# Workshop on Syzygies in Algebraic Geometry, with an exploration of a connection with String Theory

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## 1 Overview of the Field

Free resolutions are often naturally attached to geometric objects. A question of prime interest has been to understand what constraints the geometry of a variety imposes on the corresponding Betti numbers and structure of the resolution. In recent years, free resolutions have been also impressively used to study the birational geometry of moduli spaces of curves.

One of the most exciting and challenging long-standing open conjectures on free resolutions is the Regularity Eisenbud-Goto Conjecture that the regularity of a prime ideal is bounded above by its multiplicity. The conjecture has roots in Castelnuovo's work, and is known to hold in only a few cases: the Cohen-Macaulay case, for curves, and for smooth surfaces.

In the 80's, Mark Green [11], [12] conjectured that the the minimal resolution of the canonical ring a non-hyperelliptic curve $X$ of genus $g$ satisfies the $N_p$ property if and only if the Clifford index of $X$ is greater than $P$. Recall that $N_1$ says that the homogenous ideal of $X$ in $\mathbf{P}^{g-1}$ is generated by quadrics. For $p \geq 2$, $N_p$ means that the first $p-1$ steps of the minimal resolution of the homogenous ideal of $X$ are given by matrices of linear forms. Using geometry of the Hilbert schemes of K3 surfaces, Voisin [20], [21] showed that the Green conjecture holds for generic curves. In the last few years, by the work of Schreyer, Texidor, Aprodu and Farkas [17]. [19], [1], [2], we know that Green's conjecture holds also in many other cases. For higher dimension variety, the spectacular recent work of Birkar, Cascini, Hacon and Mckernan [4] showed that the canonical ring of a smooth projective variety is always finitely generated, but we also know that the canonical ring may require generators of very high degrees. An alternative approach to study the syzygies of higher dimensional varieties would be to study the asymptotic syzygies of embeddings given by very positive line bundles. For curves, a conjecture of Green and Lazarsfeld [13] would predict essentially that the shape of the minimal resolution of a very high degree embedding is controlled by the gonality of the curve. The conjecture is again known for many cases by the work of Aprodu and Voisin [3], [1]. In higher dimensions, our knowledge is quite limited. Pareschi [16] has done very nice work on the $N_p$-property of abelian varieties. Snowden [18] has obtained interesting results for Segre varieties. Ottoviani and Paoletti [14] have proved sharp results on the $N_p$-property of the Veronese embedding of $\mathbf{P}^2$. See also the work of Bruns, Conca and Römer [5], [6]. In [7], Ein and Lazarsfeld showed that if $X$ is smooth complex projective variety of dimension $n$ and $L_d$ is a line bundle of of the form $L_p = \mathcal{O}_X(K_X + (n+1+p)A + B)$ where $A$ is very ample, $B$ is nef and $p$ is a nonnegative, then $L_p$ satisfies the property $N_p$ . More generally, we can consider a line bundle of the form $L_d = \mathcal{O}_X(dA + B)$, where $A$ is an ample divisor and $B$ is a given fixed divisor. Let $r(L_d) = h^0(L_d) - 1$. Observe that with $d >> 0$ one has

$$r(L_d) \;=\; O(d^n).$$

In particular, the length of the minimal resolution $E_\bullet(X, L_d)$ for the coordinate ring of $X$ grows like a polynomial of degree $n$ in $d$. So in dimensions two and higher, statements such as the one above that are linear in $d$ ignore most of the syzygy modules that occur. It therefore seemed interesting to ask whether one can say anything about the overall shape of $E_\bullet(X, L_d)$ for $d \gg 0$. Denote by $K_{p,q}(X, L_d)$ be the space of minimal generators of $E_P(X, L_d)$ of degree $p + q$. Let $S$ be the polynomial ring of $\mathbf{P}^{r_d}$, then

$$E_p(X; L) \;= \bigoplus_q \; K_{p,q}(X; L_d) \;\otimes_k\; S(-p-q).$$

By elementary consideration, one sees that

$$K_{p,q}(X, L_d) = 0 \;\text{ for }\;\; q > n + 1.$$

In [8], Ein and Lazarsfeld obtain the following non-vanishing theorem. They show that there are constants $C_1, C_2 > 0$ with the property that if $d$ is sufficiently large then

$$K_{p,q}(X, L_d) \neq 0$$

for every value of $p$ satisfying

$$C_1 d^{q-1} \leq p \leq r_d - C_2 d^{n-1}.$$

This result gives the rough shape of the minimal resolution. Ein and Lazarsfeld further conjecture that these results are asymptotically sharp. This means that we expect a vanishing result which says that if we fix an integer $q$ such that $2 \leq q \leq n$, then we can find a positive constant $C$ with the property that if $d$ is sufficiently large then

$$K_{p,q}(X, L_d) = 0 \;\text{ for }\; p \leq C d^{q-1}.$$

In the case for $X = \mathbf{P}^n$ and $L_d = \mathcal{O}_{\mathbf{P}^n}(d)$, one has very precise conjecture. This case is particularly interesting, because one can study the question using techniques from the cohomology of homogenous vector bundles and representation theory. In particular, in the very interesting work [15] Ottaviani and Rubei study these bundles using quiver representations. Another natural problem is determine the size of the Betti numbers of the minimal resolution. In a recent preprint, Ein, Erman and Lazarsfeld [9] show that a random Betti table with a fixed number of rows, sampled according to a uniform choice of Boiz-Söderberg coefficients [10], for a fixed $q$, the distribution of $k_{p,q}$ would converge to a normal distribution after some normalization. They conjecture that as $d \to \infty$ the distribution of $k_{p,a}(X, L_d)$ would also converges to a normal distribution. This is know to hold when $X$ is a curve. Assuming that $K_X$ is ample, it would also be natural to ask whether one can understand the intrinsic geometry of $X$ by studying the syzygies of modules of the form $\bigoplus_s H^0((mK_X + sL_d)$, where $m$ is a positive integer. This would parallel the Green-Lazarsfeld gonality conjecture for curves.

The second focus in the workshop was matrix factorizations and their applications in physics. We organized a 10-lecture short course on that topic. A "matrix factorization" (of size $n$) of a function $f$ (say on some variety) is a pair of $n \times n$ matrices $A, B$ such that $AB = f \cdot \mathrm{Identity}$. A classic example is the expression of the determinant of a matrix as the product of the matrix with its cofactor matrix. Eisenbud introduced matrix factorizations in the 80's to describe the asymptotic behavior of minimal free resolutions over hypersurfaces in affine or projective space (or, equivalently, Cohen-Macaulay modules over hypersurfaces). For example, if $S$ is polynomial ring and $f \in S$, any free resolution over $S/(f)$ becomes eventually periodic of period 2, with maps given by matrices $\overline{A}, \overline{B}$ such that any lifting to matrices $A, B$ over $S$ is a matrix factorization of $f$. In recent years it was discovered by physicists that matrix factorizations were useful in several new contexts:

- To provide supersymmetric boundary conditions for Landau-Ginzburg theories;
- To describe objects of the category of D-branes related to superpotentials in Landau-Ginzburg theories;
- As a tool in the theory of noncommutative crepant resolutions of singularities.

