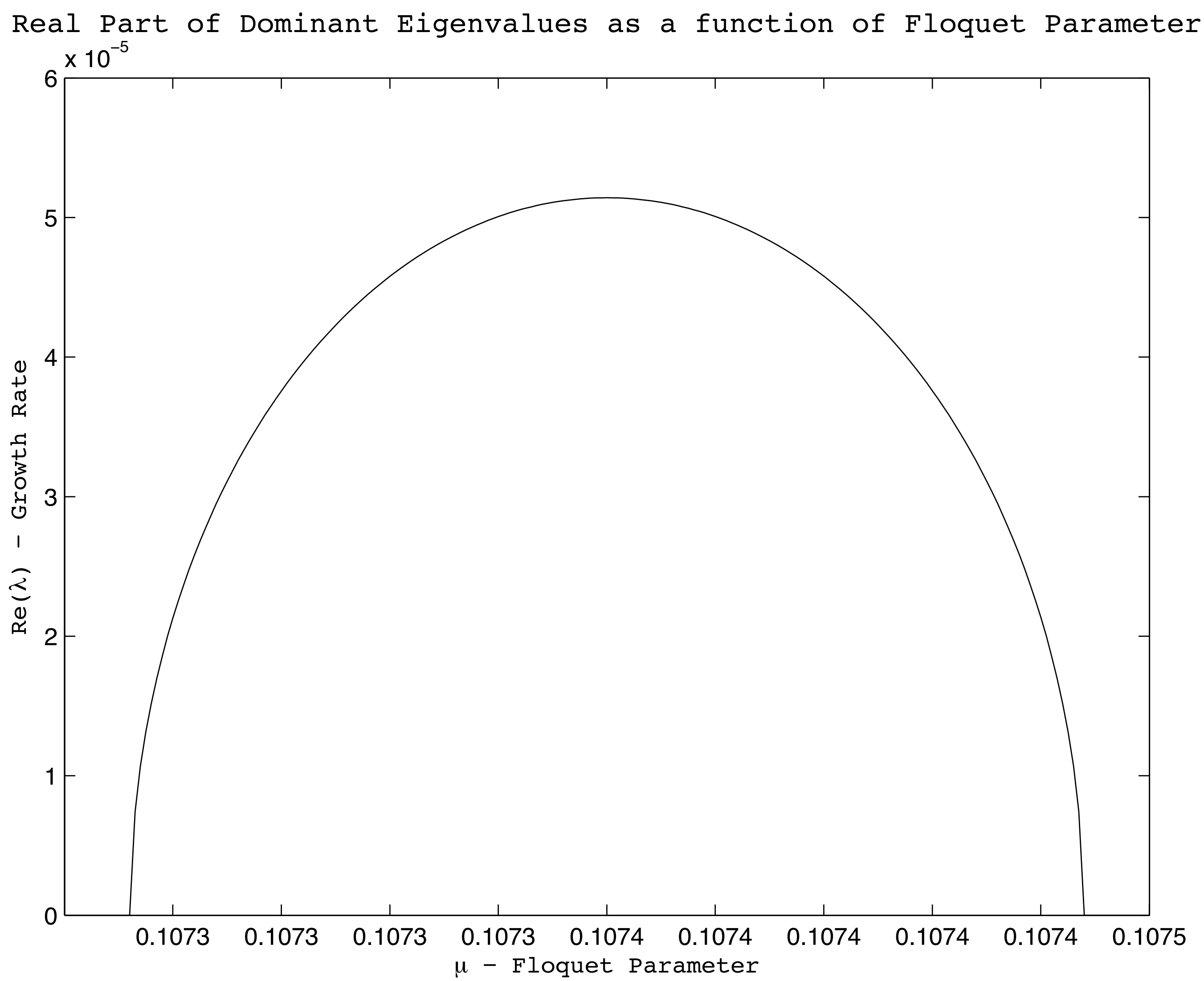
FINITE DEPTH EIGENVALUES



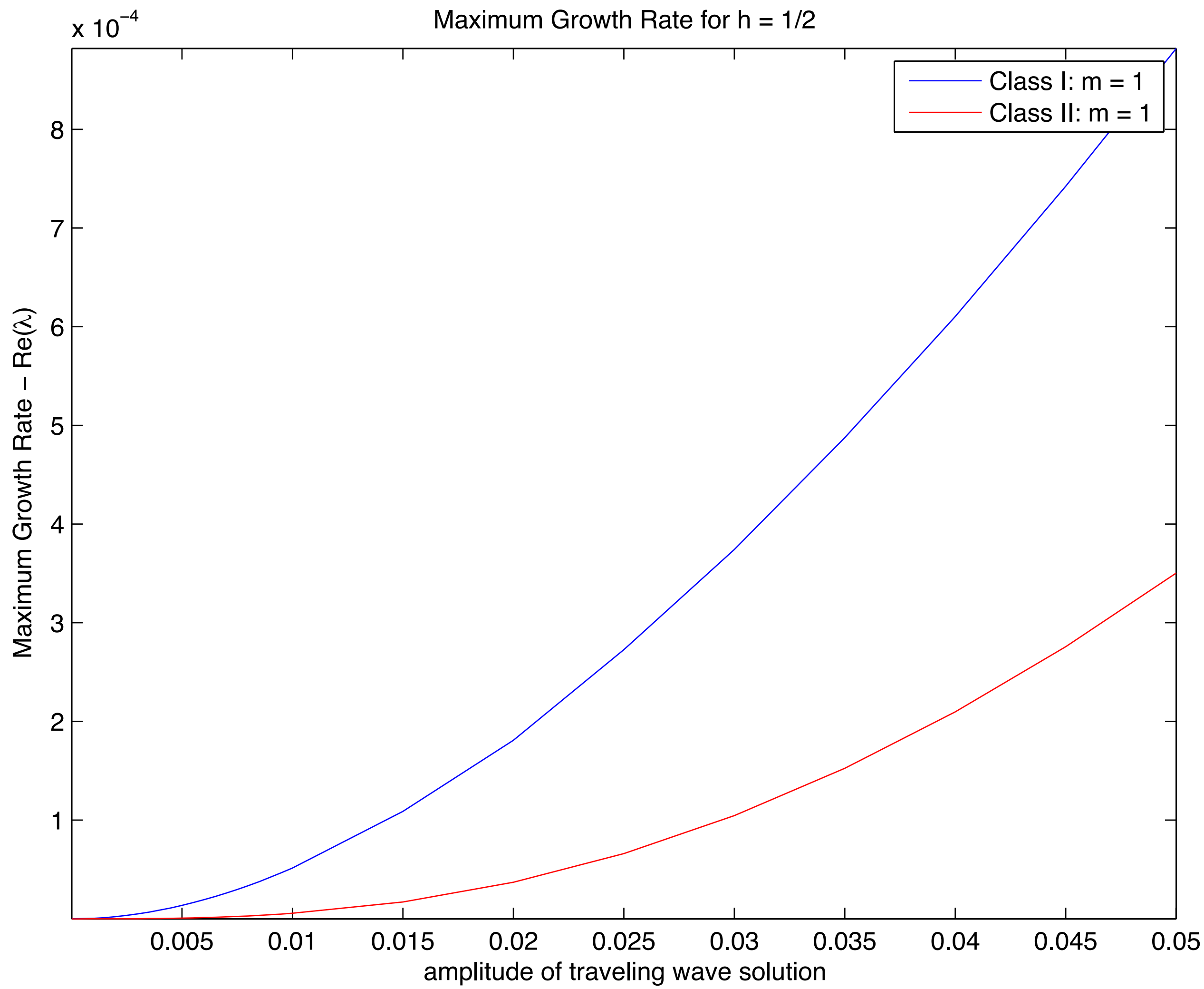
FINITE DEPTH EIGENVALUES



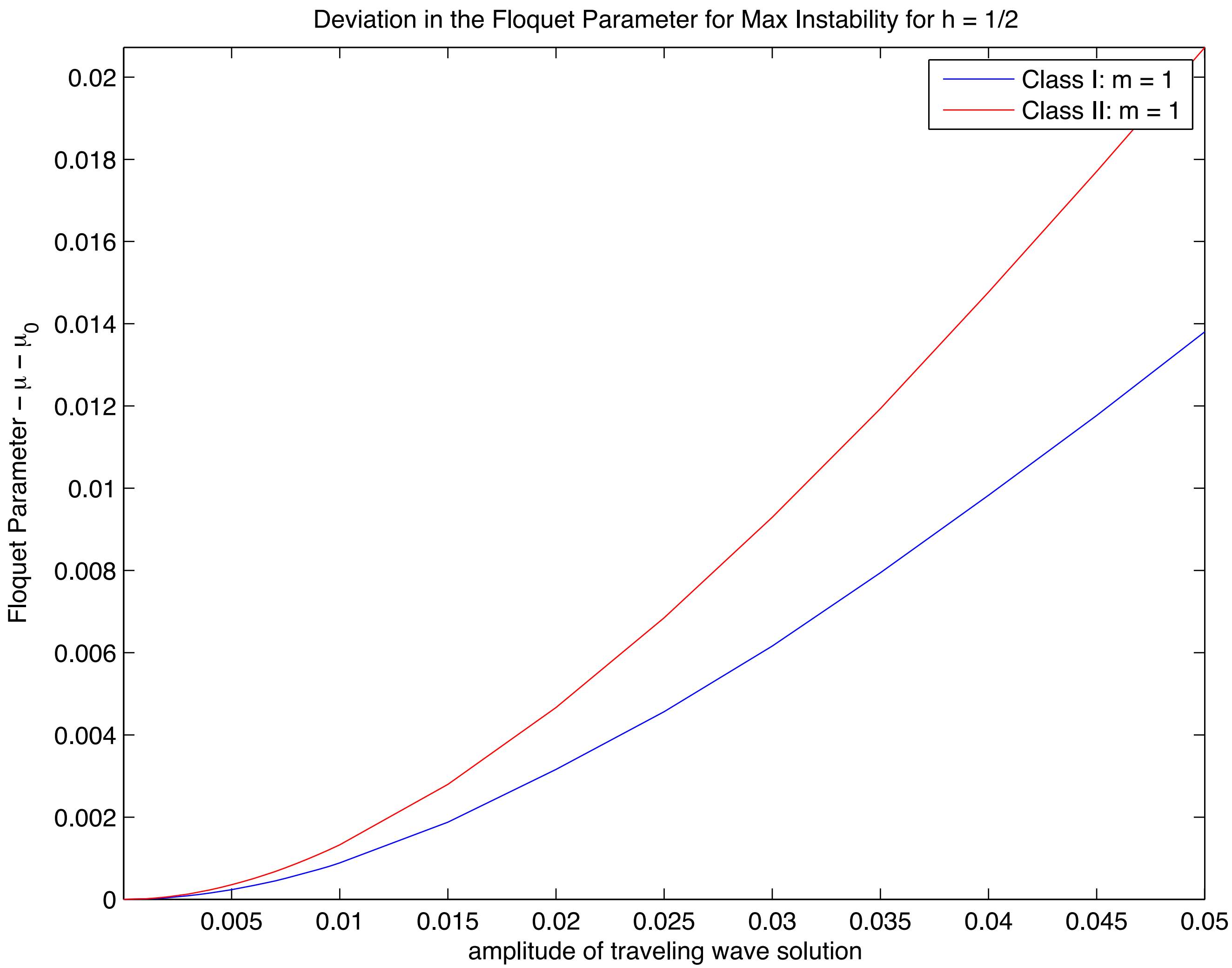
FINITE DEPTH EIGENVALUES



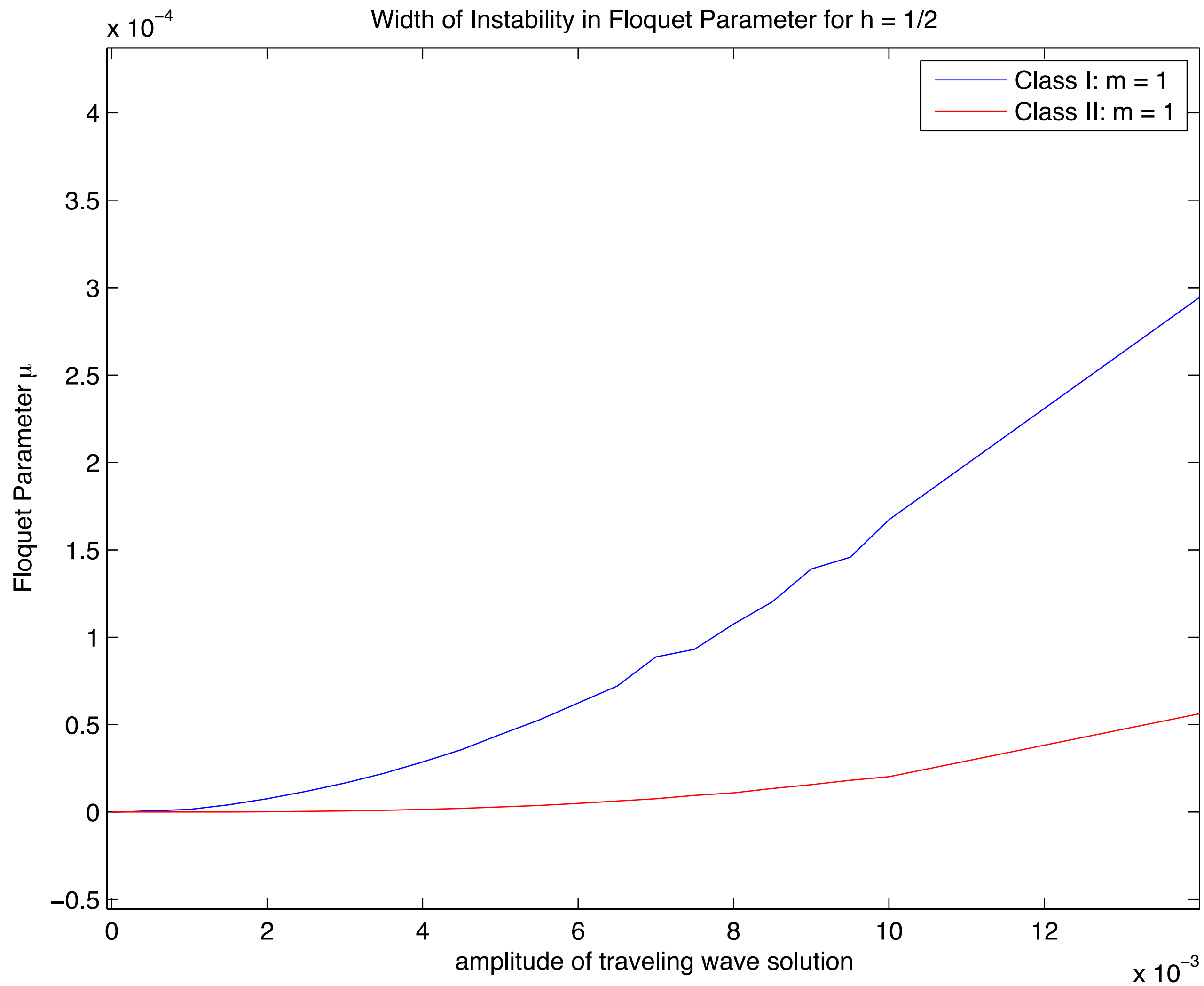
TRACKING THE INSTABILITIES



TRACKING THE INSTABILITIES



TRACKING THE INSTABILITIES



OUTLINE OF THE TALK

SEATTLE UNIVERSITY

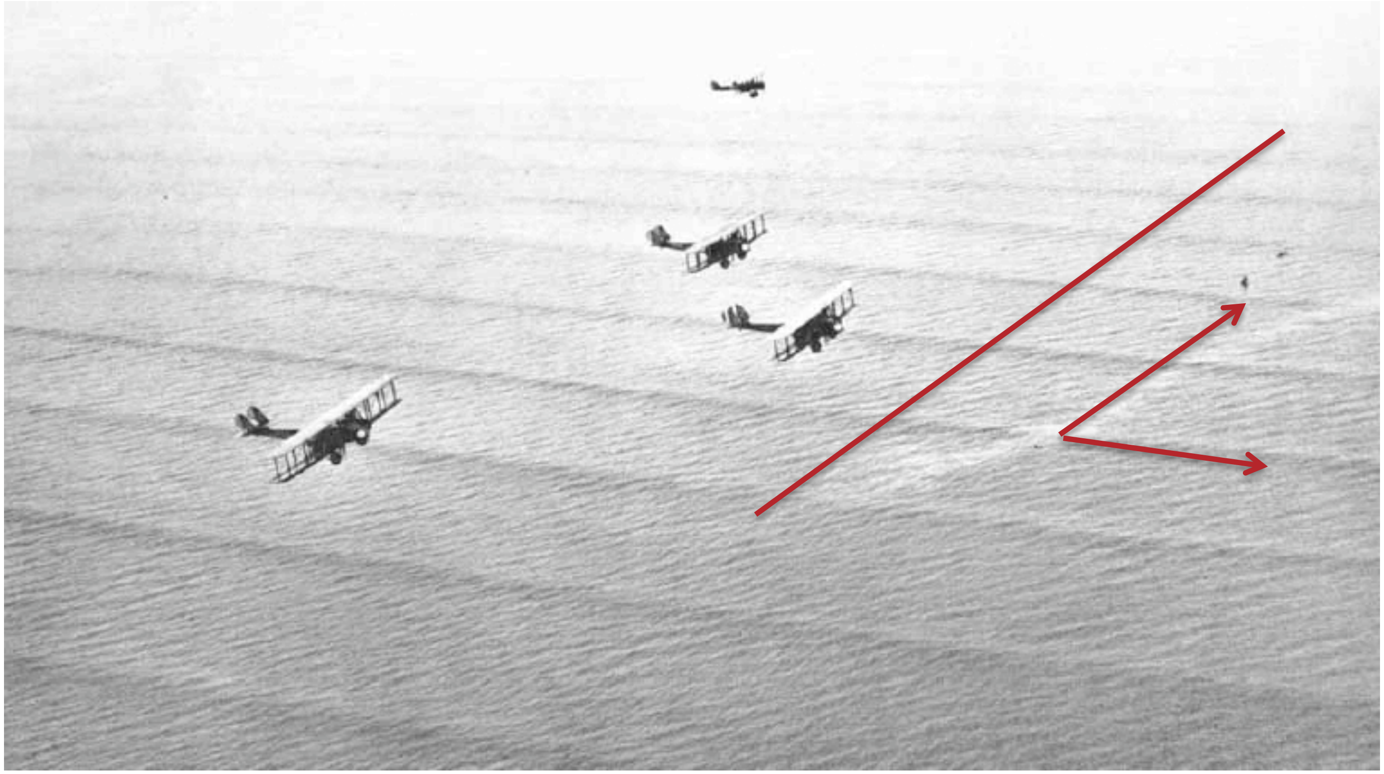
A NEW FORMULATION

TRAVELING WAVE SOLUTIONS

SPECTRAL STABILITY CALCULATIONS

TRANSVERSE STABILITY CALCULATIONS

Transverse Stability



TRANSVERSE STABILITY

We consider perturbations of the form

$$\begin{aligned} \eta(x,t) &= \eta_0(x) + \epsilon\eta_1(x)e^{i\mu x}e^{i\rho y}e^{\lambda t} + \ldots \\ q(x,t) &= q_0(x) + \epsilon q_1(x)e^{i\mu x}e^{i\rho y}e^{\lambda t} + \ldots \end{aligned}$$

