Algebraic K-Theory and Equivariant Homotopy Theory

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February 12- February 17, 2012

1 Overview of the Field

The study of the algebraic K-theory of rings and schemes has been revolutionized over the past two decades by the development of "trace methods". Following ideas of Goodwillie, Bökstedt and Bökstedt-Hsiang-Madsen developed topological analogues of Hochschild homology and cyclic homology and a "trace map" out of K-theory that lands in these theories [15, 8, 9]. The fiber of this map can often be understood (by work of McCarthy and Dundas) [27, 13]. Topological Hochschild homology (THH) has a natural circle action, and topological cyclic homology (TC) is relatively computable using the methods of equivariant stable homotopy theory. Starting from Quillen's computation of the K-theory of finite fields [28], Hesselholt and Madsen used TC to make extensive computations in K-theory [16, 17], in particular verifying certain cases of the Quillen-Lichtenbaum conjecture.

As a consequence of these developments, the modern study of algebraic K-theory is deeply intertwined with development of computational tools and foundations in equivariant stable homotopy theory. At the same time, there has been a flurry of renewed interest and activity in equivariant homotopy theory motivated by the nature of the Hill-Hopkins-Ravenel solution to the Kervaire invariant problem [19]. The construction of the norm functor from H-spectra to G-spectra involves exploiting a little-known aspect of the equivariant stable category from a novel perspective, and this has begun to lead to a variety of analyses. One of the exciting aspects of this conference was an effort to grapple with various perspectives on equivariant stable homotopy theory in the context of real applications.

2 Recent Developments in Algebraic K-Theory

Algebraic K-theory is a field of wide mathematical interest, lying in the intersection of algebraic topology, algebraic geometry, and number theory. A number of speakers at the workshop reported on exciting recent developments in the study of algebraic K-theory and related invariants which were informed by or involved equivariant homotopy theory.

2.1 Real algebraic K-theory

In the study of topological K-theory, Atiyah's Real K-theory gives rise to a G-equivariant spectrum KR, where $G = Gal(\mathbb{C}/\mathbb{R})$ [2]. The underlying non-equivariant spectrum of KR is equivalent to KU, representing periodic complex K-theory. The spectrum of G-fixed points of KR is equivalent to KO, representing periodic real K-theory. Lars Hesselholt and Ib Madsen have developed an analogous theory, called Real algebraic K-theory. Lars Hesselholt reported on these recent developments at the workshop. They associate to a pointed exact category with strict duality $(\mathcal{C}, T, 0)$ a G-equivariant spectrum $KR(\mathcal{C}, T, 0)$ that they call the Real algebraic K-theory of $(\mathcal{C}, T, 0)$. The underlying non-equivariant spectrum is equivalent to Quillen's algebraic K-theory spectrum $K(\mathcal{C}, 0)$ [28]. The spectrum of G fixed points is equivalent to the Hermitian K-theory of $(\mathcal{C}, T, 0)$. To construct the spectrum $KR(\mathcal{C}, T, 0)$, Hesselholt and Madsen have developed a new variant of Waldhausen's $S_\bullet$-construction which they call the Real Waldhausen construction [36]. They also introduce a G-equivariant spectrum $KR^\oplus(\mathcal{C}, T, 0)$, called the Real direct sum K-theory of $(\mathcal{C}, T, 0)$. This spectrum is essential for understanding the G-equivariant homotopy type of $KR(\mathcal{C}, T, 0)$. It uses a variant of Segal's Γ-category construction that Hesselholt and Madsen call the Real Γ-category construction [31]. Hesselholt and Madsen have proven the following theorem:

Theorem 1. *If $\mathcal{C}$ is split-exact, then there is a canonical weak equivalence of G-spectra*

$$KR^\oplus(\mathcal{C}, T, 0) \simeq KR(\mathcal{C}, T, 0).$$

They define the Real algebraic K-theory groups of $(\mathcal{C}, T, 0)$ to be the bi-graded family of equivariant homotopy groups:

$$KR_{p,q}(\mathcal{C}, T, 0) = [S^{p,q}, KR(\mathcal{C}, T, 0)]_G.$$

Here $S^{p,q}$ is the virtual G-equivariant sphere $S^{\mathbb{R}^{p-q}} \wedge S^{i\mathbb{R}^q}$, where $S^{i\mathbb{R}}$ denotes the sign representation. If (A, L, α) is a ring with antistructure and $(\mathcal{C}, T, 0)$ is the category of finitely generated projective right A-modules, with the induced duality structure, then the main theorem identifies the groups $KR_{p,0}(\mathcal{C}, T, 0)$ with the Hermitian K-groups of (A, L, α), defined by Karoubi.

