Minimum Rank, Maximum Nullity, and Zero Forcing of Graphs

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1 Mathematical Background

Combinatorial matrix theory, which involves connections between linear algebra, graph theory, and combinatorics, is a vital area and dynamic area of research, with applications to fields such as biology, chemistry, economics, and computer engineering. One area generating considerable interest recently is the study of the minimum rank of matrices associated with graphs.

Let F be any field. For a graph $G = (V, E)$, which has vertex set $V = \{1, \ldots, n\}$ and edge set E, $S(F, G)$ is the set of all symmetric $n \times n$ matrices A with entries from F such that for any non-diagonal entry a_{ij} in A, $a_{ij} \neq 0$ if and only if $ij \in E$. The minimum rank of G is

$$mr(F, G) = \min\{rank(A) : A \in S(F, G)\},$$

and the maximum nullity of G is

$$M(F, G) = \max\{null(A) : A \in S(F, G)\}.$$

Note that $mr(F, G) + M(F, G) = |G|$, where $|G|$ is the number of vertices in G. Thus the problem of finding the minimum rank of a given graph is equivalent to the problem of determining its maximum nullity.

The zero forcing number of a graph is the minimum number of blue vertices initially needed to force all the vertices in the graph blue according to the color-change rule. The color-change rule is defined as follows: if G is a graph with each vertex colored either white or blue, u is a blue vertex of G and exactly one neighbor v of u is white, then change the color of v to be blue. Let S be a subset of V. The derived coloring of S is the result of coloring each vertex in S blue, coloring each vertex not in S white, and then applying the color-change rule until no more changes are possible. A zero forcing set of G is a set $Z \subseteq V$ such that there are no white vertices in the derived coloring of Z. The zero forcing number of G is $Z(G) = min\{|Z| : Z$ is a zero forcing set of $G\}$. A zero forcing set of G, Z, is called a minimum zero forcing set of G if $|Z| = Z(G)$.

The relationships between $M(F,G)$ and $Z(G)$ for any graph G are discussed in papers devoted to the study of minimum rank problems. For extensive surveys on minimum rank and related problems, see [9] or [10]. It is shown in [1] that for any graph G, $M(F,G) \leq Z(G)$.

In recent years there has been extensive interest in minimum rank, maximum nullity, and related parameters which have applications as varied as communication complexity in computer science and control of quantum systems in mathematical physics. The recent survey [10] lists more than 100 related references, most from the last ten years.

2 History of Proposed Problems

The main objective of this Focused Research Group was to study the relationship between the maximum nullity and zero forcing number of complete subdivision graphs. Let $e = uv$ be an edge of $G = (V, E)$. Define G_e to be the graph obtained from G by inserting a new vertex w into $V(G)$, deleting the edge e and inserting edges uw and wv. We say that the edge e has been subdivided and call G_e an edge subdivision of G. A complete subdivision graph $s(G)$ is obtained from the graph G by subdividing every edge of G exactly once. In [3], [7], and [12], the authors investigate the maximum nullity and zero forcing number of graphs obtained by a finite number of edge subdivisions of a given graph and, among other results, establish results on the effect subdividing an edge has on the graph's zero forcing number and maximum nullity. It is known that there exists a graph G such that $M(F,G) < Z(G)$. In fact, there exists graphs G such that $M(F,G_e) < Z(G_e)$ for some edge $e \in E$. However, it was asked and conjectured in [3] and [7] for all fields F and graphs G, $M(F,s(G)) = Z(s(G))$. For certain classes of graphs, the conjecture has been proven to be true in [3] and [7].

Our secondary objective was to make progress on the Graph Complement Conjecture (GCC) for minimum rank and other Nordhaus-Gaddum type problems related to minimum rank and maximum nullity. We denote the complement of G by $\overline{G}$. The GCC, which arose from the 2006 AIM workshop [6], states: For any graph G, $mr(G) + mr(\overline{G}) \leq |G| + 2$. (see [6]). The GCC and its variants [2] are what graph theorists call Nordhaus-Gaddum type problems, in that they involve bounding the sum of a graph parameter evaluated at a graph G and its complement $\overline{G}$. Nordhaus-Gaddum type problems have been studied for many different graph parameters, including chromatic number, independence number, domination number, Hadwiger number, etc. [8].

Various members of our group attended the 2013 AMS Spring Central Sectional Meeting at Iowa State University and stayed afterwards for several days to collaborate. Despite extensive discussion of GCC, we made little to no progress on the Nordhaus-Gaddum problems. There the group also began to investigate problems on principal rank characteristic sequences (described in Section 3.2 below) at the suggestion of Pauline van den Driessche, who would become one of our group‘s participants. Most of us attended the 2013 Conference of the International Linear Algebra Society and met there for research discussions, where we made additional progress on principal rank characteristic sequences. Thus we decided to focus our efforts in the Focused Research Group on the maximum nullity and zero forcing number of complete subdivision graphs and principal rank characteristic sequences.