Since the only explicit dependence on time is through the exponential term, we will conclude that the wave is spectrally unstable if there is any value of λ which has a positive real part.

Substituting the perturbed solution into the AFM formulation will generate an eigenvalue problem for the stability of our traveling wave solutions.

$$\mathcal{L}_{\mu,\rho}\mathbf{X} = \lambda\, \mathcal{M}_{\mu,\rho}\mathbf{X}$$

Range of the Floquet Parameter

- We only need to consider the range $0 \le \mu \le 0.5$ instead of the full range
- This allows us to reduce the size of the computational domain.

Solving for the flat-water case, we have:

$$\lambda_m^{\pm} = i\left(-c(\mu+m) \pm \sqrt{g\kappa\tanh(\kappa h)}\right), \quad \kappa = \sqrt{(\mu+m)^2+\rho^2}$$

Again, we have similar necessary conditions as before for instability.

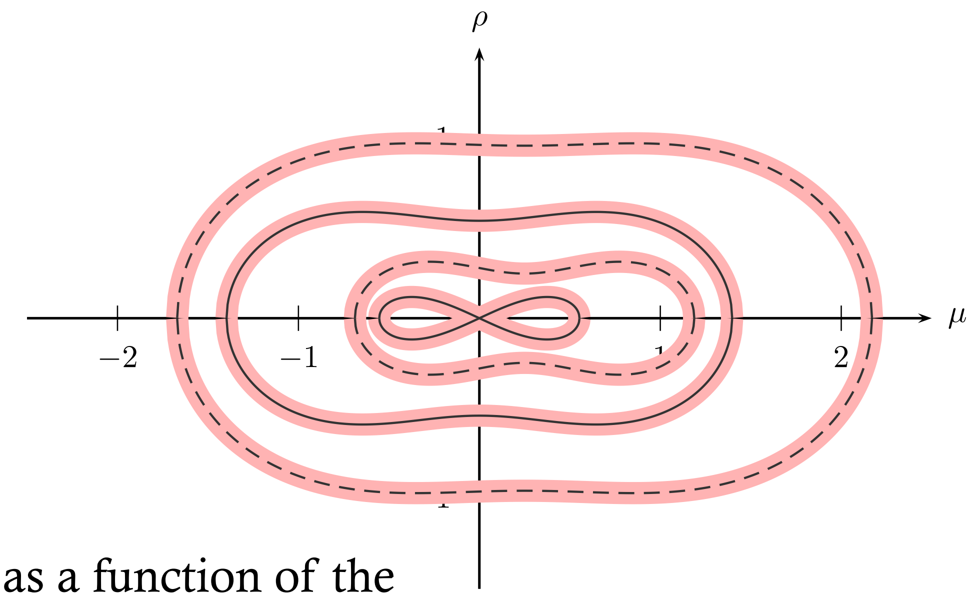
Case I:

$$n = -m$$

Case 2:

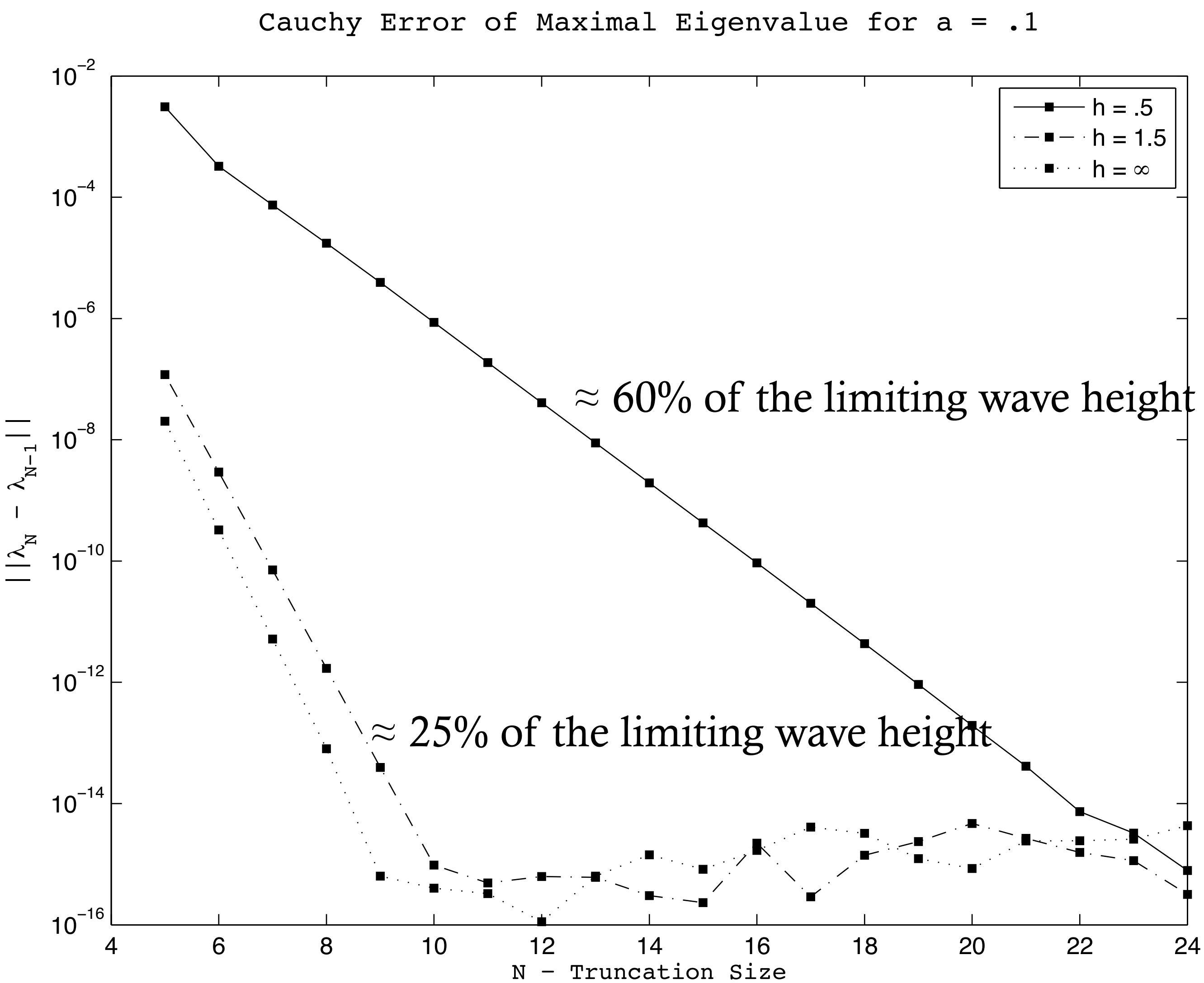
$$n = -m - 1$$

… and so on …

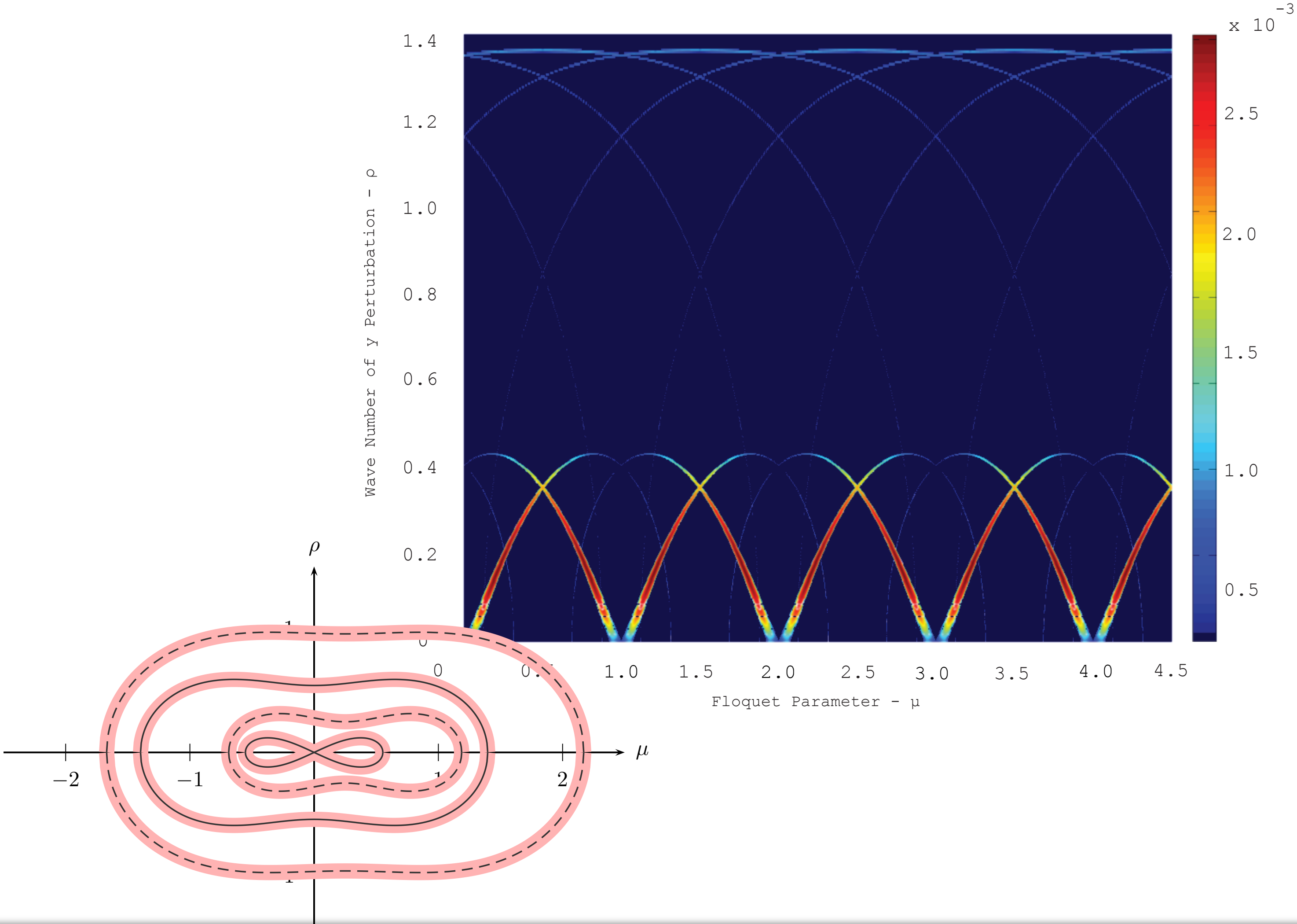


We track the location of instabilities as a function of the amplitude.