2.2 Progress towards $TC(MU)$

Thom spectra in general, and MU in particular, are vitally important spectra carrying rich structure. Andrew Blumberg described joint work with Angeltveit, Gerhardt, Hill, and Lawson which generalizes previous work of Blumberg, Cohen, Schlichtkrull [5] on the topological Hochschild homology of Thom spectra. In particular, Blumberg described several different symmetric monoidal products on G-spaces. The "commutative monoids" for these various symmetric monoidal products are the infinite loop space analogue of the flavors of E_∞ ring spectra for which the E_∞-operad is modeled by linear isometries on a possibly incomplete universe. Blumberg described how to make the earlier constructions of $THH(M_f)$ into an equivariant construction, producing a genuine S^1-equivariant commutative ring spectrum.

This new construction of THH arises by introducing an equivariant version of Hopkins' construction of the Thom spectrum of a map $f\colon X \to BGL_1 S^0$ [1]. Blumberg described how one can apply the Hill-Hopkins-Ravenel norm technology to mirror this equivariantly, landing not in maps to $BGL_1 S^0$ but rather in maps to $BGL_1 S^0_G$. Coupled with a new description of THH as the left adjoint to the forgetful functor from S^1-commutative ring spectra to ordinary commutative ring spectra, this produces a model of $THH(M_f)$ as an equivariant Thom spectrum that has the right equivariant homotopy type for all finite subgroups. Blumberg also indicated that this new model of THH could be extended to construct TC relative to ground rings other than the sphere spectrum; this resolves an old question in the area, and opens the door to new computational approaches.

2.3 Representation rings and K-theory

Gunnar Carlsson reported on the completion of a program relating algebraic K-theory and the representation theory of Galois groups. Several results in algebraic K-theory, such as Thomason's descent theorem and the Quillen-Lichtenbaum conjectures, assert that the algebraic K-groups of a field F should be assembled from the algebraic K-groups of its algebraic closure $\overline{F}$ and the action of the Galois group G_F. Carlsson's program aimed to recover the entire homotopy type of the spectrum $K(F)$ from $K(\overline{F})$, which is understood by work of Quillen and Suslin, and the representation theory of the Galois group [28, 35].

If F contains an algebraically closed field k, Carlsson constructed a K-theory spectrum $K(Rep_k(G_F))$ from the category of representations of G over k. The homotopy groups are closely related to the representation ring of G_F, and descent theory provides it with a natural map $K(Rep_k(G_F)) \to K(F)$. Carlsson's conjecture has been that, upon applying an appropriately "derived" notion of completion, this map becomes an equivalence.

The main theorem described in this talk is a proof of this result. The proof first uses a calculation for the case of a Laurent polynomial ring $k[t^{\pm 1}]$ and extensions obtained by adjoining roots of t. It then applies the Bloch-Kato theorem, which shows that algebraic K-groups of fields (appropriately completed) are generated in degrees 1 and 2, with specific descriptions of the generating elements. Finally, there is an "algebraic to geometric" spectral sequence relating the homotopy groups of a completed spectrum to the appropriate derived completion in algebra.

2.4 Localization sequences in THH

Mike Mandell reported on joint work with Andrew Blumberg establishing certain localization sequences in THH. In earlier work [6], they established certain localization sequences in algebraic K-theory, the most important example being the cofibration sequence

$$K(\mathbb{Z}) \to K(ku) \to K(KU).$$

The main result described in this talk is a corresponding localization sequence for THH [7]. If R is a discrete valuation ring with residue field k and fraction field F, there is a cofibration sequence

$$THH(k) \to THH(R) \to THH(F)$$

compatible with the corresponding cofibration sequence for algebraic K-theory.

Ausoni-Rognes and Hesselholt have conjectured that there should be a similar localization sequence involving $THH(ku)$ [4]. The most obvious approach does not work because $THH(KU)$ is not connective, so the homotopy fiber of the map $THH(ku) \to THH(KU)$ is something strange and definitely not $THH(\mathbb{Z})$.