3 Scientific Progress Made

3.1 The maximum nullity and zero forcing number of complete subdivision graphs

As stated previously, the main goal of our group was to attack the following conjecture: $M(F,s(G)) = Z(s(G))$ for every field F and graph G. Our group not only proved that the conjecture is true, we also determined the value in the main result of [4]:

Theorem: For any graph G with n vertices, m edges, and c connected components and any field F,

$$M(F,s(G)) = Z(s(G)) = m - n + c + Z(s(BF(G))).$$

The graph $BF(G)$ is obtained by contracting every maximal bridgeless subgraph of G with more than one vertex to a single vertex. This structure naturally arose in our technique since [7] showed that $M(F, s(G)) = Z(s(G))$ for any connected and bridgeless graph G. We extended this result to include the exact value and structure of some zero forcing sets.

Theorem: Given any connected bridgeless graph G, and any vertices $u, v \in V(G)$, there exists a zero forcing set of $s(G)$, B, of order $m(G) - n(G) + 2$ such that $u \in B$ and v does not force.

This became the initial step in inductively proving that for any connected graph G, $m - n + 1 + Z(s(BF(G)))$ is an upperbound for $Z(s(G))$. The proof that for any connected graph G, $m - n + 1 + Z(s(BF(G)))$ is a lowerbound for $M(F, s(G))$ is also inductive.

3.2 Principal Rank Characteristic Sequence of a Matrix

The group discussed and made substantial progress on several variations of this problem. The definition of the principal rank characteristic sequence of a real symmetric matrix was introduced in [5] as a simplification of the principal minor assignment problem formulated in [11]. The group extended this to other fields under the assumption that the matrix is symmetric or complex Hermitian.

Definition: The *principal rank characteristic sequence* of an $n \times n$ symmetric or complex Hermitian matrix A, denoted by pr(A), is pr$(A) = r_0 r_1 \cdots r_n$, where $r_0 = 1$ if A has at least one zero diagonal entry, otherwise $r_0 = 0$, and $r_k = 1$ for $1 \le k \le n$ if A has at least one $k \times k$ principal submatrix of full rank, otherwise $r_k = 0$.

Given a sequence $r_0 \cdots r_n$ of 0s and 1s, the focus in [5] is to determine whether or not there exists a real symmetric matrix A so that pr$(A) = r_0 \cdots r_n$. If so then the sequence is *attainable*. Results in [5] completely resolved this for $1 \le n \le 6$ with partial results for $n = 7$. Our group began by determining all attainable sequences for a real matrix with $n = 7$, and giving an explicit matrix example for each such attainable sequence. We also answered negatively an open question from [5]: Is every attainable sequence with $r_0 = 1, r_1 = 0$ attainable by the adjacency matrix of some graph? By an exhaustive computer search we found a sequence for $n = 7$ that cannot be attained by a 0,1 adjacency matrix, but is attained by a real circulant matrix ([5], Example 6.7).

Results of [5] show that for $n = 5$, there is a difference in sequences attainable by real matrices and those attainable by complex Hermitian and complex symmetric matrices. We further distinguished between the latter two classes by finding an example with $n = 6$ that is attainable by a complex Hermitian matrix but not by a real or complex symmetric matrix.

These differences led the group to consider matrices over $\mathbb{Z}_2$ (the integers modulo 2). We first did an exhaustive search to find attainable sequences for small orders. This led to a complete characterization of attainable sequences over this field, together with an example matrix for each such sequence. The attainable sequences over $\mathbb{Z}_2$ are more restrictive than over the reals. For example, the sequence 0101 is not attainable over $\mathbb{Z}_2$, but the real matrix $A = J_3 - 2I_3$ has pr$(A) = 0101$.

We also further discussed attainable sequences for adjacency matrices of graphs and found some results for sequences with $r_n = 1$, i.e., the adjacency matrix is invertible. Some of these were motivated by extensive numerical calculations, noting that for an adjacency matrix $r_0 = 1, r_1 = 0$.

We are writing up the results outlined above and plan to submit a paper in the near future. Towards the end of our stay at BIRS, we formulated and began investigation on a refined problem, which we call the ANS refinement. Here 0 or 1 in pr(A) is replaced by A, S or N where A means that *all* of the principal submatrices of a given order have full rank, S means that *some* but not all of the principal submatrices of a given order have full rank, and N means that *none* of the principal minors of a given order have full rank. For an $n \times n$ matrix this gives a word $\ell_1 \ell_2 \cdots \ell_n$ where $\ell_i \in$ {A,S,N}. The problem then is to determine which words are attainable. We have some preliminary results on this problem for real symmetric matrices with computations for small orders and for ℓ_1=A, ℓ_2=N. This word problem is subtly different from (but more informative than) the sequence problem with entries in $\{0, 1\}$. For example, in the latter problem it is easy to append an additional 0 to a known sequence by doing a simple copy of the last row and column. However, for the refined problem appending an additional N to an attainable word may not be possible. For example, with real symmetric matrices ANA is attainable, but ANAN is not. Research on this refined problem, which like the previous problem includes a fascinating mixture of linear algebra and graph theory, is ongoing.

4 Acknowledgement

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