CONVERGENCE

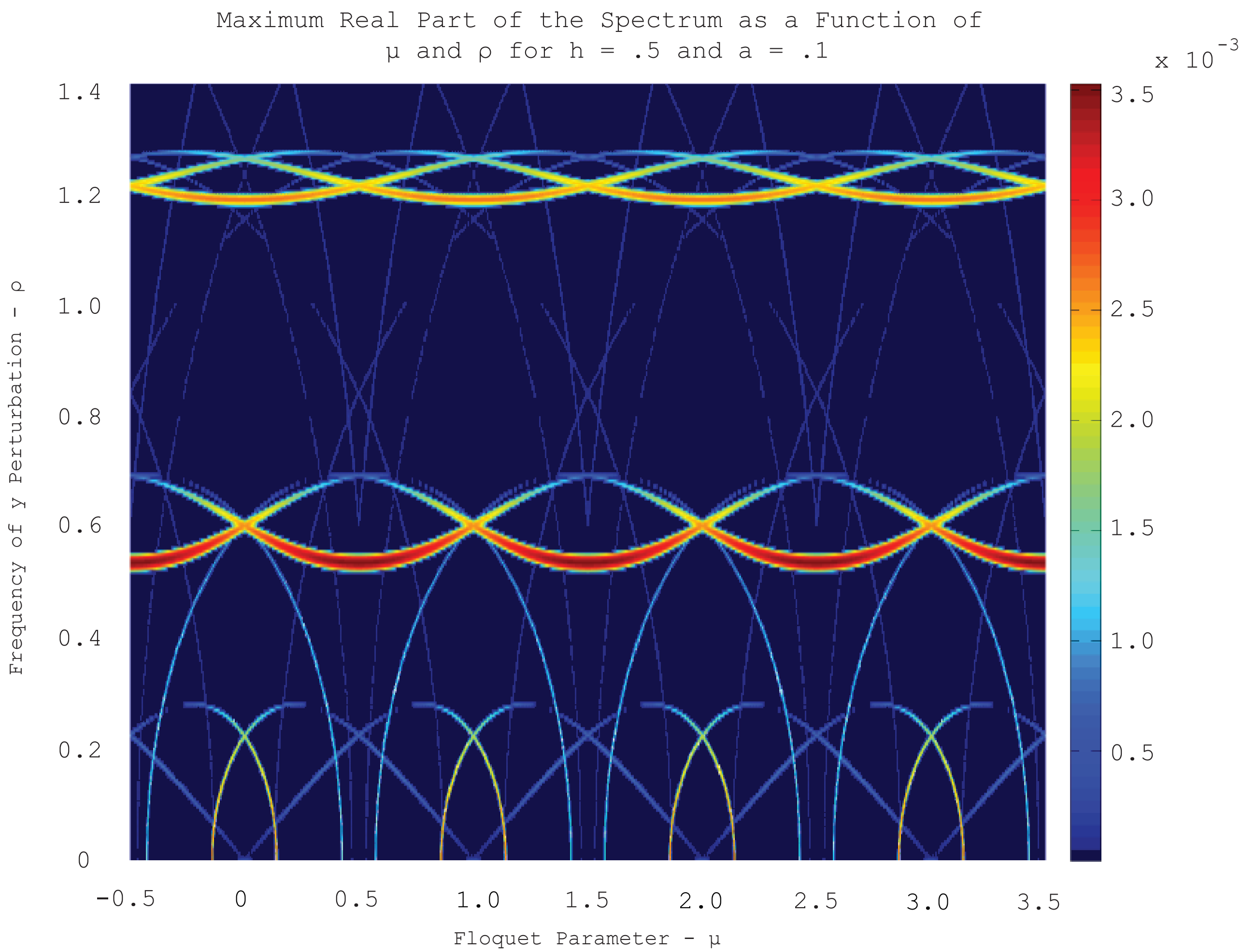


TRANSVERSE STABILITY



TRANSVERSE STABILITY

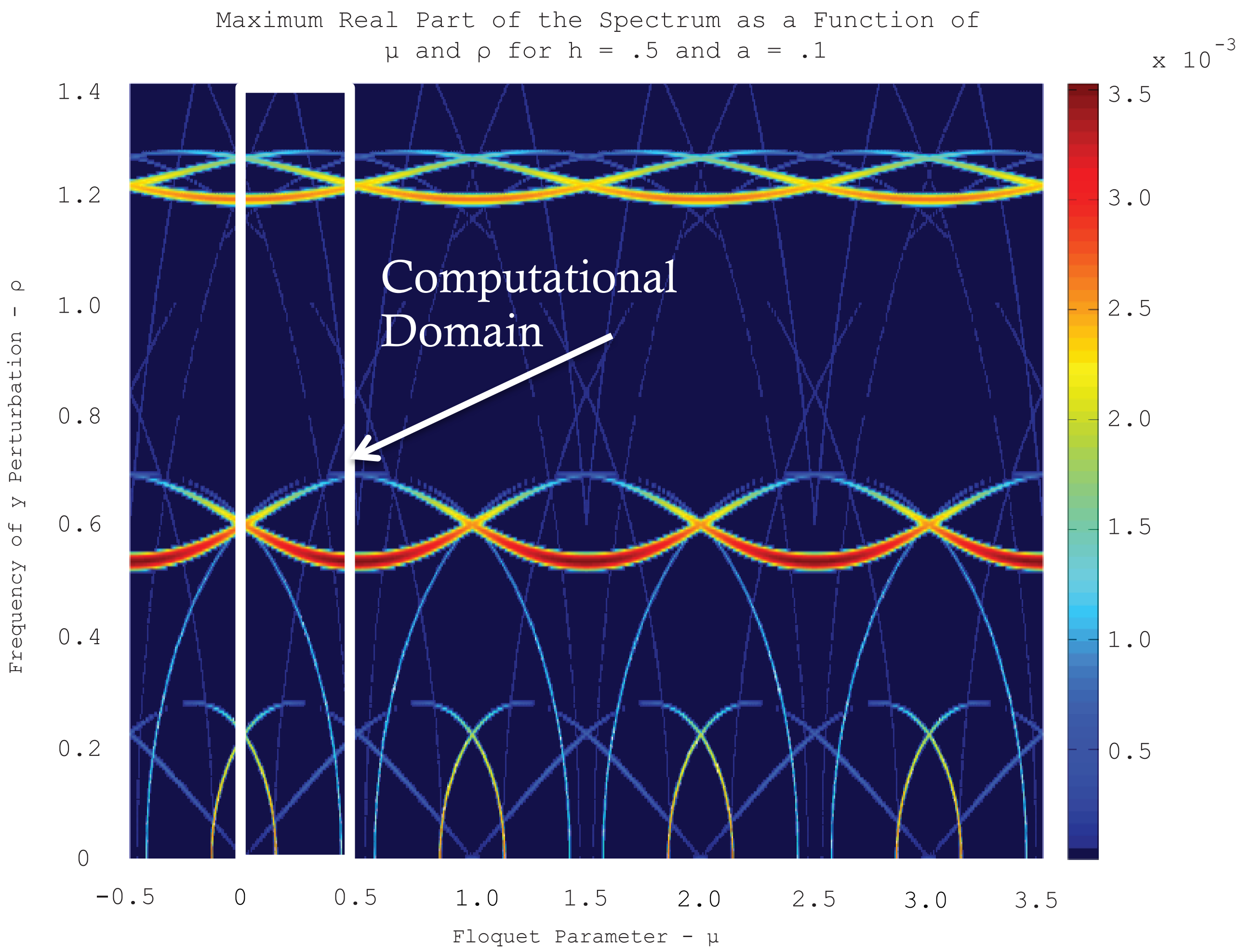
SEATTLE UNIVERSITY



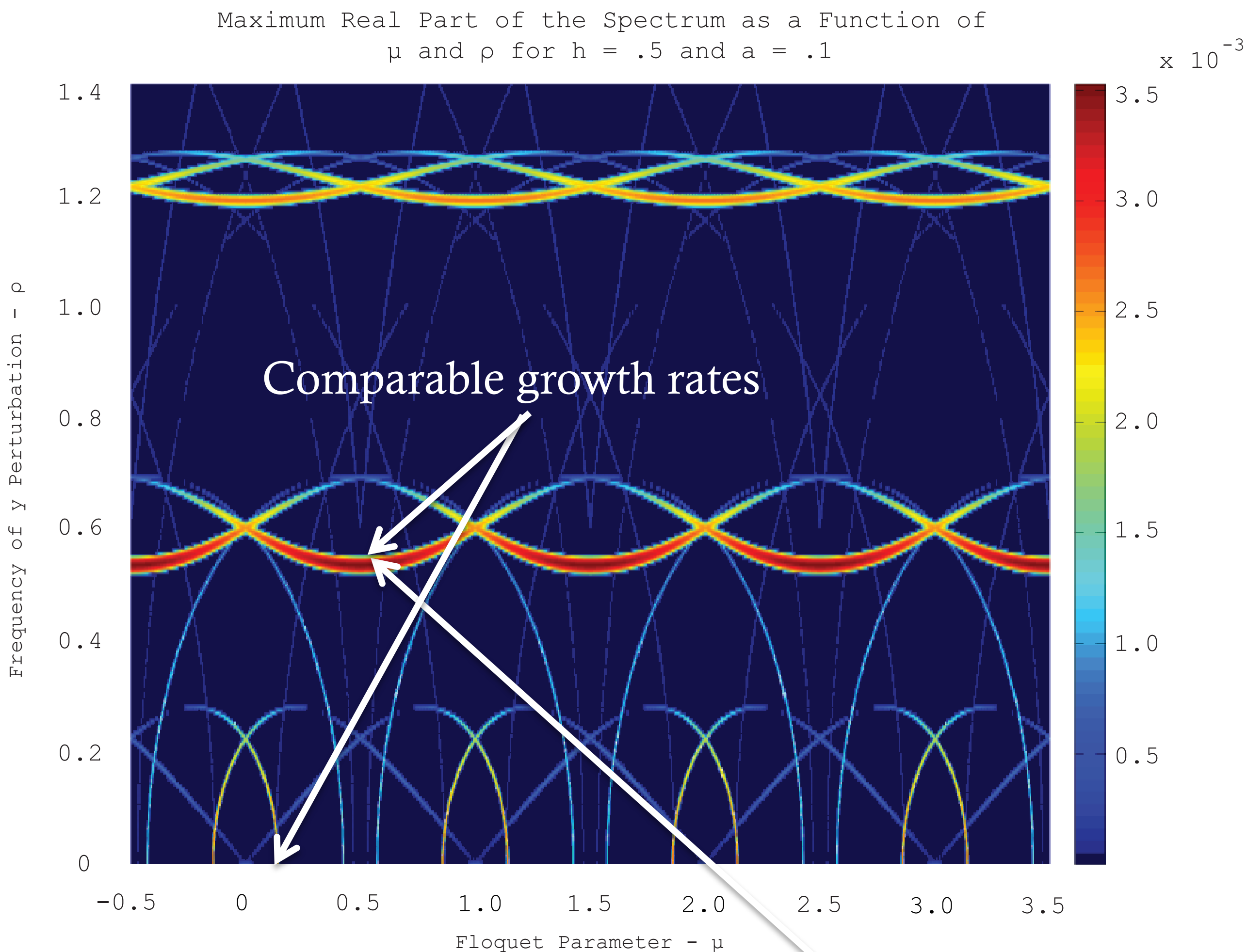
$$\begin{aligned}\eta(x,t) &= \eta_0(x) + \epsilon\eta_1(x)e^{i\mu x}e^{i\rho y}e^{\lambda t} + \dots \\ q(x,t) &= q_0(x) + \epsilon q_1(x)e^{i\mu x}e^{i\rho y}e^{\lambda t} + \dots\end{aligned}$$