Instead, Blumberg and Mandell work with THH of spectral categories. Let $\mathcal{C}$ be the category of finite cell ku-modules. For $X, Y \in \mathcal{C}$ one can define a connective spectral category $\mathcal{C}^\Gamma$ by

$$\mathcal{C}(X,Y)^\Gamma(n) = |\mathcal{C}(X, \bigvee_{S^n_\bullet} Y)|.$$

Then one can recover $THH(ku)$ by applying the Bökstedt version of the cyclic bar construction to the category $S_\bullet N^w_\bullet \mathcal{C}^\Gamma$. Here $N^w_\bullet$ is the nerve of the subcategory where the maps are all the weak equivalences, and $S_\bullet$ is the Waldhausen construction.

If we instead use the category $S_\bullet N^q_\bullet \mathcal{C}^\Gamma$, where $N^q_\bullet$ means taking the nerve of the category where all the maps become weak equivalences after inverting the Bott element, we get a spectrum $W^\Gamma THH(ku|KU)$.

The homotopy fiber of this map is the cyclic bar construction on the category $S_\bullet N^w_\bullet(\mathcal{C}^\Gamma)^q$ consisting of torsion ku-modules. By a devissage theorem, they identify this with $THH(\pi_0 ku) = THH(\mathbb{Z})$. Hence there is a cofiber sequence

$$THH(\mathbb{Z}) \to THH(ku) \to W^\Gamma THH(ku|KU).$$

Moreover, this cofiber sequence is compatible with the Dennis trace.

2.5 K-theoretic assembly maps, Rips complexes, and equivariant phantom maps

Dan Ramras described recent progress on Loday's assembly map and the integral Novikov conjecture: If G is a discrete, torsion free group, then the map

$$\alpha \colon BG_+ \wedge K(R) \to K(R[G])$$

is injective in homotopy groups. He began by recasting the problem into a geometric one, analogous to Segal's description of the K-theory of a space [32]. This allowed more geometric tools and approaches to be brought to bear. In particular, Ramras considered several families of groups that have buildings with particularly nice geometric structure. The additional geometry allowed, for this family of groups, descent-style arguments showing the Novikov conjecture. Ramras finished with several conjectures, based on of Rips complexes, which would establish the Novikov conjecture in a wide variety of cases.

3 Recent Developments in Equivariant Stable Homotopy Theory

Complementing the talks on algebraic K-theory proper were a series of talks on new work in the foundations of equivariant stable homotopy theory, as well as applications of recent foundational work to other areas.

3.1 G-spectra as presheaves of spectra

Bert Guillou and Peter May have developed a model of the category of G-spectra as a category of enriched presheaves of spectra. Both researchers presented on these results at the workshop. By a result of Schwede and Shipley, any stable model category is equivalent to a category of presheaves enriched in a chosen category of spectra [30]. However, the domain category can be rather mysterious. Guillou and May give an explicit construction of the domain category in their case by applying an infinite loop space machine, $\mathbb{K}$, to an elementary category of finite G-sets enriched in permutative categories, $G\mathcal{E}$. They prove the following.

Theorem 2. *Let G be finite. The the category $\mathbb{K}(G\mathcal{E})$ is equivalent to $G\mathcal{B}$.*

Here $G\mathcal{B}$ denotes an enriched version of the Burnside category of G. This new model extends a description of the homotopy category given in [20], recasting equivariant stable homotopy theory in terms of elementary point-set level categories of G-spans and nonequivariant spectra.

This work requires a number of ingredients of independent interest, such as the theory of classifying G-spaces for equivariant bundles. Guillou and May also define and give examples of genuine permutative G-categories, and more generally E_∞ G-categories. Further contributions of the work include:

1. equivariant infinite loop space theory and infinite loop space machines,
2. the equivariant Barratt-Priddy-Quillen theorem,
3. the tom Dieck splitting theorem for suspension G-spectra,
4. equivariant algebraic K-theory, and
5. pairings of permutative G-categories.