## 2 Recent developments on Free Resolutions and related topics

In this section we provide short summaries of recent research conducted by participants in the workshop. They give an excellent overview of some of the current research in the area. The summaries are ordered alphabetically by the last name of the author.

**Christine Berkesch's** research focuses on homological invariants in combinatorial algebraic geometry, specifically within the areas of hypergeometric systems in $D$-module theory and free resolutions in commutative algebra.

Hypergeometric series include many familiar functions, such as trigonometric and Bessel functions. Their importance stems not only from their appearances throughout mathematics, but physics and engineering as well. Their differential annihilators can be studied via algebraic analysis, and due to work of Gelfand, Kapranov, and Zelevinsky, they fit nicely into the context of toric geometry. Berkesch is working with J. Forsgård and L. Matusevich in order to obtain a complete and computationally explicit understanding of the solution spaces of these equivariant hypergeometric systems, or GKZ-systems. She is also establishing with L. Matusevich and U. Walther a framework to translate essential properties between GKZ-systems and the classical hypergeometric systems studied by Euler, Gauss, Appell, and Horn, among others. The methods for these projects involve integrating homological and combinatorial tools with complex analysis and tropical geometry. In addition, Berkesch is considering with E. Miller and S. Griffith a generalization of GKZ-systems due to Kapranov, which replaces the torus with an arbitrary reductive group. The goal of this project is to construct a theory of holonomic families and Euler–Koszul homology in this setting, generalizing the work of Matusevich, Miller, and Walther.

While computer algebra systems can compute individual examples of these homological objects, classifying their collective properties is a powerful approach to understanding many structural and enumerative invariants in algebraic geometry. In this area, Berkesch has focused on describing the numerics of free resolutions for several rings and algebraic varieties, including local rings, projective hypersurfaces, and, most recently, toric varieties. She is currently working with D. Erman and G.G. Smith to understand complexes over a Cox ring that have irrelevant homology. While various methods are employed, the answers sought parallel recent ground-breaking results of Eisenbud and Schreyer for projective space, which not only confirmed the long-standing Multiplicity Conjectures of Herzog, Huneke, and Srinivasan, but also exhibited a beautiful duality between the numerics of free resolutions and the cohomology of coherent sheaves.

**Mats Boij** together with Jonas Söderberg initiated the study of the cone of Betti tables of graded Cohen-Macaulay modules over the polynomial ring. The original purpose was to prove the Multiplicity Conjecture by Huneke and Srinivasan, and they proposed a conjectural structure of the cone which would imply this conjecture. These conjectures were proven by David Eisenbud and Frank-Olaf Schreyer and later Boij and Söderberg could extend these results to the case of modules that are not necessarily Cohen-Macaulay. In order to get a better understanding of the situation when the polynomial ring comes with a different grading Boij has has been working together with Gunnar Fløystad on the bigraded case and together with Gregory G. Smith on the cone of Hilbert functions in other gradings.

Together with Juan Migliore, Rosa Maria Miró-Roig, Uwe Nagel and Fabrizio Zanello, Boij has studied graded level algebras. In particular they wrote "*On the shape of a pure O-sequence*" which addresses several aspects of monomial level algebras.

Together with Juan Migliore, Rosa Maria Miró-Roig and Uwe Nagel, Boij has studied the Weak Lefschetz property of Artinian complete intersections.

Together with Fabrizio Zanello, Boij worked on consequences of Green's Hyperplane Restriction Theorem on $h$-vectors of Gorenstein algebras.

**Jesse Burke's** research interests are in complete intersection rings. One particular focus has been using matrix factorizations to study the singularity category of a complete intersection. Using a theorem of Orlov, the singularity category of a complete intersection is equivalent to the singularity category of a suitably defined non-affine hypersurface $Y$. In detail, $Y$ is the hypersurface in a smooth scheme $X$ defined by the vanishing of a regular global section $W$ of a line bundle $\mathcal{L}$ on $X$. The singularity category of $Y$ is, in turn, equivalent to the homotopy category of "twisted" matrix factorizations associated to the triple $(X, \mathcal{L}, W)$. An object in the latter category consists of a pair of algebraic vector bundles $E_1, E_0$ on $X$ and morphisms $E_1 \to E_0$ and $E_0 \to E_1 \otimes \mathcal{L}$ such that each of the two evident compositions is multiplication by $W$.

Using this machinery, Mark Walker and Burke have redeveloped the theory of stable supports for modules over complete intersections in a more geometric manner. Additionally, from such a matrix factorization a free resolution may be constructed. Exploring these resolutions is another topic of interest for Burke.

Finally, Burke is interested in Boij-Söderberg theory. In recent work with Christine Berkesch, Daniel Erman and Courtney Gibbons, Boij-Söderberg theory was extended from the case of a polynomial ring to a simple class of hypersurface rings.

**Andrei Căldăraru's** research is concentrated in two main areas. Both directions of research are influenced by the study of open problems in physics, especially mirror symmetry and string theory. The first direction of research is focused on the interactions of string theory with birational geometry (a fairly classical subject in algebraic geometry). The second direction involves applying ideas and techniques of a very new field, derived algebraic geometry, to the study of problems in algebraic geometry, some new (inspired by string theory), and some old (the Hodge-de Rham degeneration).

One of the main ideas that string theory has brought to modern algebraic geometry is that we should not always study individual spaces separately, but rather study them up to the equivalence relation on smooth projective varieties given by the notion of so-called derived equivalence. There is another important equivalence relation in algebraic geometry, that of birationality, and this relationship has been extensively studied for over two centuries. A major question is to understand the relationship between these equivalence relations. In joint work with Lev Borisov, Căldăraru constructed the first known example of two non-birational Calabi-Yau varieties of dimension three which are derived equivalent. He is currently interested in understanding the gauged non-linear sigma models from physics, which appear to explain when such examples can appear.

In a second direction, Căldăraru has been working on understanding fundamental properties of the Hochschild homology and cohomology of algebraic varieties. These are fundamental invariants which, in the case of the spaces that appear in mirror symmetry (Calabi-Yau's) are mirror to the usual, singular homology and cohomology. Căldăraru introduced the so-called Mukai pairing on Hochschild homology, which is the analogue of the usual Poincaré pairing on cohomology, and showed that it satisfies many remarkable properties. At the moment Căldăraru is studying interpretations of Hochschild (co)homology from the perspective of derived algebraic geometry, in particular their relationship with derived intersection theory. In a related area, Căldăraru is expecting that ideas from derived intersection theory can be applied to give geometrical interpretations of results of Deligne-Illusie on the degeneration of the Hodge-de Rham spectral sequence and similar results by and Barannikov-Kontsevich-Sabbah on the twisted de Rham complex.

**Izzet Coskun** discovered positive, geometric rules for computing restriction coefficients for all classical flag varieties. The symplectic and orthogonal flag varieties embed in the ordinary flag variety by inclusion. The induced map in cohomology carries information important in representation theory, combinatorics and algebraic geometry. The image of a Schubert class under this map is a non-negtaive sum of Schubert classes, whose coefficients are called restriction coefficients. Using degenerations, a positive, geometric rule for computing these coefficients was discovered.