TRANSVERSE STABILITY

SEATTLE UNIVERSITY



TRANSVERSE STABILITY

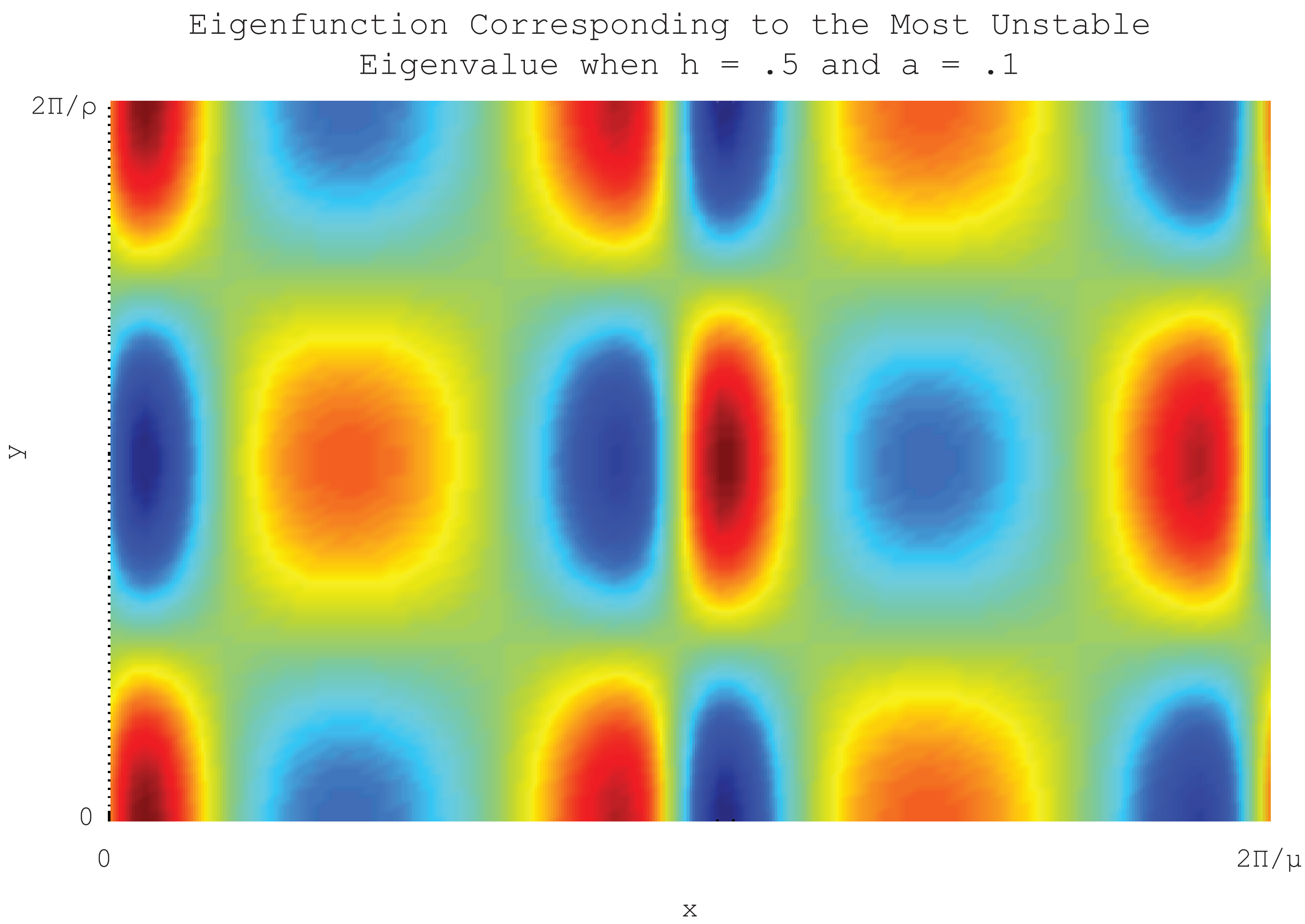


Maximal Instability at $\mu = .5$

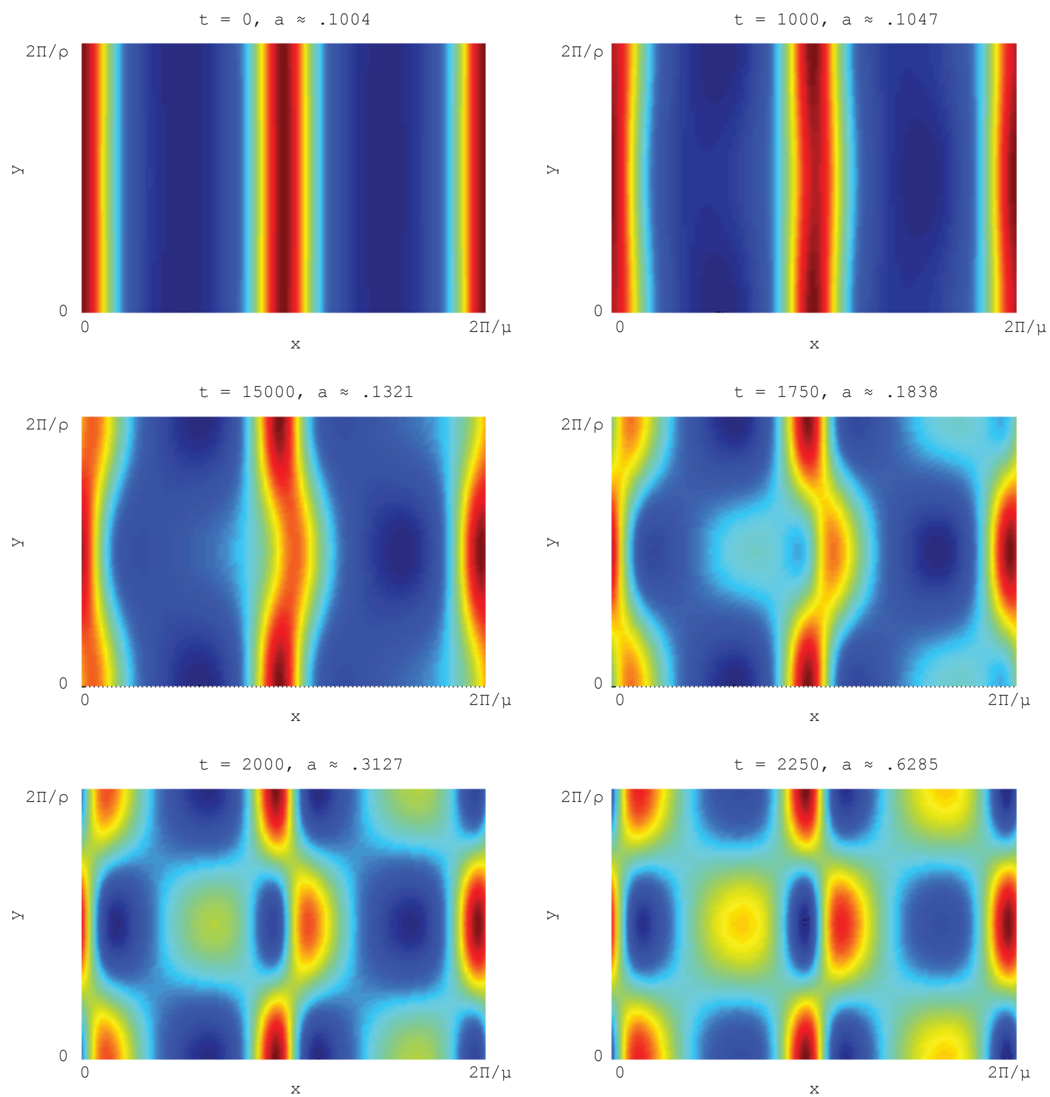
Skip to conclusions

UNSTABLE EIGENFUNCTION

SEATTLE UNIVERSITY



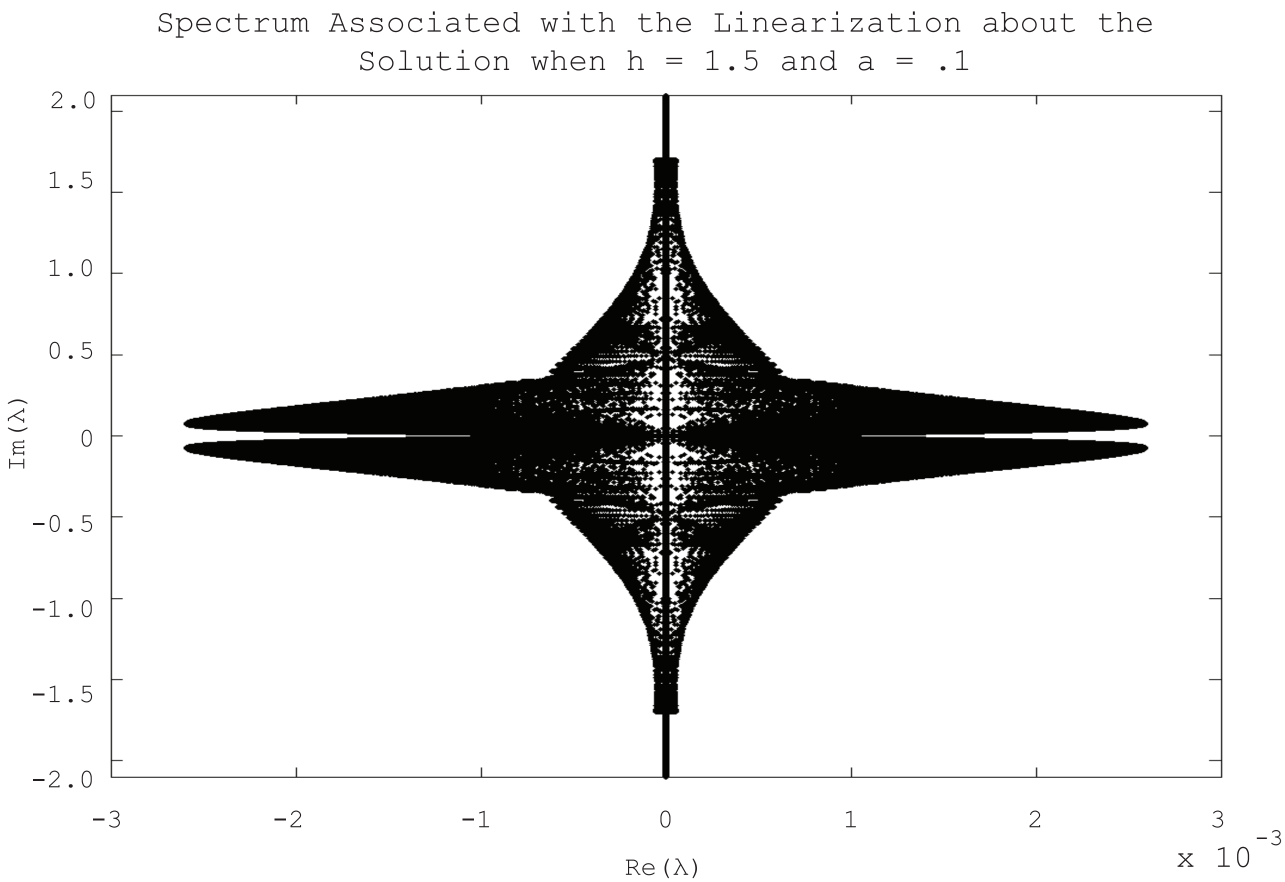
LINEAR TIME EVOLUTION



Shallow Water (h = 0.5)

