EFFECTIVE COMPUTATIONS IN ARITHMETIC MIRROR SYMMETRY

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1 Overview of the Field and Recent Developments

Suppose we have an algebraic variety X over a finite field $\mathbb{F}_p$ (e.g, the set of common zeros of a finite set of polynomials in n variables with coefficients in $\mathbb{F}_p$). The *congruent zeta function* (also known as the Hasse-Weil zeta function) is a generating function for the number of $\mathbb{F}_{p^d}$-valued points on X, that is, the number of n-tuples in $\mathbb{F}_{p^d}^n$ on the variety. Formally, the congruent zeta function is defined as:

$$\mathrm{Zeta}(X/\mathbb{F}_p, t) := \exp\left(\sum_{d\geq 1} \frac{\#X(\mathbb{F}_{p^d})t^d}{d}\right).$$

Dwork showed that this zeta function is actually rational [12], and can be factored in terms of polynomials with integer coefficients:

$$\mathrm{Zeta}(X/\mathbb{F}_p, t) := \frac{\prod_{j=1}^{n} P_{2j-1}(t)}{\prod_{j=0}^{n} P_{2j}(t)},$$

Here n is the dimension of the variety X. Furthermore, $P_0(t) = 1 - t$, $P_{2n}(t) = 1 - p^n t$, and for each $1 \leq j \leq 2n-1$, the polynomial $P_j(t)$ has degree b_j, the jth Betti numbers of the variety. (That is, $b_j = \dim H^j_{dR}(X)$.)

Mirror symmetry for Calabi-Yau threefold mirror pairs predicts that the Hodge numbers $h^{1,1}$ and $h^{2,1}$ are interchanged. The possible implications of this exchange for the arithmetic structure of the varieties were first explored in the physics literature in 2000 by Candelas, de la Ossa, and Rodriguez Villegas (see [4]). In particular, because the Hodge numbers control the Betti numbers, it follows that mirror symmetry will be reflected in the congruent zeta functions of mirror pairs. Shabnam Kadir studied this phenomenon for specific mirror pairs of Calabi-Yau threefolds in [17], using a two-parameter deformation of a diagonal hypersurface in a weighted projective space. Candelas and de la Ossa described techniques for computing the congruent zeta function of the pencil of Fermat threefolds in [3], and Kloosterman developed techniques for computing the zeta function of arbitrary deformations of a Fermat hypersurface in [19]. Recently, Goto, Kloosterman, and Yui have studied the congruent zeta function for a class of diagonal hypersurfaces in weighted projective spaces (see [15]). The higher regulator maps of Beilinson-Bloch have image a lattice of maximal rank whose volume is related to the value of the Hasse-Weil zeta function [2]. These higher regulator maps are investigated for anticanonical hypersurfaces in toric Fano varieties in connection with local mirror symmetry in the recent work of Doran and Kerr [11].

New research by computational number theorists provides a framework for understanding the congruent zeta function for varieties other than deformations of diagonal hypersurfaces. Much of the number theory literature focuses on properties of curves, but forthcoming work by Sperber and Voight describes an algorithm for computing the zeta function for hypersurfaces described by sparse polynomials in toric varieties (see [23]). These computational advances offered us an unprecedented opportunity to explore the arithmetic relationships between varieties that have been predicted by mirror symmetry.

2 Scientific Progress Made

2.1 Alternate mirror pencils

The first mirror symmetry construction, due to Greene and Plesser, used a one-parameter family of Calabi-Yau threefolds with a $(\mathbb{Z}/5\mathbb{Z})^3$ action to construct the mirror family to all quintic hypersurfaces in $\mathbb{P}^4$. This construction gives an expansion of the mirror map near the Fermat quintic. In [16], Greene, Plesser, and Roan used one-parameter families of quintic hypersurfaces with other abelian group actions to describe the mirror correspondence near other quintic hypersurfaces. This construction offers a way to study the relationship between the family of all quintic hypersurfaces and its mirror family at specific points in moduli space. More recently, [10] computed the Picard-Fuchs equations of the one-parameter families of Calabi-Yau hypersurfaces used in the [16] mirror construction; they showed that while the Picard-Fuchs equation of the holomorphic period is always the same (and identical to the Picard-Fuchs equation for the mirror family, see also [1]), the Picard-Fuchs equations of non-holomorphic differential forms are sensitive to the particular discrete group action. We conjecture that the congruent zeta functions will encode this information about the common structure of these quintic pencils. In this way, mirror symmetry provides a powerful tool for making predictions about the arithmetic properties of Calabi-Yau varieties.