The Hilbert scheme of points $\mathrm{Hilb}_n(P^2)$ parameterizes length $n$ subschemes of $P^2$. $\mathrm{Hilb}_n(P^2)$ is a smooth, irreducible variety of dimension $2n$ that contains the locus of $n$ unordered points as a dense open subset. $\mathrm{Hilb}_n(P^2)$ is a very important parameter space that plays a crucial role in algebraic geometry, combinatorics and mathematical physics. In joint work with, Daniele Arcara, Aaron Bertram and Jack Huizenga, Coskun studied the birational geometry of $\mathrm{Hilb}_n(P^2)$, describing the ample and effective cones, the stable base locus decomposition and the explicit sequence of flips and divisorial contractions between the models. Most interestingly, modular interpretations of the models that occur in MMP were discovered. For small $n$, all the models are moduli spaces of Bridgeland stable objects in the derived category. In fact, there is a very precise correspondence between Mori walls and Bridgeland walls in the stability manifold. In joint work with Aaron Bertram, these results have been extended to other surfaces such as $P^1 \times P^1$ and Hirzebruch surfaces.

**Steven Dale Cutkosky** has made a study of asymptotic properties of ideals. Although various properties of ideals such as regularity can be very complicated, they do tend to behave well for large powers. An example is the regularity of homogeneous ideals, which becomes a linear polynomial for large $n$, as shown by Cutkosky, Herzog and Trung, and independently by Kodiyalum. Finally, this is a reflection of the fact that the Rees algebra of powers of ideals is a finitely generated algebra. There are other interesting numerical functions of ideals for which one can also ask if they have a good expression for high powers of an ideal. An example is the regularity of saturated powers. Geometrically, saturation corresponds to the notion of sections over a punctured neighborhood of a point of an ideal sheaf. The algebra of saturated powers of an ideal is in general not finitely generated, so we can not expect as good an answer in this case. However, the limit of the regularity of the saturation of $I^n$ divided by $n$ always exists, as was shown by Cutkosky, Ein and Lazarsfeld. However, Cutkosky gave an example that shows that even for a homogeneous prime ideal of a smooth curve in $P^3$, the limit can be irrational.

Cutkosky has shown with Ha, Srinivasan and Theodorescu that the lengths of the quotients of the saturated powers by the ordinary powers of a homogeneous ideal has a limit, and that this limit can be irrational. More recently, Cutkosky has extended this theorem to local rings, and generalized saturation. If $I$ and $J$ are ideals in a ring $R$, then the generalized saturated power $I_n(J)$ is the saturation $I^n : J^\infty$, which corresponds to the geometric notion of taking section of $I^n$ on the complement of the vanishing locus of the ideal $J$. Specifically, if $R$ is a local ring with some very mild conditions (for instance a local ring of a point on an algebraic variety over a field of characteristic zero, or a local ring of a complex analytic variety) and $I, J$ are ideals in $R$, then the limit of multiplicities of quotients of the generalized saturated powers of $I$ by the ordinary powers has a limit.

Cutkosky's recent research also includes simplification of algebraic mappings and resolution of singularities.

**Hailong Dao** works on a number of questions in commutative algebra. One of his most recent works involves new invariants and classifications of subcategories of modules over a commutative noetherian ring. With Takahashi, he classified all resolving categories of modules over a complete intersection $R$ using the "grade consistent functions" from $\mathrm{Spec} R$ to non-negative integers.

Another recent topic is about non-commutative desingularizations. He proved that an isolated hypersurface singularity in dimension three which is a UFD admits no non-commutative crepant resolutions in the sense of Van den Bergh. He and Huneke gave a simple proof of a characterization for all such resolutions over certain $cA_n$ singularities. Most recently, he and Iyama, Takahashi, Vial proved that over certain algebras $R$, the existence of a faithful module whose endomorphism ring has finite global dimension forces $R$ to have only rational singularities, extending work by Stafford-Van den Bergh.

Another topic is about solving simple equations in the semi-ring of vector bundles over the punctured spectrum of a regular local ring. He showed that some weak form of multiplicative cancellation holds, for example the equation $XY = nY$ for $Y \neq 0$ and an integer $n$ yields $X = n$.

**Daniel Erman** has been working on three recent projects on the structure of free resolutions and their applications to algebraic geometry.

First, David Eisenbud and Daniel Erman provide a robust categorical foundation for the duality theory introduced by Eisenbud and Schreyer to prove the Boij-Söderberg conjectures describing numerical invariants of syzygies. The new foundation extends the reach of the theory substantially. More explicitly, Eisenbud and Erman construct a pairing between derived categories that simultaneously categorifies all the functionals used by Eisenbud and Schreyer.

Second, Lawrence Ein, Daniel Erman and Rob Lazarsfeld prove a Law of Large Numbers type result for high degree syzygies. They present a conjecture to the effect that the ranks of the syzygy modules of a smooth projective variety become normally distributed as the positivity of the embedding line bundle grows. Then to render the conjecture plausible, they prove a result suggesting that this is in any event the typical behavior from a probabilistic point of view.

Third, Daniel Erman and Melanie Matchett Wood prove a generalized Bertini theorem over a finite field, thus extending a result of Poonen from the ample case to the semiample case. Free resolutions, Castelnuovo–Mumford regularity, and other homological techniques play an essential role in the proof.

**Gunnar Fløystad** has been recently working on the Boij-Söderberg theory. Pure free resolutions are graded free resolutions over the polynomial ring $S$ of the form

$$S(-d_0)^{\beta_0} \leftarrow S(-d_1)^{\beta_1} \leftarrow \cdots \leftarrow S(-d_r)^{\beta_r}.$$

Their Betti diagrams have proven to be of fundamental importance in the study of Betti diagrams of graded modules over the polynomial ring. Their significance were put to light by the Boij-Söderberg conjectures which describes such Betti diagrams, up to multiplication by a rational number. Theses conjectures were later proven in full generality by D.Eisenbud and F.-O. Schreyer.

More generally, a complex $F^\bullet$ of free modules over the polynomial ring $S$, for instance a free resolution, comes with a triplet of numerical homological invariants: i) Its graded Betti numbers, ii) the Hilbert functions of its homology modules, and iii) the Hilbert functions of the cohomology modules, where these modules are defined as the modules of the dual complex $Hom_S(F^\bullet, \omega_S)$. The current research of Fløystad is inspired by the following general problem: Up to rational multiple, what sets of triplets of numerical homological invariants can occur?