3.2 Equivariant commutative ring spectra

Bjørn Dundas reported on work (in part by his student Stolz) aimed at providing foundational underpinnings to study the redshift conjecture and the answering the question: "What are the slices of the equivariant THH spectrum?" Rognes' redshift conjecture asserts that K-theory increases chromatic (telescopic) complexity; this is supported by calculation in the cases for $n = 0$ and $n = 1$ [3]. The conjecture suggests studying iterated K-theory and consequently iterated THH and TC. In previous work, Dundas (with Brun, Carlsson, and Douglas) has studied iterated THH for commutative ring spectra in terms of tensoring with higher tori, and associated "TC-like" limit constructions [10, 11]. In this talk, Dundas described a model structure on commutative ring spectra (constructed by Stolz) which provides a formal home for interpreting the equivariant nature of these tensor constructions and, more generally, the equivariant nature of smash powers of ring spectra. This work leads to interesting equivariant filtrations on smash powers.

3.3 Global equivariant homotopy theory

Both Anna-Marie Bohmann and Stefan Schwede reported on work aimed at constructing a "global" equivariant stable homotopy theory.

Anna-Marie Bohmann's report was devoted to conceptual foundations for these categories. She motivated global equivariant homotopy theory as describing a family of compatible G-equivariant homotopy types as G varies, with the goal of understanding "change of groups" phenomena.

In Bohmann's version, a global spectrum is a compatible family of equivariant spectra. To make sense of this, one needs compatibility of the universes for maps of groups, and Bohmann described a categorical framework for this work, based on universe-indexed spectra such as those employed by Lewis-May-Steinberger and Elmendorff-Kriz-Mandell-May [20, 14]. Given an appropriately compatible family of G-equivariant universes for all groups G, a global equivariant spectrum consists of a section of a certain functor of categories. She also described the roles of several canonical examples in the global equivariant world, including the sphere spectrum, the equivariant K-theory spectrum, and equivariant bordism theories.

By contrast, Stefan Schwede described a notion of G-equivariant spectra based a new model structure on the orthogonal spectra introduced in Mandell-May-Schwede-Shipley [25]. Orthogonal spectra are equivalent to certain "enriched" functors on equivariant vector spaces; this is based on previous observations published by Shimakawa, and played an important role in the solution of the Kervaire invariant problem [33, 19].

Stefan also reported on some calculational work in global equivariant homotopy theory. Global equivariant homotopy groups take values in the category of global functors. These have the feature that, unlike Mackey functors or abelian groups, they are not rationally semisimple, and so rational equivariant homotopy types do not naturally decompose as products of Eilenberg-Mac Lane objects.

The bulk of Schwede's talk focused on a particular example: the homotopy groups of the symmetric powers of the sphere spectrum. Schwede completely computed π_0 as a global functor (showing the fantastically simple solution in the global context), and he used this to produce explicit examples of nontrivial extensions naturally occurring in rational global homotopy theory.

3.4 An algebraic model for rational G-spectra

Brooke Shipley and John Greenlees both reported on their joint work on developing models for rational G-spectra. The category of rational spectra, with no group action, is Quillen equivalent to the category of $\mathbb{Q}$-DG modules [30]. By previous work of Schwede and Shipley we also know that, given certain technical conditions, any rational stable homotopy theory with a single generator is Quillen equivalent to the category of DG modules over some $\mathbb{Q}$-DGA (or over a DG category in the case of a set of generators). This result applies to free rational G-spectra, but it is only an existence result and we would like a small, explicit algebraic model.

The first talk, by Brooke Shipley, focused on the category of free $\mathbb{Q}$-G-spectra, which is Quillen equivalent to $H\mathbb{Q}[G]$-module spectra, where $H\mathbb{Q}[G] = H\mathbb{Q} \wedge G_+$. If G is finite then $\pi_* H\mathbb{Q}[G]$ is concentrated in degree 0 and we simply get $\mathbb{Q}$-DG modules with a G-action.

If G is an arbitrary connected compact Lie group, we can use Koszul duality in spectra:

$$H\mathbb{Q}[G] \rightsquigarrow F(BG_+, \mathbb{Q})$$

The latter is commutative and the homotopy is a polynomial algebra concentrated in even degrees, hence formal. From this we get the following result:

Theorem 3 (Greenlees-Shipley). *For any connected compact Lie group G we have a Quillen equivalence*

$$\text{free-}\mathbb{Q}\text{-}G\text{-}Sp \simeq_Q \text{torsion DG } H\mathbb{Q}^*(BG)\text{-modules.}$$

In the nonconnected case, let N be the identity component of G and $W = G/N$ the component group. Then we can combine the above result with the simpler behavior for finite groups to get the following:

Theorem 4 (Greenlees-Shipley). *Define $\tilde{B}N = EG/N$, which has a W-action. Then we have a Quillen equivalence*

$$\text{free-}\mathbb{Q}\text{-}G\text{-}Sp \simeq_Q \text{torsion DG } H^*(\tilde{B}N)\langle W\rangle\text{-modules,}$$

where $H^*(\tilde{B}N)\langle W\rangle$ denotes the twisted group ring.