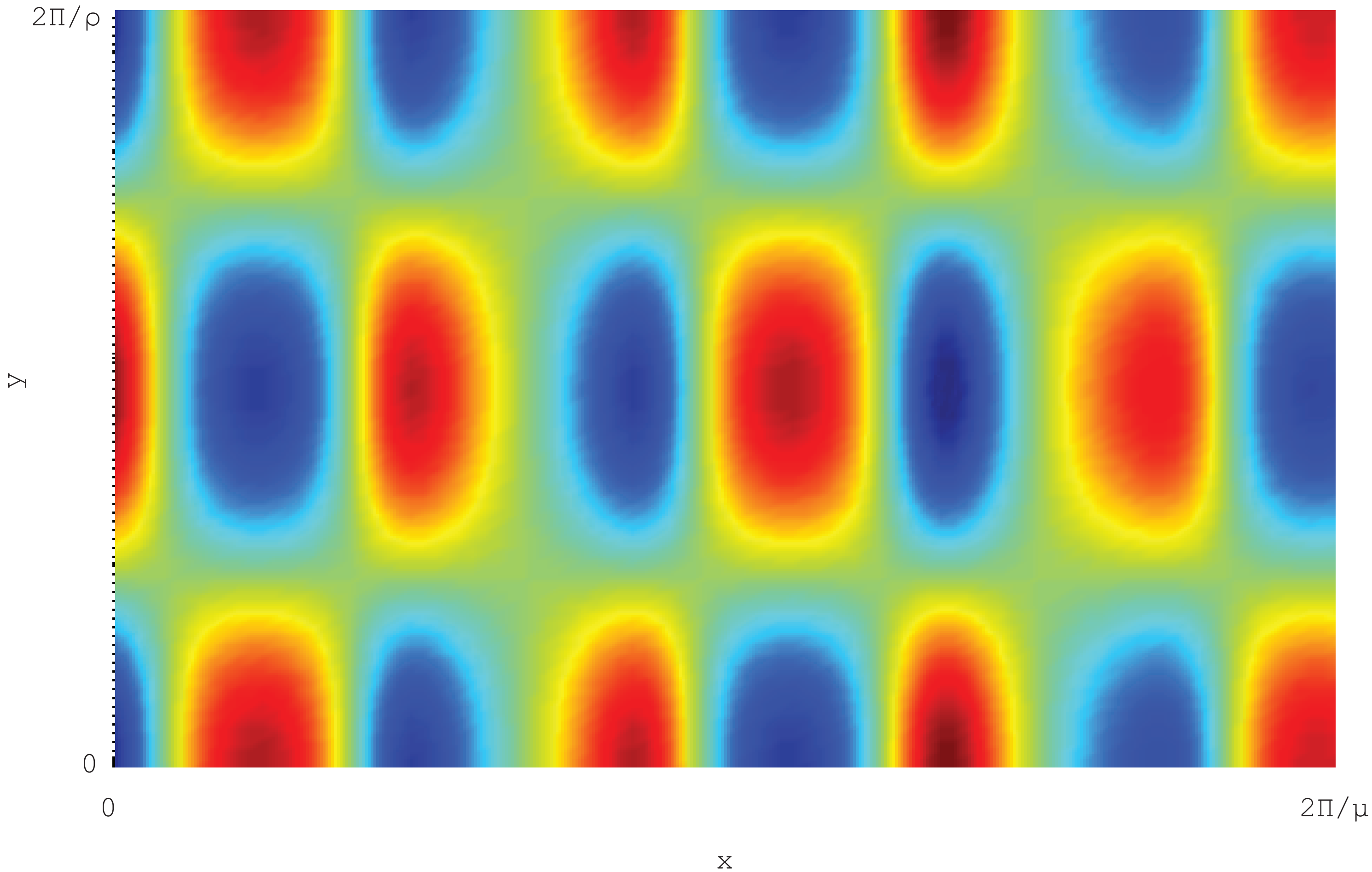
TRANSVERSE STABILITY

SEATTLE UNIVERSITY

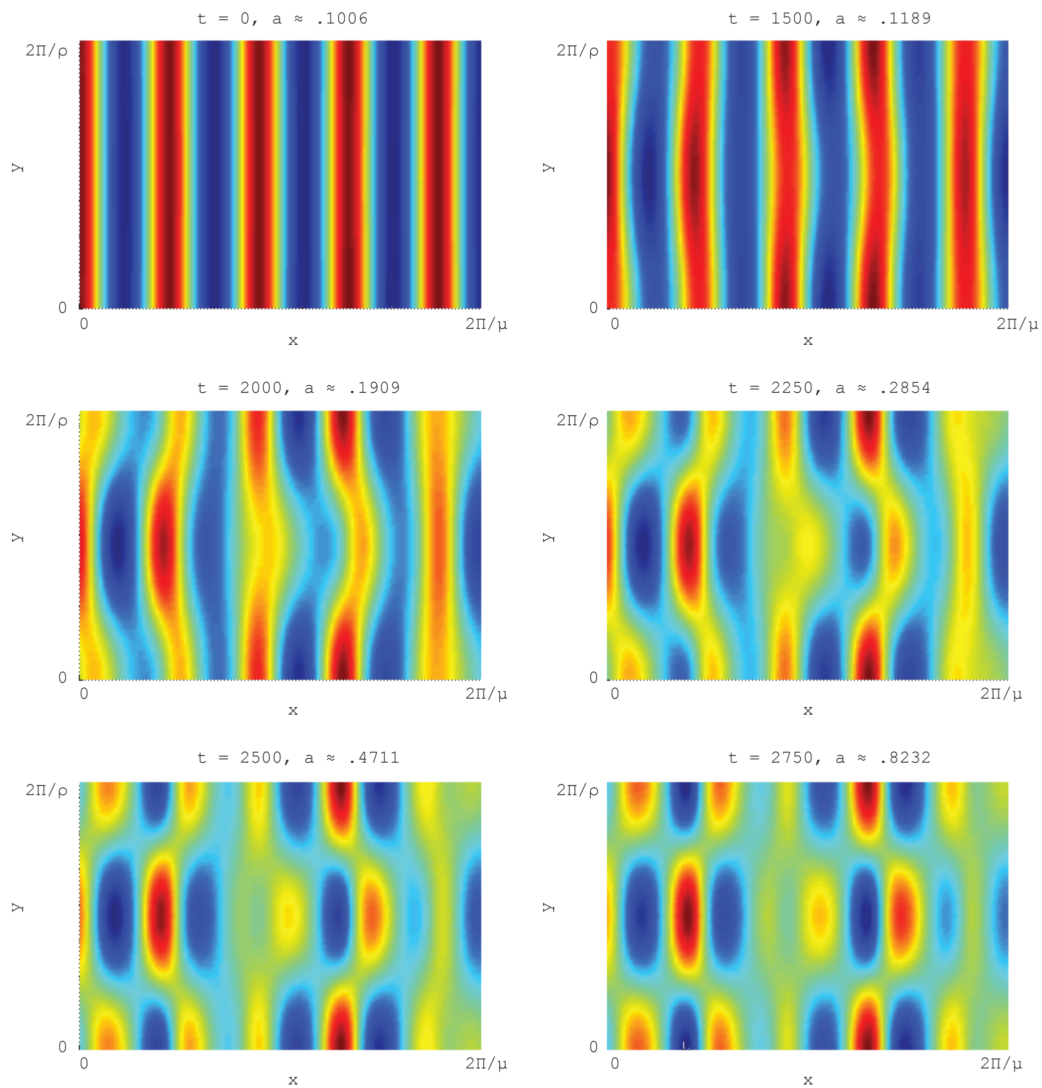


UNSTABLE EIGENFUNCTION

Eigenfunction Corresponding to the Most Unstable Eigenvalue when h = 1.5 and a = .1

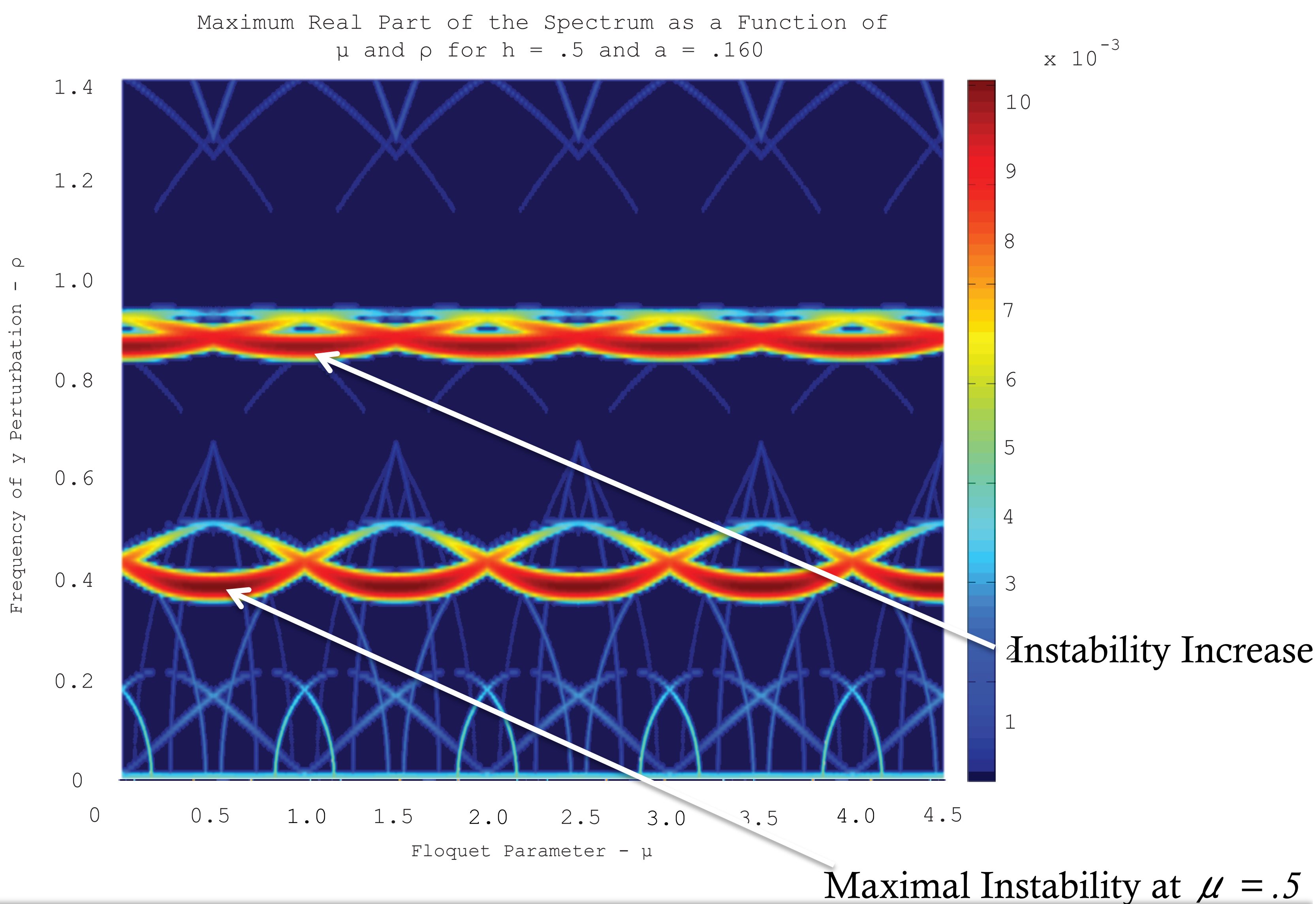


LINEAR TIME EVOLUTION



Deep Water (h = 1.5)

TRANSVERSE STABILITY



CONCLUSION

SEATTLE UNIVERSITY

- Using the AFM formulation, we developed a new single equation for traveling wave solutions to Euler's Equations.
- We are able to capture the Benjamin-Feir Instability at precisely the depth predicted by the theory (See Bridges & Mielke 1995).
- We see that waves in shallow water ($h < 1.363$) are unstable with respect to narrow bands of perturbations.
  - We find these instabilities for very small amplitudes which are not oblique.
  - These instabilities are not captured by many commonly used shallow water equations with the exception of Serre Equations (Carter & Cienfuegos).
- Even for small amplitude solutions in deep water, the Benjamin-Feir instability might not be dominant.
- For transverse perturbations, our results are in good general agreement with previously known results.

THANK YOU!

Thank you for your attention.

OUTLINE OF THE TALK

LAGNIAPPE

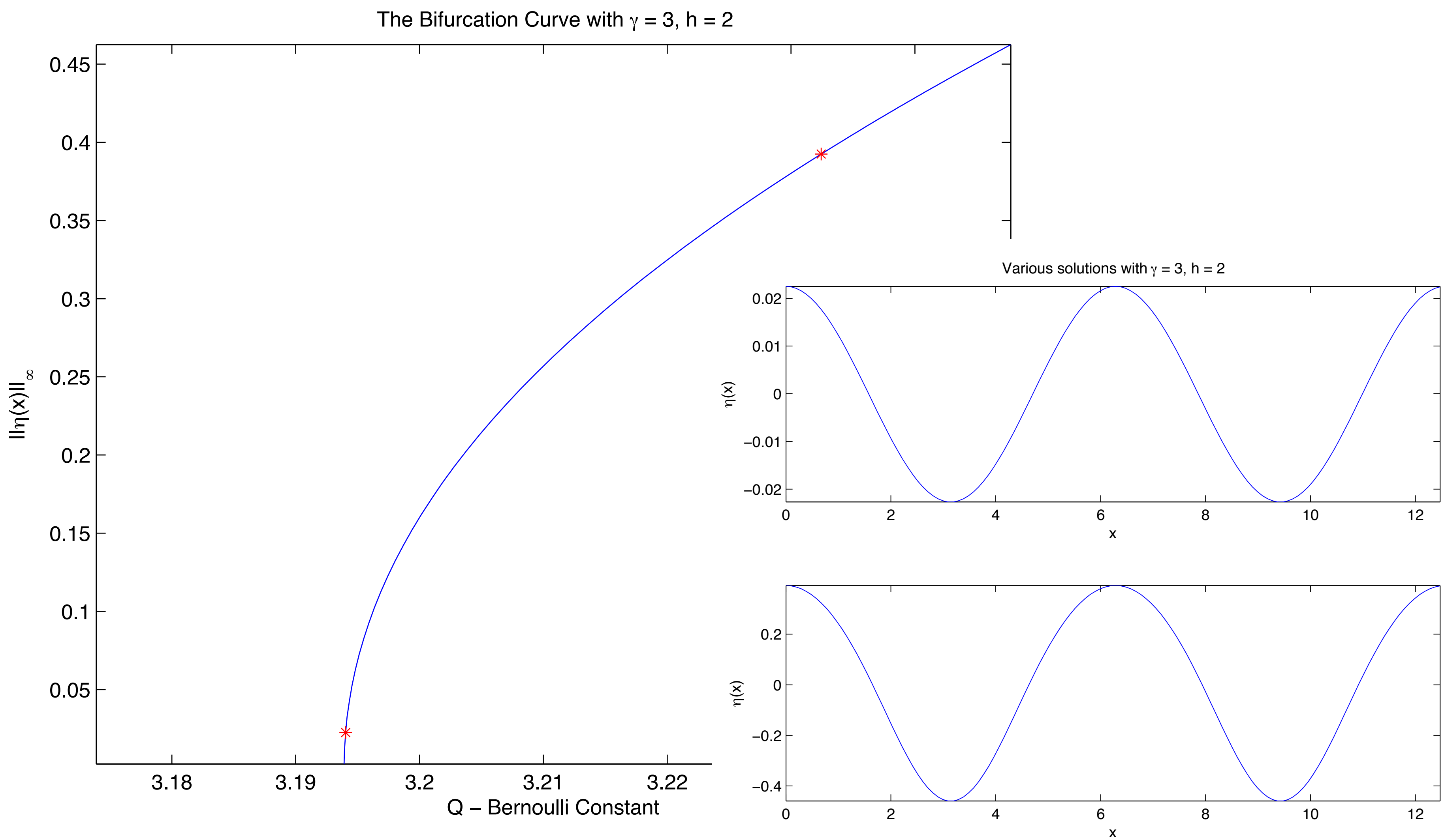
CONSTANT VORTICITY

The full water-wave for traveling waves with constant vorticity can be reduced to solving the following single equation for the free-surface variable $\eta(x)$.