Fløystad is currently working on the following approach: On the category of graded $S$-modules, there is the standard duality functor $\mathbf{D} = Hom_S(-, \omega_S)$. Restricting to a subcategory of graded $S$-modules, the squarefree $S$-modules, there is also Alexander duality $\mathbf{A}$. The composite functor $\mathbf{A} \circ \mathbf{D}$ *rotates the three homological invariants*, and has order three on the derived category of squarefree modules, up to translation. He then studies complexes of free squarefree modules $F^\bullet$ such that (when considered as singly graded modules), both $F^\bullet$, $\mathbf{A} \circ \mathbf{D}(F^\bullet)$ and $(\mathbf{A} \circ \mathbf{D})^2(F^\bullet)$ are pure. This is called a *triplet of pure complexes*. (That $F^\bullet$ is a pure resolution of a Cohen-Macaulay squarefree module, the classical case, corresponds to the second and third complex being linear.) Such complexes have unique sets of Betti numbers (just as in the classical case) up to scalar multiple. He has partly been able to construct such triplets of complexes using recent constructions of Berkesch, Erman, Kumini, and Sam of tensor complexes. Inspired by their approach he is furthermore working on transferring the conjectured existence of triplets of pure complexes, to a conjecture on the existence of certain classes of Tate resolutions over the exterior algebra.

**Hans-Chrsitian Graf v. Bothmer** recently had a joint project with Sosna. The bounded derived category of coherent sheaves $D^b(X)$ on a smooth projective variety $X$ (always over $\mathbf{C}$ in the following) may be viewed as a categorification of the Grothendieck group of $X$ or the Chow ring of $X$, both of which tend to be very intricate objects in their own right already. Moreover, according for example to Kontsevich, Orlov and Tabuada there the intuition that $D^b(X)$ should be a version of the non-commutative motive of $X$, with decompositions of the (classical) Chow motive $h(X)$ of $X$ being reflected in a suitable sense by semi-orthogonal decompositions of $D^b(X)$. Recently (see, for example Rouquier, Kawamata and Kuznetsov) a number of results as well as conjectures try to link semi-orthogonal decompositions in derived categories to the birational geometry of $X$, including very subtle features such as the rationality or irrationality of $X$ which do not seem to be detected by sheaf-cohomological (non-categorical) data. However, the best understood examples considered so far mainly consist of varieties close to the toric and rational-homogeneous ones as well as some Fano hypersurfaces. It seems however, that the optimism radiated by existing conjectures is not reflected in the data one can sample from these varieties.

It follows from Serre duality that the derived category of a Calabi-Yau manifold is indecomposable, that is, it does not admit any non-trivial semi-orthogonal decomposition. Furthermore, according to Okawa, varieties of general type with globally generated canonical bundle do not have exceptional objects. However, on surfaces of general type $X$ with $p_g = q = 0$ every line bundle is exceptional and one may hope that interesting semi-orthogonal decompositions exist which may yield a nontrivial testing ground for existing conjectures. The classical Godeaux surface is such an example. Guletskii and Pedrini have shown that the Chow motive of the classical Godeaux surface splits as a direct sum of Lefschetz motives $h(X) \simeq 1 \oplus 9\mathbf{L} \oplus \mathbf{L}^2$. The Grothendieck group of $X$ is $\mathbf{Z}^{11} \oplus \mathbf{Z}/5$, hence $X$ does not admit a full exceptional sequence, since the existence of the latter would imply that the Grothendieck group is free. One may therefore conjecture that $D^b(X)$ has an exceptional sequence of length 11 corresponding to the "trivial commutative part of the motive" and some nontrivial genuinely non-commutative semi-orthogonal complement to this sequence. This expectation turns out to be correct and is the main result of Böhning, v. Bothmer and Sosna; they prove that if $X$ is the classical Godeaux surface, then there exists a semi-orthogonal decomposition $D^b(X) = \langle \mathcal{A}, \mathcal{L}_1, \ldots, \mathcal{L}_{11} \rangle$ where $(\mathcal{L}_1, \ldots, \mathcal{L}_{11})$ is an exceptional sequence of maximal length consisting of line bundles on $X$ and $\mathcal{A} \neq 0$ is the right orthogonal to this sequence. The existence of the above decomposition answers Kuznetsov's Non-vanishing Conjecture about the Hochschild homology of an admissible subcategory, in the negative. In fact, the Hochschild homology of $\mathcal{A}$ is zero, but $\mathcal{A}$ itself is not. Böhning, v. Bothmer and Sosna also produce explicit nonzero objects in $\mathcal{A}$.

**Remke Kloosterman** is mostly interested in the syzygies of the ideal of the singular locus of projective varieties. In the case of (reducible) cuspidal plane curve $C$ of degree $d$ he showed that the maximal degree for a syzygy of $\sqrt{I(C_{sing})}$ is $5/6d$ and that the number of syzygies of maximal degree equals half the degree of the Alexander polynomial of $C$. He used this to find a new upper bound for the degree of the Alexander polynomial of a cupsidal plane curve.

Moreover, he showed that the equisingular and equianalytic deformation space of a singular plane curve of degree at least 13 with nonconstant Alexander polynomial is not $T$-smooth, i.e., the tangent space to these deformation spaces have bigger dimension than expected.

In the case of a degree $d$ nodal hypersurface $X \subset PP^{2k}$ he determined the minimal number of nodes $n(k,d)$ such that $h^{2k}(X) > 1$ holds: A general hyperplane section $X_H$ of $X$ is a smooth hypersurface in $PP^{2k-1}$ such that the Hodge structure on $H^{2k-2}(X_X, ZZ)_{prim}$ contains a onedimensional sub-Hodge structure of type $((k-1)/2,(k-1)/2)$. In particular, $X_H$ corresponds to a point of the Noether-Lefschetz locus $NL_{d,2k-1}$. By the work of Green, Voisin and Otwinowska one can associate an ideal with the tangent space of the Noether-Lefschetz locus of $X_H$. By comparing the syzygies of this ideal and the ideal of of the singular locus of $X$ one can show that if $X$ has $n(k,d)$ nodes then $X_H$ lies on the largest component of the Noether-Lefschetz locus. By the work of Otwinowska it follows that $X_H$ contains a linear space of dimension $k-1$ and that $X$ containes a linear space of dimension $k$. From this it follows that $n(k,d) = (d-1)^k$. This extends a result of Cheltsov for the case $k = 2$.

**Yusuf Mustopa** is an algebraic geometer with interests in commutative and noncommutative algebra. His recent projects focus on the following topics and their interrelations: the geometry of curves, syzygies of algebraic subsets of projective space, vector bundles, and the representation theory of associative algebras.

One major part of his current program is the study of Ulrich bundles, which are the “best-behaved” arithmetically Cohen-Macaulay bundles. These been the subject of intense investigation in recent years, and they occur naturally in a wide variety of algebraic and algebro-geometric topics, including determinantal descriptions of hypersurfaces, the computation of resultants, Boij-Söderberg theory, and generalized Clifford algebras. In previous joint work with Emre Coskun and Rajesh Kulkarni, Yusuf has applied recent work on the geometry of curves to study Ulrich bundles on quartic surfaces and del Pezzo surfaces. This body of results includes the existence of a linear Pfaffian representation of any smooth quartic surface, the existence of low-dimensional irreducible representations of the Clifford algebra of a ternary quartic form, and a characterization of Chern classes of Ulrich bundles on del Pezzos in terms of vector bundles on curves. In current joint work with Kulkarni, he is using techniques from the theory of generalized Clifford algebras to exhibit smooth threefolds of any given degree d in projective 4-space admitting stable Ulrich bundles of ranks both arbitrarily high and relatively low.

