On $\mathcal{PT}$ symmetric operators in Krein spaces

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joint work with T. Azizov (Voronezh)

5. November 2012

Main Equation

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Great interest: S. Albeverio, C. Bender, M. V. Berry, S. Böttcher, S. F. Brandt, J. Brody, E. Caliceti, F. Cannata, J.-H. Chen, P. Dorey, C. Dunning, A. Fring, H. B. Geyer, S. Graffi, U. Günther, G. S. Japaridze, H. Jones, O. Kirillov, D. Krejčiřík, S. Kuzhel, P. Mannheim, P. Meisinger, K. A. Milton, A. Mostafazadeh, M. C. Ogilvie, K. C. Shin, P. Siegl, J. Sjöstrand, F. Stefani, T. Tanaka, R. Tateo, M. Znojil...

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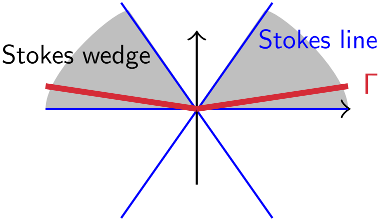

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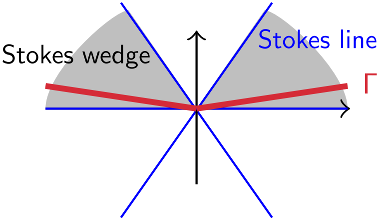

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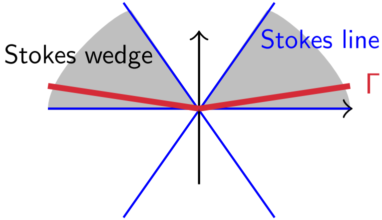

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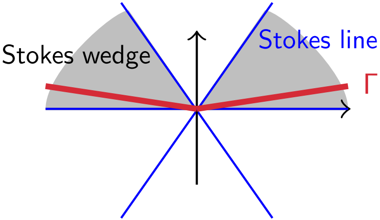

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where

$$(\mathcal{P}f)(z) = f(-\overline{z}) \quad \text{and} \quad (\mathcal{T}f)(z) = \overline{f(z)}.$$

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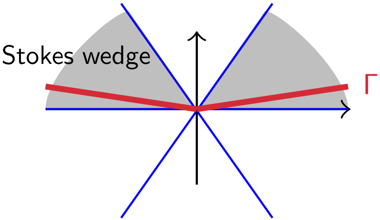

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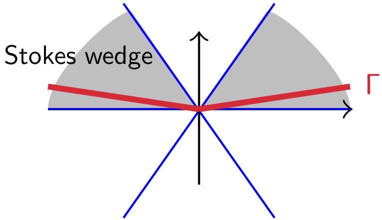

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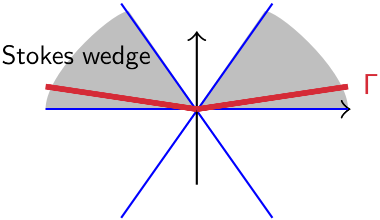

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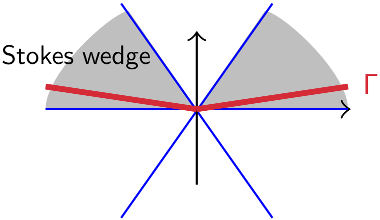

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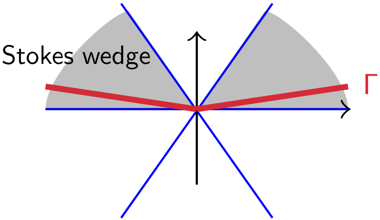

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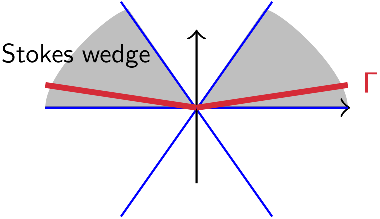

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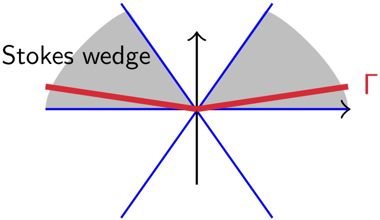

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- Is the spectrum real?

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Then: y solves (1) for $x \neq 0$ if and only if w solves

$$\begin{aligned} -e^{-2i\phi} w''(x) - e^{4i\phi} x^4 w(x) &= \lambda w(x) \qquad x > 0 \\ -e^{2i\phi} w''(x) - e^{-4i\phi} x^4 w(x) &= \lambda w(x) \qquad x < 0 \end{aligned}$$

$$\begin{aligned} y \text{ cont. at zero} \quad &\Leftrightarrow \quad w(0+) = w(0-) \\ y' \text{ cont. at zero} \quad &\Leftrightarrow \quad e^{-i\phi} w'(0+) = e^{i\phi} w'(0-) \end{aligned}$$

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3. Study operator with some matching condition in zero on $\mathbb{R}$

Half line operators + Dirichlet boundary condition

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Limit point, limit circle

The two sol. of (2) satisfy asymptotically (e.g. Eastham '89)

$$y^{\pm}(x) \sim \left[e^{-4i\phi} s(x)\right]^{-1/4} exp\left(\pm \int_0^{\infty} \mathrm{Re}\, s(t)^{1/2} dt\right)$$

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Theorem

- *If $\phi \notin \{0, \frac{\pi}{3}, \frac{2\pi}{3}, \pi, \frac{4\pi}{3}, \frac{5\pi}{3}\}$ then (2) is in Limit Point Case (i.e. one sol. $\notin L^2(\mathbb{R}^+)$).*

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- *Similarly for Equation* (3)*.*

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Missing: Boundary condition at $\pm\infty$ (cf. [AT'10] and [AT'12]).