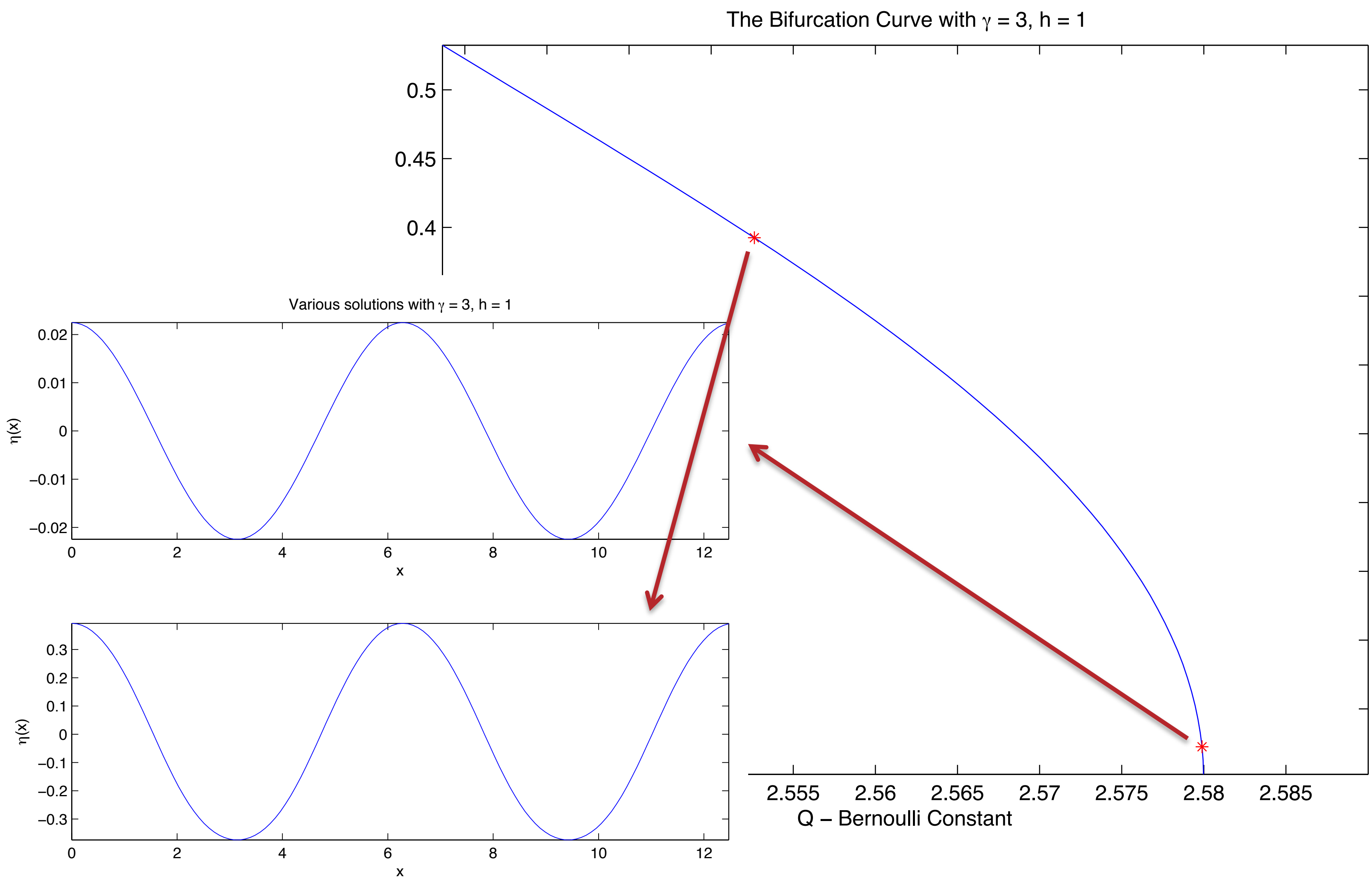
$$\int_0^{2\pi} e^{-ikx}\left(k\sqrt{(\tilde{Q}-2g\eta)(1+\eta_x^2)}\,\sinh(k(\eta+h))-\gamma\cosh(k(\eta+h))\right)\,dx=0.$$

To solve the above equation, we use a numerical continuation scheme where we choose γ, and solve for $\eta(x)$, and $\tilde{Q}$, by controlling some appropriate orthogonally condition or norm on the solution.