The other major part centers around the syzygy bundle associated to a subvariety of projective space, which governs the fine structure of the equations cutting out the subvariety. A classical result of Ein-Lazarsfeld says that the syzygy bundle associated to a smooth curve of genus at least 2 embedded in projective space by a complete linear series of sufficiently high degree is slope-stable. Lazarsfeld and Yusuf have recently generalized this to smooth projective surfaces, and they are presently working with Ein to extend it to varieties of dimension 3 or higher.

**Wenbo Niu’s** recent research focuses on singularities of a generic link of an algebraic variety. Given an algebraic variety $X$, one can construct a variety $Y$ linked to $X$ so that $X$ and $Y$ are irreducible components of a complete intersection. The idea of linkage can be traced back to the nineteenth century. The modern form of linkage was first introduced by Peskine and Szpiro. The theory of generic linkage, in which, roughly speaking, $Y$ is assumed to be taken as general as possible, was built and developed by Huneke and Ulrich in the past twenty years. Since $X$ and $Y$ are linked via a complete intersection, the properties of $X$ can be compared to the ones of $Y$, which in turn provides a way to better understand $X$ and $Y$. By the efforts of many mathematicians, including besides those mentioned above, Hartshorne, Migliore, Rao and others, many important features, such as divisor classes, Cohen-Macaulayness, normality, Serre’s conditions $S_k$ and $R_k$, have been studied for linkage. Meanwhile in the past twenty years the theory of singularities of algebraic varieties has been developed vastly. In particular, singularities arising from the minimal model program have played a central role and many modern techniques have been introduced. In his recent project, Wenbo is interested in understanding how the singularities of $X$ can be compared to the singularities of $Y$. By using resolution of singularities and multiplier ideal sheaves he is able to produce a machinery to study the singularities of $Y$. As a consequence, he gives a criterion when $Y$ has rational singularities. He also shows that log canonical threshold increases and log canonical pairs are preserved in generic linkage.

**Giorgio Ottaviani** worked in the field of algebraic geometry, especially about moduli of vector bundles, homogeneous vector bundles and group actions, small codimension subvarieties. The techniques of homogeneous vector bundles can be used to study the syzygies of homogeneous varieties, starting from the basic case of the Veronese variety.

Ottaviani is recently interested in tensor decomposition and tensor rank . The rank of a tensor $t$ is the minimum number of decomposable tensors $t_i$ needed to express $t = \sum_i t_i$. This is called a decomposition of $t$. When $t$ is a symmetric tensor, this amounts to the classical Waring decomposition of a polynomial as a sum of powers of linear forms. On the geometric side, a tensor of rank $k$ lies in the $k$-secant variety of the variety of decomposable tensors. Decomposable tensors fill the Veronese variety, (respectively the Grassmann variety, the Segre variety) in the symmetric (resp. skew-symmetric, general) case. The equations of the $k$-secant varieties of these classical varieties are still unknown in general. Such equations can be found, for small values of $k$, by minors of certain contraction maps, called flattenings or, more generally, Young flattenings. A nice example is the 3-secant variety of the cubic Veronese embedding of the plane, parametrizing polynomials which are cubes, like $x_0^3$. Its 3-secant variety is a hypersurface in $P^9$, which parametrizes polynomials which are sum of three cubes, like the Fermat cubic $x_0^3+x_1^3+x_2^3$. Its equation is a polynomial of degree $4$, called the Aronhold invariant, which can be expressed as the generator of the pfaffian ideals of a $9 \times 9$ skew-symmetric contraction matrix. Equations of this kind are useful to effectively perform the tensor decomposition.

A very recent application, joint with J.M. Landsberg, is a lower bound for the complexity of the algorithm of matrix multiplication.

**Stepan Paul's** current research involves trying to understand the structure of multi-graded betti tables using recently developed tools in Boij-Söderberg theory, and seeking to apply new results in different contexts. Multi-graded polynomial rings arise naturally as the total coordinate rings for smooth toric varieties, carrying a grading by the Picard group. A recent pre-print from David Eisenbud and Daniel Erman establishes a foundation for starting to extend Eisenbud-Schreyer duality (in the standard $\mathbf{Z}$-graded case) to the toric case. Stepan Paul has been thinking about how to build on this foundation, keeping in mind the slightly pathological case where the toric variety in question has a non-simplicial cone of effective divisor classes.

Multi-graded betti tables also become useful when investigating the coordinate ring for a Veronese embedding of a projective variety. For example, the degree-$d$ Veronese embedding of $\mathbf{P}^n$ into $\mathbf{P}^N$, where $N = \binom{d+n}{n} - 1$, induces a $\mathbf{Z}^n$-grading on $R = \mathbf{k}[x_1, \ldots, x_N]$. Stepan Paul has been considering how a suitable multi-graded version of Boij-Söderberg theory might answer open questions about the free resolution of the coordinate ring as an $R$-module.

**Mihnea Popa** currently works on problems related to generic vanishing theory and derived categories of coherent sheaves. He has been thinking about the invariance of Hodge numbers under derived equivalence, a problem inspired by mirror symmetry and string theory, and was able to prove it together with C. Schnell in the case of smooth projective threefolds. He is currently trying to prove the appropriate generalization to the case of stringy Hodge numbers for singular threefolds, using the framework of derived categories of smooth stacks introduced by Kawamata. On a related note, he is also thinking about the non-negativity of stringy Hodge numbers, a problem posed by Batyrev. In a different direction, he has proved together with C. Schnell results generalizing the classical framework of generic vanishing theory to the setting of $\mathcal{D}$-modules and local systems, and is applying them to study the singular cohomology algebra of irregular varieties.

**Maria Evelina Rossi** has been working on the classification of Artinian Gorenstein local rings. A celebrated paper by H. Bass in 60's outlines the ubiquity of Gorenstein rings: they play a fundamental role in many theories and constructions in Commutative Algebra and Algebraic Geometry. The common aim of some recent papers is to prove structure's theorems and to classify, up to analytic isomorphisms, Artinian local rings which are Gorenstein of given multiplicity. B. Poonen and A. Iarrobino proved that there exists a finite number of isomorphism classes of Artinian algebras of multiplicity at most $6$. J. Elias and G. Valla proved the existence of codimension $2$ complete intersection algebras of multiplicity $10$ with infinitely many isomorphism classes. In this direction other results come from D. A. Cartwright, D. Erman, M. Velasco, and B. Viray. Recently, jointly with Elias, we proved that the classification of Artinian Gorenstein local rings with socle degree $3$ can be reduced to the classification of $K$-standard graded algebras with socle degree $3$ or, equivalently, to the projective classification of cubic forms in $\mathbf{P}^n$. As a consequence, a complete classification arises in the case of embedding dimension three. Part of these unexpected results have been extended to compressed level $K$-algebras.

In the one-dimensional case the problem becomes much more difficult, even if one considers local complete intersections of codimension two.