Because all K3 surfaces have the same Hodge diamond, mirror symmetry constructions for K3 surfaces are more subtle: the constructions depend on choosing a *lattice polarization*, which specifies a sublattice of the Picard group of algebraic curves on the K3 surface. We wish to investigate the Picard-Fuchs equations and congruent zeta functions for families of (ADE singular/lattice-polarized) quartic hypersurfaces defined in analogy with the quintic families studied in [16], and study the relationship between the congruent zeta function and the Picard group structure.

As the Zeta function is a generating function for the number of $\mathbf{F}_q$-valued points on a variety (or family of varieties), it is essential to know how to compute the number of points efficiently. In [4], this is computed for the quintic Fermat pencil using Dwork characters. In [22], classical theorems by Delsarte and Koblitz and p-adic methods are used for the computations. These were extended to study the point count for the following alternate mirror pencils for:

$$x_1^4 + x_2^4 + x_3^4 + x_4^4 - 4\psi x_1 x_2 x_3 x_4 = 0$$

This Zeta function is well-understood. In [17], for example, it is written as

$$\mathrm{Zeta}(X/\mathbb{F}_p, t) = \frac{1}{(1-t)P(\psi,t)(1-p^2t)}$$

where P has degree 22 and can be factored as

$$P(\psi,t) = R_0(\psi,t)Q(\psi,t)(1-pt)$$

and $\deg(R_0) = 3$ and $\deg(Q) = 18$. As mentioned earlier, the Greene-Plesser construction gives us a mirror family which is also a K3 surface, Y. It is also known that

$$\text{Zeta}(Y/\mathbb{F}_p, t) = \frac{1}{(1-t)R_0(\psi,t)(1-pt)^{19}(1-p^2t)}.$$

We used the p-adic methods mentioned above to understand the point count for the following alternate pencils:

1. $x_1^3x_2 + x_2^3x_3 + x_3^3x_1 + x_4^4 - 4\psi x_1x_2x_3x_4 = 0$
2. $x_1^3x_2 + x_2^3x_3 + x_3^3x_4 + x_4^3x_1 - 4\psi x_1x_2x_3x_4 = 0$
3. $x_1^3x_2 + x_2^3x_1 + x_3^4 + x_4^4 - 4\psi x_1x_2x_3x_4 = 0$
4. $x_1^3x_2 + x_2^3x_1 + x_3^3x_4 + x_4^3x_3 - 4\psi x_1x_2x_3x_4 = 0$
5. $x_1^2x_2^2 + x_2^2x_3^2 + x_3^2x_4^2 + x_4^2x_1^2 - 4\psi x_1x_2x_3x_4 = 0$

Understanding these point-counts p-adically allows for quick computations using either Pari-GP or Sage for various primes and parameters. Already it is evident from the computations that all these Zeta functions have a common factor, $R_0(\psi, t)$, which is related to the invariant period under the group action.

Naïve point counting using Magma also yielded some interesting patterns. For example, if p is not congruent to 1 mod 4 or 7, the point counts for the Fermat quartic and the Klein-Mukai quartic $x_1^3x_2 + x_2^3x_3 + x_3^3x_1 + x_4^4$ are exactly the same. If we extend this analysis to compare point counts for different values of the parameter ψ in the Fermat pencil and the Klein-Mukai pencil (1), we find that the point counts are identical precisely $\frac{p+1}{2}$ times.

Adjusting the p-adic point-counting methods to weighted projective space was another reasonable generalization. In particular, the group worked on generalizing a classic theorem of Delsarte [9] and Furtado Gomida [14], originally for projective hypersurfaces, to weighted projective spaces.

Working with the alternate mirror families highlights the need for a detailed understanding of the interaction between discrete group actions on the pencils and the corresponding Zeta functions. In [19], Kloosterman proves that the Zeta function of certain one-parameter families of hypersurfaces in weighted projective space can be factored in such a way that each factor corresponds to a (weak) equivalence class in the p-adic cohomology under the discrete group action. This cohomological approach, also used by Katz [18], Lauder [20], Kadir [17], and Sperber and Voight [23], among others, provides another route for analyzing Zeta functions of the alternate mirror families.

2.2 Lattice polarized K3 surfaces in $\mathbb{WP}(1,1,4,6)[12]$

Recent work by McOrist, Morrison, and Sethi builds on work of Clingher-Doran and Clingher-Doran-Lewis-Whitcher in order to investigate the properties of the string theory construction known as *F-theory* described by certain families of K3 surfaces with high Picard rank (see [21, 6, 7, 8]). These families may be explicitly realized in two ways: as resolutions of singular hypersurfaces in projective space, and as hypersurfaces in toric varieties.

Let us consider the K3 surfaces polarized by $\mathrm{M} = \mathrm{H} \oplus \mathrm{E}_8 \oplus \mathrm{E}_8$, where H is the standard rank-two hyperbolic lattice and E_8 is the unique even, negative-definite and unimodular lattice of rank eight.