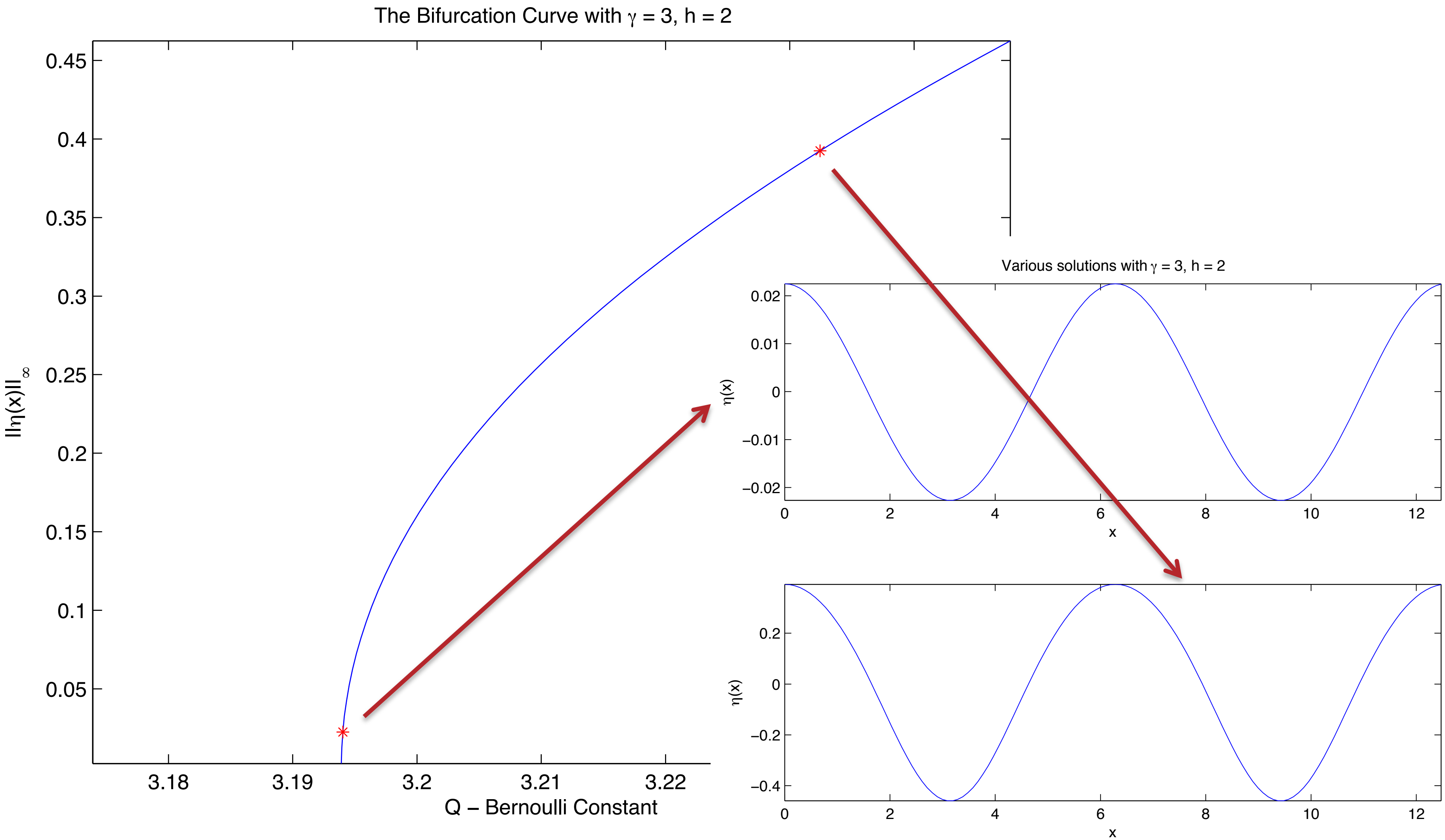
PLOTS OF SOLUTIONS



PLOTS OF SOLUTIONS

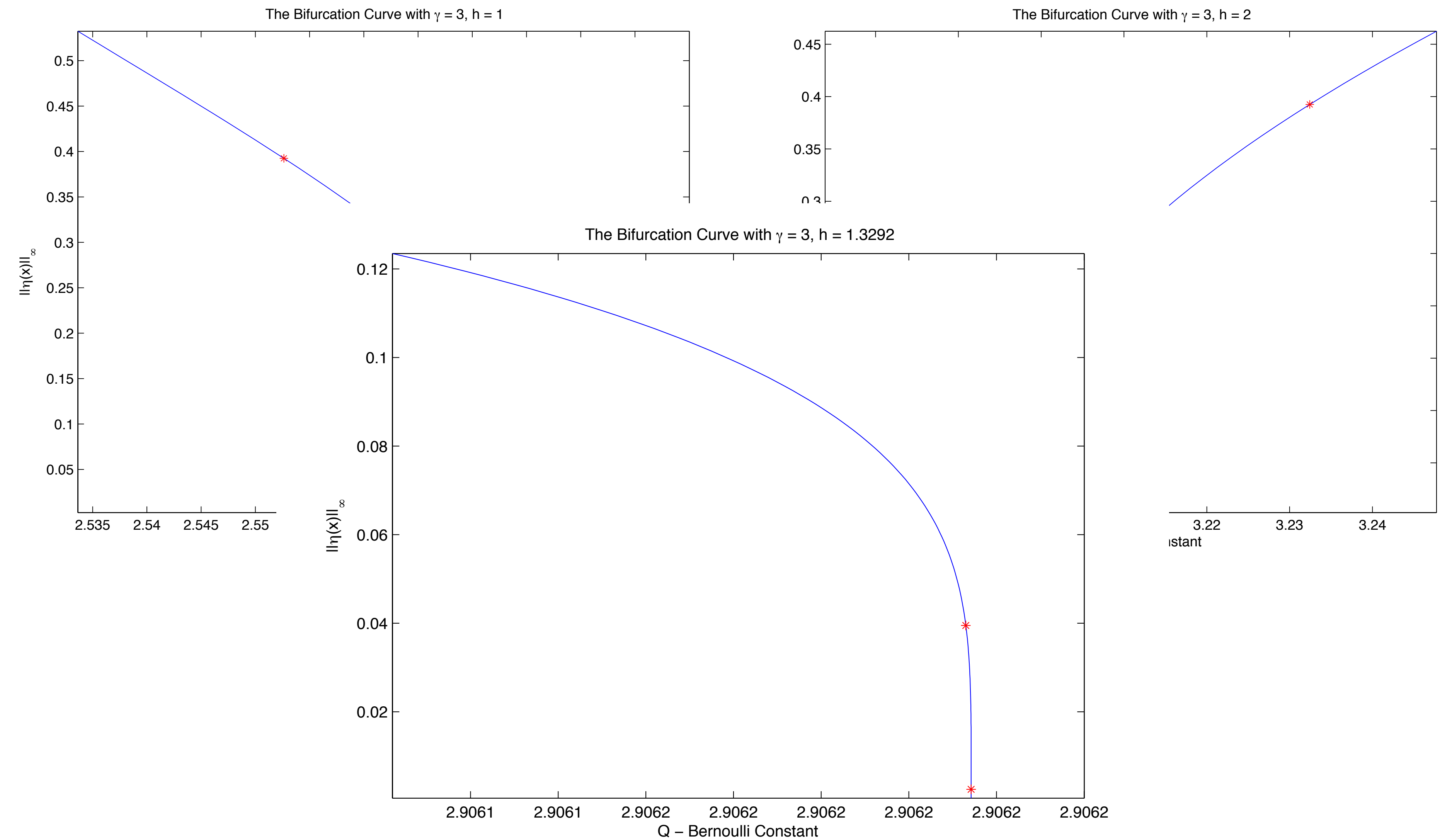


PLOTS OF SOLUTIONS

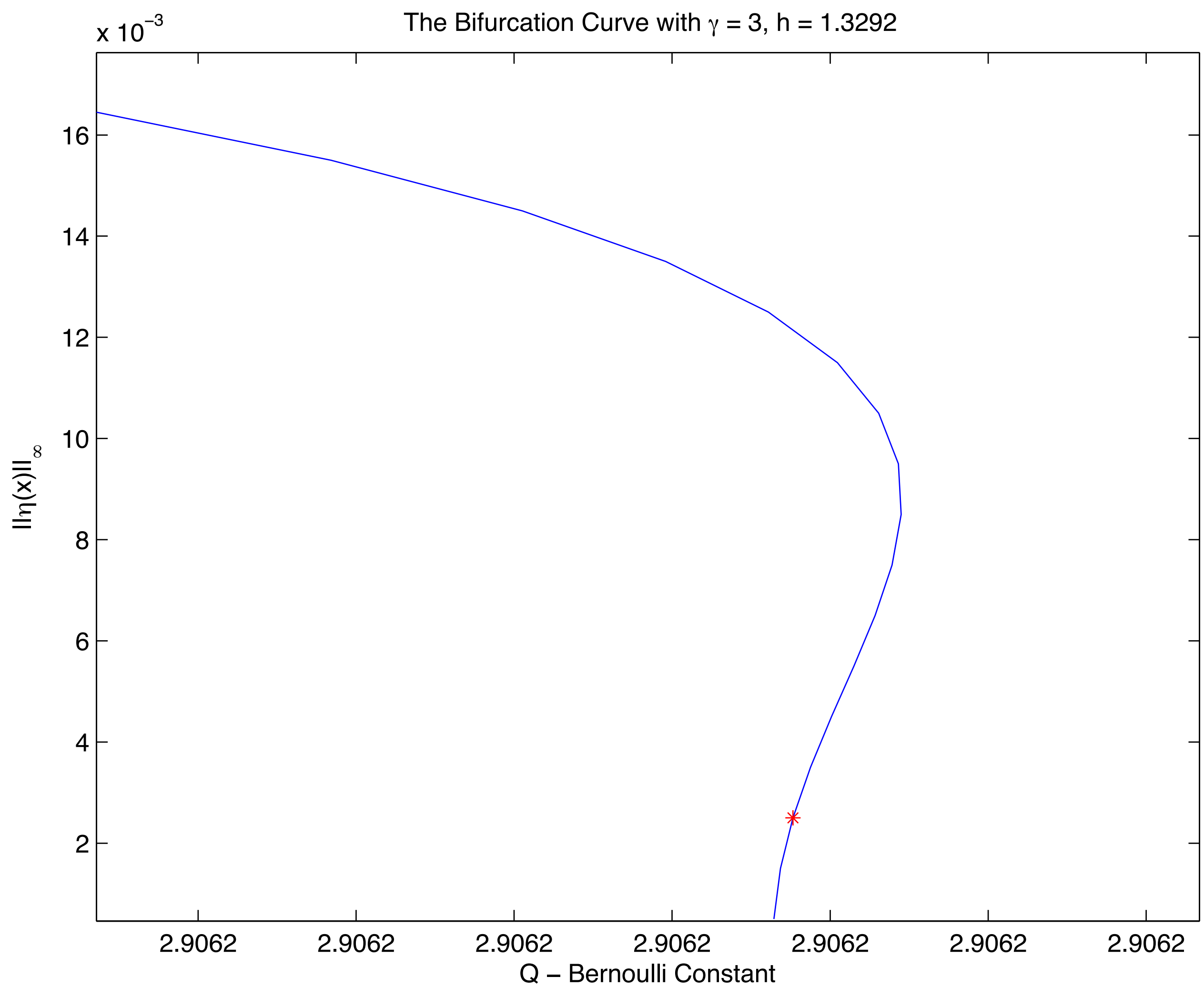


PLOTS OF SOLUTIONS

SEATTLE UNIVERSITY



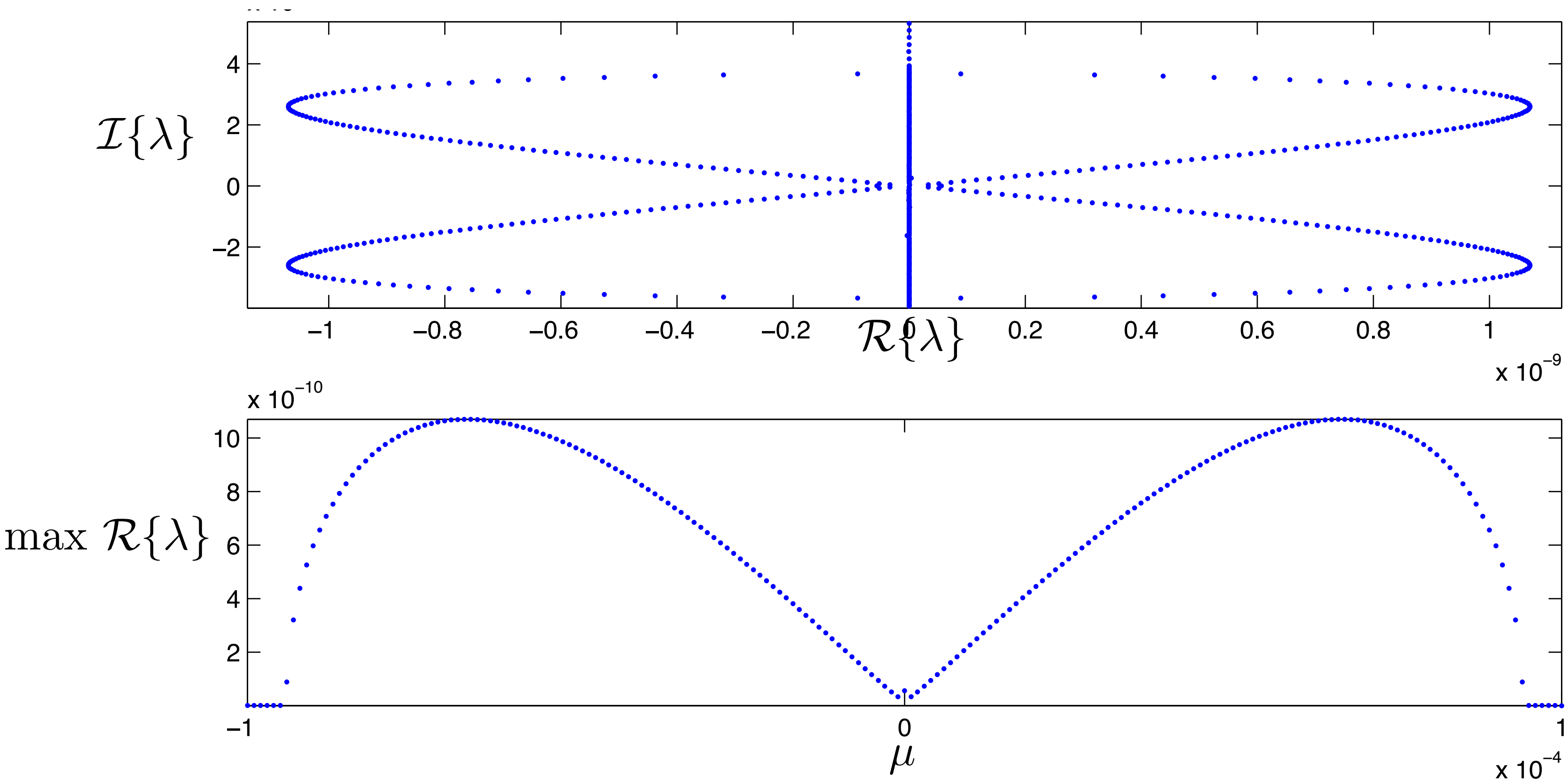
Bifurcation Curve



STABILITY CALCULATIONS

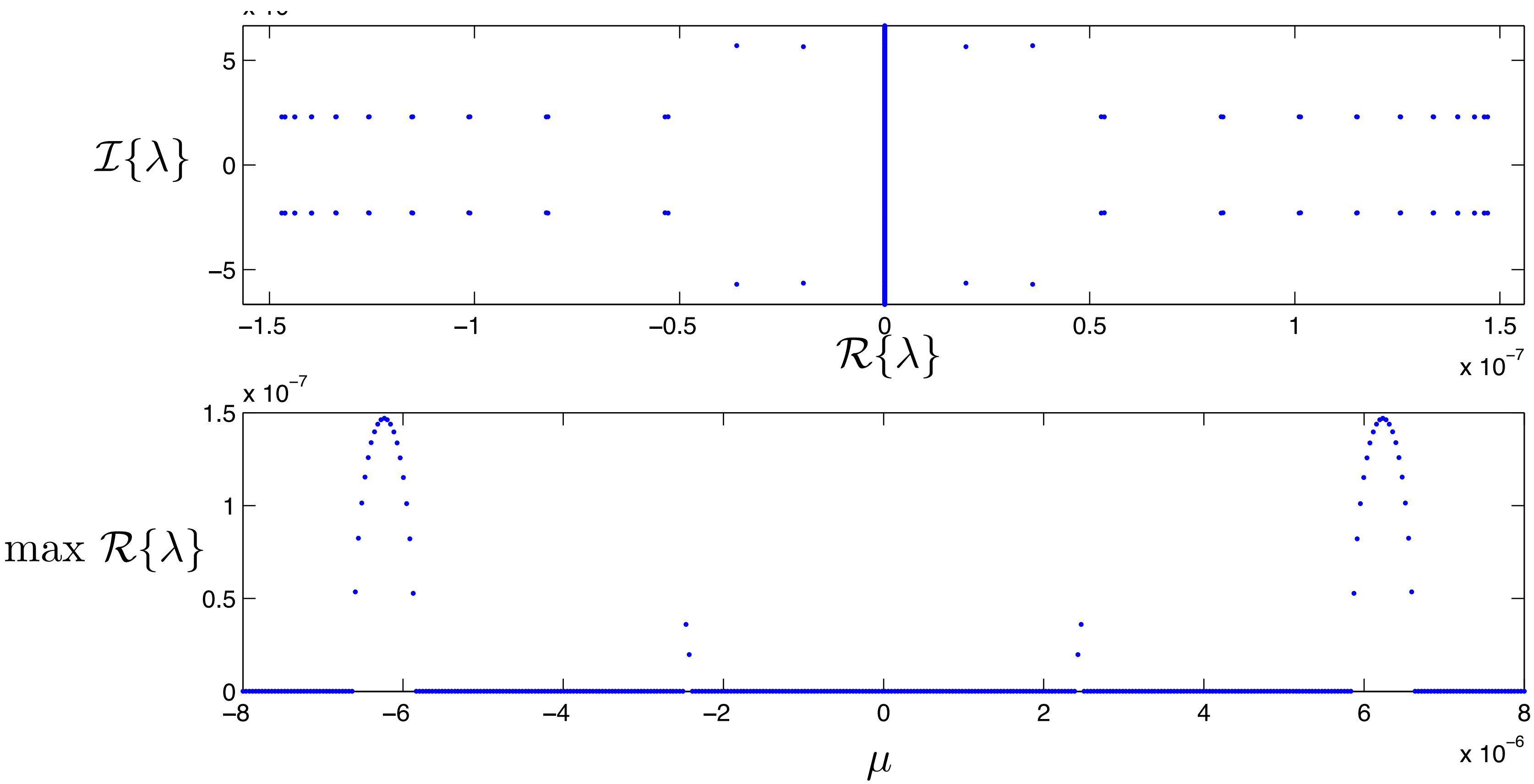
SEATTLE UNIVERSITY

$$h = 1.8, \gamma = 0$$



STABILITY CALCULATIONS

$$h = 1.8, \gamma = 2$$



REMARKS / CONCLUSIONS

SEATTLE UNIVERSITY

There's a lot going on here.

Using the AFM formulation, traveling wave solutions to Euler's Equations can be found by solving a single equation for the single unknown free surface.

Even for small amplitude solutions, the bifurcation curves are "WONKY".

We see that Benjamin-Feir cutoff ($h < 1.363$) is changed as constant vorticity is added to the equation.

These numerical computations give up a starting point for theoretical results.