The motivating goal behind this research come from classical problems:

- the study of the irreducibility and the smoothness of the punctual Hilbert scheme parameterizing 0-dimensional subschemes of fixed degree in $\mathbf{P}^n$
- the study of the rationality of the Poincare' series of $K$-algebras
- the study of free resolutions over Gorenstein algebras
- the study of the Hilbert function of complete intersection curve singularities.

**Mike Roth's** recent research uses ideas from asymptotic algebraic geometry to prove results in diophantine approximation. As an example, motivated by the Bombieri-Lang conjecture one should study how local positivity of a line bundle influences the local accumulation of rational points. Together with David McKinnon, Roth introduces an approximation constant measuring how well an algebraic point on a variety can be approximated by rational points. and shows that this constant has great formal similarity with the Sesadri constant. Most importantly they show that the classic approximation theorems on the line — the theorems of Liouville and K.F. Roth — generalize to inequalities between these invariants valid for all projective varieties.

The research of **Steven Sam** can be roughly classified into three different topics, listed below.

In Boij–Söderberg theory the goal is to construct modules with linear resolutions on special classes of varieties and study the cone of Betti tables over quadric hypersurface rings. Boij–Söderberg theory is a significant new approach to free resolutions over polynomial rings and cohomology tables of sheaves over projective space. In joint work with Berkesch, Erman, and Kummini, he produced the following three results: (1) an interpretation of a certain partial ordering arising in this theory, (2) a functorial construction for pure resolutions, which are at the heart of the theory, and (3) a description of the situation for regular local rings and local hypersurface rings.

To extend this theory to other varieties, the former is related to the existence of pure resolutions and the latter is related to the existence of Ulrich modules, and hence to elimination theory via work of Eisenbud–Schreyer. This theory gives a numerical decomposition of modules and sheaves into basic ones, and a "duality" between free resolutions and cohomology tables. These are mysterious and demand a conceptual explanation. A basic step towards this is to understand the situation for other varieties. The current work on quadric hypersurfaces has led to connections with classical invariant theory and twisted commutative algebras, which is current joint work with Snowden and Weyman.

Sam has been also working on Discriminants. The goal is to understand the relationship between "discriminantal degeneracy loci" and Abelian varieties connected to Vinberg's theory of $\theta$-representations. This is part of a larger project that originates with groundbreaking work of Bhargava on counting number fields and elliptic curves of bounded discriminant, rank, etc. While many orbit spaces of $\theta$-representations admit natural moduli interpretations, the "sporadic" examples are opaque. In work with Gruson and Weyman, he established a general framework for discovering moduli interpretations of Vinberg's $\theta$-representations using ideas from degeneracy loci and free resolutions. This connects to many topics in classical algebraic geometry and sheds new light on constructions and theorems.

Sheaf cohomology has been used successfully many times to study equations for "nice" varieties. Non-normality is a major technical hurdle, and appears in important examples like nilpotent orbits in semisimple Lie algebras. This project aims to produce tools and examples to understand the "degree of non-normality" which is a first step toward overcoming such obstacles. The main focus is on varieties arising in linear algebraic or representation-theoretic situations and to use tools from representation theory. He found a pattern for the nature of the non-normality of the "Kalman varieties" which will hopefully provide a framework for understanding other examples of interest.

**Hal Schenck** uses algebraic and computational methods to study problems at the interface of algebra, geometry, and combinatorics. His research has three main themes: Toric varieties, Geometric modelling: implicitization and approximation theory, Hyperplane arrangements.

Toric varieties are objects defined by the discrete geometric data of a fan, and here Schenck studies the interplay between the fan and various algebraic objects associated to the fan (such as the Chow ring), as well as properties of the homogeneous coordinate ring associated to a divisor. For example, he recently gave a description of certain classes in equivariant Chow cohomology for a nonsimplicial fan. The result differs substantially from the simplicial case. Interestingly, this ties into work in applied mathematics, as the equivariant Chow ring is an algebra over a polynomial ring on the ambient space of the fan, and corresponds to continuous splines on the fan. In a second paper, the Cartan-Eilenberg spectral sequence is used to obtain sufficient conditions for freeness of splines on the fan, a topic of much interest in approximation theory.

A second applied topic involves rational surface modelling and implicitization. Work of Simis-Vasconcelos on approximation complexes and Rees algebra techniques show that obtaining the syzygies of some incomplete linear system is a crucial step in determining the implicit equations. In recent work with Seceleanu and Validashti, Schenck studies this question for a four dimensional basepoint free subspace of sections of bidegree $(2,1)$ on $\mathbf{P^1} \times \mathbf{P^1}$. The result is a complete classification of all possible minimal free resolutions (there are six types), as well as a beautiful dictionary between syzygies on the sections, and singularities of the resulting surface in $\mathbf{P^3}$.

On the final topic of hyperplane arrangements, Schenck (in joint work with Terao and Yoshinaga) has recently proven an inductive criterion for splitting of the module of vector fields on $\mathbf{P^3}$ which are tangent to an arrangement of smooth plane curves with quasihomogeneous singularities. There should be many generalizations possible here.

**Frank-Olaf Schreyer** has been recently working on the three projects listed below.

Joint work with Alessandro Chiodo, David Eisenbud, and Gavril Farkas is on syzygies of torsion bundles and the geometry of the level $\ell$ modular variety over $M_g$. They formulate, and in some cases prove, three statements concerning the purity or, more generally, the naturality of the resolution of various modules one can attach to a generic curve of genus g and a torsion point of $\ell$ in its Jacobian. These statements can be viewed an analogues of Greens Conjecture and we verify them computationally for bounded genus. They then compute the cohomology class of the corresponding non- vanishing locus in the moduli space $R_{g,\ell}$ of twisted level $\ell$ curves of genus $g$ and use this to derive results about the birational geometry of $R_{g,\ell}$. For instance, they prove that $R_{g,3}$ is a variety of general type when $g > 11$ and the Kodaira dimension of $R_{11,3}$ is greater than or equal to 19. In the last section we explain probabilistically the unexpected failure of the Prym-Green conjecture in genus 8 and level 2.

Schreyer has a joint project in progress with Madhusudan Manjunath and John Wilmes on the Toppling Complex. Let $G$ be a undirected connected graph possibly with multiple edges on vertices $x_1, \ldots, x_n$. Let $\Delta_G$ be its laplacian and let $J_G$ be the corresponding binomial ideal in the polynomial ring $K[x_1, , x_n]$. The authors describe the minimal free resolution of the toppling ideal $J_G$ in terms of oriented connected partitions of the graph with a unique sink.

Frank-Olaf Schreyer and Laura Costa study Ulrich bundles on ACM varieties of codimension 2. They establish the existence of Ulrich bundles on determinantal varieties.

**Eric Sharpe's** research is in physics. A generalization of mirror symmetry, known as "(0,2) mirror symmetry," is being studied. Existence of this generalization has long been conjectured, and would be a powerful computational tool in studies of compactifications of heterotic string theory, specified by a compact Kähler manifold $X$ together with a holomorphic vector bundle $\mathcal{E} \rightarrow TX$ satisfying the constraints

$$\det \mathcal{E}^* \;\cong\; K_X, \quad \mathrm{ch}_2(TX) \;=\; \mathrm{ch}_2(\mathcal{E}).$$

(0,2) mirror symmetry exchanges pairs $(X, \mathcal{E})$, $(X', \mathcal{E}')$, and reduces to ordinary mirror symmetry between $X$ and $X'$ in the special case that $\mathcal{E} = TX$, $\mathcal{E}' = TX'$.