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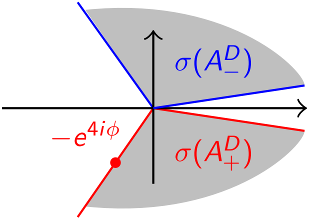


Limit point, circle and spectral properties

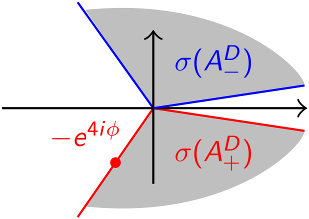
Explanation for Stokes wedges and lines:

$$\Gamma \text{ in Stokes wedge} \cong \text{Limit Point Case}$$
$$\Gamma \text{ in Stokes line} \cong \text{Limit Circle Case}$$

Discussion Limit Point Case:

Theorem

The spectrum of A_+^D and A_-^D consists of discrete eigenvalues of finite algebraic multiplicity with no finite acc. point and is in



Moreover, if $\phi < \frac{\pi}{4}$ then

$$\sigma(A_+^D) \cap \sigma(A_-^D) = \emptyset.$$

Plan + Overview

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2. Study operator with Dirichlet boundary conditions on the semi axis
3. *Study operator with some matching condition in zero on* $\mathbb{R}$:

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3. *Study operator with some matching condition in zero on* $\mathbb{R}$:
   1. $\mathcal{PT}$ symmetry
   2. Selfadjointness in a Krein space
   3. Spectrum

$\mathcal{PT}$ symmetric operators

Define

$$(\mathcal{P}f)(x) = f(-x) \quad \text{and} \quad (\mathcal{T}f)(x) = \overline{f(x)}, \qquad f \in L^2(\mathbb{R}).$$

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Definition

A closed densely defined op. H in $L^2(\mathbb{R})$ is *$\mathcal{PT}$ symmetric* if for all $y \in \operatorname{dom} H$ we have

$$\mathcal{PT}y \in \operatorname{dom} H \quad \text{and} \quad \mathcal{PT}Hy = H\mathcal{PT}y.$$

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- Define the *Adjoint* A^+ with respect to $[\cdot,\cdot]$.
- A $[\cdot,\cdot]$-*selfadjoint* if $A^+ = A$.

Full line operator A + conditions at zero

Define operator A

$$Aw := \begin{cases} -e^{-2i\phi}w''(x) - e^{4i\phi}x^4 w(x) = \lambda w(x), & x > 0 \\ -e^{2i\phi}w''(x) - e^{-4i\phi}x^4 w(x) = \lambda w(x), & x < 0 \end{cases}$$

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with domain

$$\operatorname{dom} A := \left\{ w, Aw \in L^2(\mathbb{R}) : \begin{array}{c} w|_{\mathbb{R}^\pm}, w'|_{\mathbb{R}^\pm} \in AC(\mathbb{R}^\pm), \\ w(0+) = w(0-) \\ w'(0+) = \alpha w'(0-) \end{array} \right\}$$

Then y on Γ is continuous. y' on Γ is continuous $\Leftrightarrow \alpha = e^{2i\phi}$.

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Theorem

- *A is $\mathcal{PT}$-symmetric if and only if $|\alpha| = 1$.*
- *A is $[\cdot,\cdot]$-selfadjoint if and only if $\alpha = e^{4i\phi}$.*

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Lemma

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If $\lambda \notin \sigma_p(A_+^D) \cup \sigma_p(A_-^D)$, *then*

$$\lambda \in \sigma_p(A) \quad \Leftrightarrow \quad \frac{u'_{\lambda,+}(0)}{u_{\lambda,+}(0)} = e^{4i\phi} \frac{u'_{\lambda,-}(0)}{u_{\lambda,-}(0)},$$

where $u_{\lambda,+}$, $u_{\lambda,-}$ *are non-zero sol. of* (2), *resp. (3).*

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Moreover, A and $A_+^D \times A_+^D$ are 1-dim extensions of the (Krein space) symmetric operator $A \cap (A_+^D \times A_+^D)$ and we obtain

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Theorem

Let $\alpha = e^{4i\phi}$ and $\phi < \frac{\pi}{4}$. Then

$$\boxed{\rho(A) \neq \emptyset.}$$

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Let $\alpha = e^{4i\phi}$ and $\phi < \frac{\pi}{4}$.

1. *A is $\mathcal{PT}$ symmetric.*
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Next: Realness of spectrum. $\mathcal{PT}$-symmetric case ($|\alpha| = 1$)...

Thank You !