The second talk, by John Greenlees, discussed the case where we no longer assume that the G-action is free. John started with a conjecture:

Conjecture 1. *For any compact Lie group G we have*

$$\mathbb{Q}\text{-}G\text{-}Sp \simeq_Q DG\mathcal{A}(G)$$

for some "nice" abelian category $\mathcal{A}(G)$ of injective dimension equal to the rank of G.

There are several applications of this.

1. It enables us to do calculations by using an Adams short exact sequence or spectral sequence with finitely many rows.
2. It lets us construct of G-spectra algebraically.
3. It has applications to other theories such as G-equivariant elliptic cohomology.

The idea is as follows: $\mathcal{A}(G)$ should be some category of sheaves over the category of subgroups $Sub(G)$ with fiber over H capturing H-geometric isotropy information.

When G is a torus, then they have verified their conjecture.

Theorem 5 (Greenlees-Shipley)**.** *There is a Quillen equivalence*

$$\mathbb{Q}\text{-}G\text{-}Sp \simeq_Q DG\mathcal{A}(G)$$

for G a torus.

3.5 The Gap theorem at 3

Ravenel described ongoing, and largely conjectural, work with Hill and Hopkins concerning the 3-primary Arf-Kervaire problem, and specifically the survival of the family $\beta_{3^i/3^i}$ in the Adams-Novikov spectral sequence. The Hill-Hopkins-Ravenel solution to the Kervaire invariant one problem (at the prime two) used several equivariant techniques which port over directly to the odd primary case. In particular, there is a natural slice filtration described by Hill-Hopkins-Ravenel for any finite group, and the norm machinery allows the construction of commutative ring spectra for larger groups.

Ravenel described a large snag: we do not have a 3-primary analogue of the spectrum $MU_{\mathbb{R}}$ of Real bordism. This was the start of the Kervaire solution, as from this C_2-equivariant spectrum one can build a C_8-equivariant spectrum that detects the Kervaire classes and for which their non-existence follows from straightforward computations. The desired properties of a C_3-analogue, called MU_A, are fairly simple:

1. The underlying spectrum should be $MU \wedge MU$ with a kind of "reduced regular" action.
2. The geometric fixed points should carry the "universal formal group law in which the 3-series is zero".

Assuming the existence of such a spectrum, and basic properties connecting it to MU, Ravenel sketched out a proof of the 3-primary analogue of the "Gap Theorem": the homotopy group π_{-2} of any regular representation suspension of MU_A (or its norm to C_9) is torsion free. In particular, coupled with a periodicity theorem (provable via homotopy fixed point arguments), we see that only finitely many of the classes $\beta_{3^i/3^i}$ survive the Adams-Novikov spectral sequence.

3.6 Equivariant A_∞ bundle theory

John Lind described work using "rigid" models of infinite loop space theory to study bundle theory [21]. Based on work of Blumberg, Cohen, and Schlichtkrull [5], there now exist various categories of "spaces" with a symmetric monoidal product such that monoids and commutative monoids model A_∞ and E_∞ spaces. This is akin to the situation with modern categories of spectra (and there is a strong mathematical analogy in the technology used). Lind is applying this technology to study principal fibrations where the structure group acting is an A_∞ space; this is already interesting non-equivariantly, as it allows us to talk about bundles of spaces that are "groups up to coherent homotopy" without having to fixed an equivalent group. This talk focused also on the extension to the equivariant setting (for both finite and compact Lie groups). Lind described versions of the standard classification results in this context and sketched applications to equivariant twisted cohomology theories and modeling iterated algebraic K-theory classes.