One part of this effort involves understanding an analogue of quantum cohomology, involving the sheaf cohomology groups $H^*(X, \wedge^* \mathcal{E}^*)$. These have a natural product structure

$$H^p(X, \wedge^q \mathcal{E}^*) \times H^{p'}(X, \wedge^{q'} \mathcal{E}^*) \;\longrightarrow\; H^{p+p'}(X, \wedge^{q+q'} \mathcal{E}^*),$$

and a trace $H^{\mathrm{top}}(X, \wedge^{\mathrm{top}} \mathcal{E}^*) \longrightarrow \mathbf{C}$, which are used to define a 'quantum' sheaf cohomology ring, analogous to quantum cohomology rings.

Recent work computes quantum sheaf cohomology rings in the special case that $X$ is a toric variety and $\mathcal{E}$ is a deformation of the tangent bundle $TX$, checking and extending results in the physics literature.

The results agree with the ordinary quantum cohomology relations described by Batyrev in the special case $\mathcal{E} = TX$.

**Ian Shipman** works in algebraic geometry and representation theory. He is currently exploring two circles of ideas. First, suppose that a reductive group $G$ acts on a smooth quasi-projective algbraic variety $X$. Given a $G$-equivariant line bundle $L$ (a linearization) on $X$, GIT provides an open set $X^s$ of semistable points together with a stratification of the complement $X \setminus X^s$. The geometry of this stratification, and the action of $G$ on it has been used in two recent preprints by Halpern-Leistner and Ballard-Favero-Katzarkov to relate the derived categories of $G$ equivariant coherent sheaves on $X^{ss}$ and $X$. In an preprint in preparation, Shipman and Dan Halpern-Leistner will use, under certain circumstances, this relationship to construct derived autoequivalences and to explain relations in the group of derived autoequivalences. Furthermore, this picture can be sometimes be used to construct derived equivalences among the various GIT quotients of $X$. In the setting of the abelian McKay correspondence, there is a variation of GIT connecting an abelian quotient singularity to its crepant resolutions and this gives rise to derived equivalences. Shipman and Halpern-Leistner are working to construct the crepant resolutions (when they exist) by first constructing different t-structures on the noncommutative resolution of the singularity, then obtaining the resolution as a moduli space of simple objects, following Bridgeland.

Shipman is also interested in the connection between the representation theory of finite dimensional algebras and the geometry of various varieties parameterizing their modules. Of particular interest is the degeneration order. A general linear group $G$ acts on the standard affine variety parameterizing framed modules of a fixed dimension over a finite dimensional algebra. The orbits for this action are in bijection with the isomorphism classes of modules. A module is said to degenerate to another if the closure of the orbit corresponding to the first module contains the orbit corresponding to the second module. There is a representation theoretic criterion for when this happens, involving an auxilliary module. However bounds on the dimension of this module, or constraints on its structure are unknown. With Ryan Kinser, Shipman is developing a technique involving the loop group $G((t))$ of $G$ to address such questions.

**Gregory Smith** has been working recently with Victor Lozovanu. In higher-dimensional algebraic geometry, vanishing theorems are indispensable for uncovering the deeper relations between the geometry of a subvariety and its defining equations. Given scheme-theoretic equations for a nonsingular subvariety, Victor Lozovanu and Gregory G. Smith prove that the higher cohomology groups for suitable twists of the corresponding ideal sheaf vanish. From that result, they obtain linear bounds on the multigraded Castelnuovo-Mumford regularity of a nonsingular subvariety, and new criteria for the embeddings by adjoint line bundles to be projectively normal. Their techniques also yield a new Griffiths-type vanishing theorem for vector bundles. To be more precise, let $X$ be a nonsingular complex projective variety with canonical line bundle $K_X$. A subvariety $Y \subseteq X$ with ideal sheaf $\mathcal{I}_Y$ is *defined scheme-theoretically* by the divisors $D_1, \ldots, D_r$ on $X$ if $Y = D_1 \cap \cdots \cap D_r$ and the map $\bigoplus_{j=1}^{r} \mathcal{O}_X(-D_j) \to \mathcal{I}_Y$ determined by the $D_j$ is surjective. For a nonsingular subvariety $Y \subseteq X$ of codimension $e$, the following is their main theorem: Let $L$ be a line bundle on $X$ and let $m$ be a nonnegative integer. If $Y$ is defined scheme-theoretically by the nef divisors $D_1, \ldots, D_r$ and $L \otimes \mathcal{O}_X(-(m+1)D_{s_1} - D_{s_2} - \cdots - D_{s_e})$ is a big, nef line bundle for all subsets $\{s_1, s_2, \ldots, s_e\} \subseteq \{1, \ldots, r\}$, then we have $H^i\big(X, \mathcal{I}_Y^{m+1} \otimes K_X \otimes L\big) = 0$ for all $i > 0$.

Under the assumption that each $D_j$ belongs to the linear system for some power of a single globally generated line bundle, Lozovanu and Smith recover the vanishing theorem of Bertram, Ein, and Lazarsfeld. In particular, this additional hypothesis induces an ordering on the subsets $\{s_1, \ldots, s_e\}$ and it is enough to consider the unique maximal subsets.

Part of **Mark Walker's** current research interests concern the singularity category (also known as the stable category) of complete intersections and the theory of affine and non-affine matrix factorizations.

Using a theorem of Orlov, the singularity category of a complete intersection is equivalent to the singularity category of a suitably defined non-affine hypersurface $Y$. In detail, $Y$ is the hypersurface in a smooth scheme $X$ defined by the vanishing of a regular global section $W$ of a line bundle $\mathcal{L}$ on $X$. The singularity category of $Y$ is, in turn, equivalent to the homotopy category of "twisted" matrix factorizations associated to the triple $(X, \mathcal{L}, W)$. An object in the latter category consists of a pair of algebraic vector bundles $E_1, E_0$ on $X$ and morphisms $E_1 \to E_0$ and $E_0 \to E_1 \otimes \mathcal{L}$ such that each of the two evident compositions is multiplication by $W$.

Using this machinery, Jesse Burke and Walker have redeveloped the theory of stable supports for modules over complete intersections in a more geometric manner. Additional applications include devising a notion of Chern classes for modules over a complete intersection, taking values in a suitably defined version of Hochschild homology for twisted matrix factorizations. Among other things, Walker hopes to use these Chern classes to establish the vanishing of Dao's so-called "$\eta$ invariant" for pair of modules over a complete intersection with an isolated singularity.

**Xin Zhou's** research interest lies in the study of asymptotic behaviors of syzygies. The first line of research he pursued is the asymptotic nonvanishing behavior of syzygies. Ein and Lazarsfeld, in their paper "Asymptotic syzygies of algebraic varieties", describes the nonvanishing of almost all syzygy groups asymptotically, when the embedding is defined by higher and higher multiples of an ample line bundle. They proved a noneffective result for general varieties and an effective result for projective spaces. In "Effective non-vanishing of asymptotic adjoint syzygies", using a variant of their method, Xin Zhou proves an effective result for adjoint syzygies of an arbitrary variety which specializes to the effective result for projective spaces of Ein and Lazarsfeld.

The other focus of Xin Zhou's research is the asymptotic behavior of syzygy functors of Veronese embeddings. For Veronese embeddings, the syzygy groups are functorial in the underlying vector space. Hence, they have functorial decompositions as Schur functors. In "Asymptotics of syzygy functors" (in prepration), Mihai Fulger and Xin Zhou start by counting the Schur functors in $\otimes S^d$, $\mathrm{Sym}^p\mathrm{Sym}^d$ and $\wedge^p\mathrm{Sym}^d$. Then they apply these results to the study of the functors $\mathbf{K}_{p,1}(d)$, $\mathbf{K}_{p,0}(b;d)$ and show that the functors have maximal scales of growth possible with respect to the parameter $d$, fixing the other parameters. They also study other interesting asymptotic behaviors present when varying the first parameter following a restriction method for Koszul cohomology suggested by Rob Lazarsfeld.

# 3 Presentation Highlights

The following six one-hour presentations on topics in the area of Syzygies in Algebraic Geometry took place at the workshop. They cover recent important developments in that area.

**Marian Aprodu** gave a one-hour talk on *Vector bundles and syzygies*, with emphasis on Green's Conjecture for curves on arbitrary $K3$ surfaces. He discussed recent developments in the theory of syzygies using vector bundle techniques. The talk was partly based on joint works with **G**avril Farkas.

**Aldo Conca** gave a one-hour talk on *Koszul algebras and their syzygies*. In a joint paper with Avramov and Iyengar it was shown that the syzygies of Koszul algebras behave very much as the syzygies of algebras with quadratic monomial relations; for example, the highest degree of an $i$'th syzygy of a Koszul algebra is at most $2i$. Conca presented new results concerning this analogy. He also discussed some open problems concerning the syzygies of modules over a Koszul algebra.

**Steven Cutkosky** gave a one-hour talk on *Multiplicities Associated to Graded Families of Ideals*. He had proved that limits of multiplicities associated to graded families of ideals exist under very general conditions. Most of the results hold for reduced excellent equicharacteristic local rings, with perfect residue fields. He discussed a number of applications, including a "volume = multiplicity" formula, generalizing formulas of Lazarsfeld and Mustata and of Ein, Lazarsfeld and Smith, and a proof that the epsilon multiplicity of Ulrich and Validashti exists as a limit for ideals in rather general rings, including analytic local domains. He presented a generalization of this to generalized symbolic powers of ideals, proposed by Herzog, Puthenpurakal and Verma. He also gave an asymptotic "additivity formula" for limits of multiplicities, and a formula on limiting growth of valuations, which answers a question posed by the author, Kia Dalili and Olga Kashcheyeva. The proofs are inspired by a philosophy of Okounkov, for computing limits of multiplicities as the volume of a slice of an appropriate cone generated by a semigroup determined by an appropriate filtration on a family of algebraic objects.

**Robert Lazarsfeld** gave a one-hour talk on *Asymptotic syzygies of algebraic varieties*. He discussed joint work with **L**awrence Ein and others concerning the asymptotic behavior of the syzygies of algebraic varieties as the positivity of the embedding line bundle increases.

**Giorgio Ottaviani** gave a one-hour talk on *The syzygies of Veronese embeddings*. The resolution of the Veronese embedding of $P^n$ is a basic algebraic object, not yet completely understood. It is a prototype for the behaviour of Betti numers of more general varieties, as shown by Ein and Lazarsfeld. It is well known that the resolution of the Veronese embedding can be in principle computed by the cohomology of certain homogeneous bundles. The category of homogeneous bundles on $P^n$ is equivalent to the category of representations of a certain quiver with commutativity relations. This gives a combinatorial point of view for the computation of the cohomology of the relevant homogeneous bundles. Ottaviani discussed these ideas having in mind the example of the Veronese embedding of $P^2$.

**Frank-Olaf Schreyer** gave a one-hour talk on *Syzygies of torsion bundles and the geometry of the level $\ell$ modular variety over $M_g$*. The talk was on joint work with Alessandro Chiodo, David Eisenbud, and Gavril Farkas. The authors formulate, and in some cases prove, three statements concerning the purity or, more generally, the naturality of the resolution of various modules one can attach to a generic curve of genus g and a torsion point of $\ell$ in its Jacobian. These statements can be viewed an analogues of Green's Conjecture and we verify them computationally for bounded genus. Schreyer focused on the unexpected failure of the Prym-Green conjecture in genus 8 and level 2, which they can establish probabilistically. It is expected that the Prym-Green conjecture fails for all genera which are powers of 2 and a theoretical explanation of this fact which has been discovered computationally is still to be found.

The other focus in the workshop was a Short Course on Matrix Factorizations and String Theory. The Short Course consisted of ten one-hour lectures.

The first two lectures in the Short Course were algebraic. **Ian Shipman** gave a talk on Orlov's theorem and related results. Orlov's theorem describes a precise relationship between two triangulated categories related to a projective hypersurface: the derived category of coherent sheaves, and the equivariant stable derived category of its affine cone. Shipman discussed semiorthogonal decompositions, the the equivalence between the equivariant stable derived category and a category of matrix factorizations, and the proof of the theorem. **Andrei Caldararu** gave a talk on curved algebras, curved dg-algebras, and curved $A_\infty$ algebras. He also discussed the twisted complexes construction, and how matrix factorizations can be viewed as analogues of complexes of projective modules for a certain curved algebra. The remaining eight lectures in the Short Course were:

- **David Berenstein**: "From quivers and superpotentials to algebras and representation theory"
- **Paul Aspinwall**: "The topological B-model and superpotentials"
- **Eric Sharpe**: "Boundary terms in 2d theories and matrix factorization"
- **Dave Morrison**: "D-brane algebras"
- **David Berenstein**: "Conjectures about superpotential algebras"
- **Sheldon Katz**: "Computation of superpotentials for D-branes"
- **Paul Aspinwall**: "Matrix factorization on the quintic"
- **Dave Morrison**: "Matrix factorizations in physics (Summary talk)".

# 4 The Workshop

The physicists and mathematicians working on the edge of physics discovered new generalizations and results about the matrix factorizations; and the language of the mathematicians and the physicists slowly diverged. The goal of our workshop was to bring the mathematics culture and the physics culture closer together, and to promote the exchange of new results between the two groups. We organized a 10-lecture short course on matrix factorizations. The following eight physicists and mathematical physicists participated in the workshop: Paul S. Aspinwall, David Berenstein, Andrei Caldararu, Sheldon Katz, David R. Morrison, Stepan Paul, Eric Sharpe.

The Workshop provided an opportunity to exchange open problems, share ideas, and explore in new directions on free resolutions in Algebraic Geometry and Commutative Algebra (and related topics).

We had a lively exchange of ideas and methods which will foster further research.

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