3.7 Modeling stable n-types

Angelica Osorno described a joint project with Niles Johnson studying the relationship between homotopy n-types and higher category theory. A homotopy n-type is a space X whose homotopy groups vanish in degrees above n for all choices of basepoint. It is a classical result that groupoids model homotopy 1-types, in the sense that the classifying space and fundamental groupoid functors establish an equivalence between their homotopy categories. In higher category theory this is known as the "homotopy hypothesis" and has long been a motivating principle.

Johnson and Osorno have extended this result to an equivalence between stable homotopy 1-types and Picard groupoids. A Picard groupoid is a symmetric monoidal groupoid in which every object has a weak inverse under the monoidal structure. Using an algebraic description of Picard groupoids, they have identified the Postnikov data associated to a stable 1-type:

1. the group π_0 is the set of isomorphism classes,
2. π_1 is the automorphism group of the unit object, and
3. the unique k-invariant is determined by the twist automorphism.

Their ongoing work has also explored the case for $n = 2$, where they expect stable homotopy 2-types to be modeled by Picard bigroupoids. In this direction, they have already identified a Picard bigroupoid which acts as the homotopy cofiber of a map between Picard groupoids.

3.8 Fusion categories and field theories

Chris Douglas, Chris Schommer-Pries, and Noah Snyder have explored the relationship between fusion categories and 3-dimensional topological field theories. Chris Douglas reported on this work at the workshop. Fusion categories are monoidal categories that have the nice properties of the category of representation of a finite group:

1. each object has a dual,
2. there are finitely many simple objects, and
3. any object decomposes into a finite sum of simple objects.

In particular, fusion categories are a type of tensor category. Any fusion category gives rise to a 3-dimensional topological field theory.

A key question about the algebraic structure of a fusion category is whether the double dual operation is trivial, as it is in the representation category of a finite group. The following is known:

Theorem 6 (Etingof-Nikshych-Ostrik)**.** *The quadruple dual is trivial.*

Etingof, Nikshych, and Ostrik also conjecture that the double dual is trivial. While this interesting question remains open, Douglas reported on new perspectives on this question provided by the work of Douglas, Schommer-Pries and Snyder. This question corresponds to the question of whether the 3-manifold invariants of the associated field theory depend on a spin structure. Douglas then connected the problem to various other structures on 3-manifolds, linking the problem to classically known homotopy computations.

4 Scientific Progress Made

By bringing together the experts in equivariant homotopy and algebraic K-theory, the workshop established several large projects which helped define the scope of the field for the next several years. In particular, several dominant themes arose:

1. Understand G-equivariant infinite loop space machines and more generally what is meant by a G-symmetric monoidal categories.

2. Use new constructions and approaches in equivariant homotopy to compute algebraic K-groups.

The first point was spearheaded by the talks of Guillou and May. Their talks outlined a construction of a G-equivariant infinite loop space machine on G-permutative categories (along the way, discussing what is meant by a G-permutative category). This meshed with a philosophy expounded by Hill for G-symmetric monoidal categories as symmetric monoidal categories for which we have "products indexed by G-sets". The talk and philosophy underscore several big, outstanding questions in equivariant homotopy: how to reconcile G-equivariant as diagrams indexed by the category G and other notions of G-objects with symmetric monoidal structures. This should give rise to new interpretations of previously confusing topics (such as the difference between Green and Tambara functors), and allow a very natural explanation of the Hill-Hopkins result about equivariant localizations [18].

The second point comes from the specifics of constructions of THH as an S^1-equivariant spectrum. Blumberg, Dundas, Hesselholt, and Mandell all spoke about such constructions and the computational ramifications. Together they provide a picture of equivariant homotopy which is computationally approachable. Tethered to the models described by the first point, we see new way to interpret the homotopy groups provided by trace methods. The modern constructions spell out the connection quite clearly and cleanly.

Based on the work presented at this conference and some of the collaborations initiated, we are optimistic that the new foundations of equivariant stable homotopy theory wil facilitate and support continued progress in the use of trace methods to understand algebraic K-theory.

5 Outcome of the Meeting

This meeting gathered together experts from around the world in the areas of equivariant stable homotopy theory and algebraic K-theory. Recent advancements in these areas were presented at the workshop, and the talks all sparked lively discussion. Time was also set aside for participant discussion, and extensive collaboration took place during the week. A number of participants commented how unique and valuable it was to have this meeting of experts. We felt that the meeting was a tremendous success, far exceeding our hopes and goals for the week.

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