Theorem 1 ([6])**.** Let Y be an M-polarized K3 surface. Then:

1. Y is isomorphic to the minimal resolution of a quartic surface in $\mathbb{P}^3$ given by
$$y^2zw - 4x^3z + 3axzw^2 + bzw^3 - \frac{1}{2}(dz^2w^2 + w^4) = 0;$$
2. the parameters a, b, and d in the above equation specify a unique point $(a, b, d) \in \mathbb{WP}(2,3,6)$ with $d \neq 0$;
3. Y canonically corresponds to a pair of elliptic curves $\{E_1, E_2\}$;

4. the modular parameters of Y and $\{E_1, E_2\}$ are related by

$$\pi = j(E_1)j(E_2) = \frac{a^3}{d} \qquad \text{and} \qquad \sigma = j(E_1) + j(E_2) = \frac{a^3 - b^2 + d}{d};$$

Because M-polarized K3 surfaces are tightly related to pairs of elliptic curves, we can use this family to probe the relationship between the Zeta functions of elliptic curves, which are well understood, and the Zeta functions of higher-dimensional varieties. In particular, the Picard rank of an M-polarized K3 surface will increase from the general rank of 18 when the corresponding elliptic curves are isogenous. We expect that these jumps in Picard rank will be reflected in factorizations of the Zeta function for appropriately chosen parameter values.

An alternative realization of the M-polarized K3 surfaces is more tractable for computational considerations: we may construct the family of surfaces as the mirror family to the anticanonical hypersurfaces in $\mathbb{WP}(1, 1, 4, 6)$, using either a Greene-Plesser style quotient of a highly symmetric family of hypersurfaces or the Batyrev-Borisov polar polytope construction.

Proposition 1 ([8])**.** An anticanonical hypersurface in the space polar to the weighted projective $\mathbb{WP}(1, 1, 4, 6)$ is an M-polarized K3 surface defined by

$$\lambda_0 x_0^{12} + \lambda_1 x_1^{12} + \lambda_2 x_2^2 + \lambda_3 x_3^3 + \lambda_4 x_0^6 x_1^6 + \lambda_5 x_0 x_1 x_2 x_3.$$

This equation is related to the normal form given in Theorem 1 by

$$a^3 = \frac{1}{12^6 \Lambda_0^2 \Lambda_1},\ b^2 = \frac{(6 \cdot 12^2 \Lambda_0 - 1)^2}{12^6 \Lambda_0^2 \Lambda_1},\ d = 1,\ \text{with } \Lambda_0 = \frac{\lambda_2^3 \lambda_3^2 \lambda_4}{\lambda_5^6},\ \Lambda_1 = \frac{\lambda_0 \lambda_1}{\lambda_4^2}.$$

The parameters $\lambda_0, \ldots, \lambda_5$ are redundant. We set $\lambda_0 = \lambda_1 = \lambda_2 = \lambda_3 = 1$, $\lambda_4 = -2\phi$, and $\lambda_5 = -12\psi$, to obtain the 2-parameter family

$$x_0^{12} + x_1^{12} + x_2^2 + x_3^3 - 2\phi x_0^6 x_1^6 - 12\psi x_0 x_1 x_2 x_3. \tag{1}$$

In these coordinates, we have

$$\begin{aligned} a^3 &= 12^6 \psi^{12} \\ b^2 &= \left(\phi + 12^3 \psi^6\right)^2 \\ d &= 1. \end{aligned}$$

Let $X_{\phi\psi}$ be the hypersurface in $\mathbb{WP}(1, 1, 4, 6)$ defined by Equation 1, and let $Y_{\phi\psi}$ be the M-polarized hypersurface mirror to $X_{\phi\psi}$ defined by the same equation. We anticipate that the congruent zeta function of $X_{\phi\psi}$ will take the following form:

$$Z(X_{\phi\psi}(\mathbb{F}_p), T) = \frac{1}{(1-T)R_0(\phi, \psi, T)Q(\phi, \psi, T)(1-p^2T)} \tag{2}$$

Here R_0 is a polynomial of degree 4 in T corresponding to the periods of the holomorphic $(2, 0)$-form, and Q is a polynomial of degree 18 in T.

For general ϕ and ψ, the Néron-Severi group of $Y_{\phi\psi}$ is an 18-dimensional lattice fixed by the Frobenius action, so the denominator of $Z(Y_{\phi\psi}(\mathbb{F}_p), T)$ must contain the factor $(1 - pT)^{18}$. Because $X_{\phi\psi}$ and $Y_{\phi\psi}$ are related by a Greene-Plesser mirror construction, it follows from [17, Theorem 7.1] that $Z(Y_{\phi\psi}(\mathbb{F}_p), T)$ must take the form

$$Z(Y_{\phi\psi}(\mathbb{F}_p), T) = \frac{1}{(1-T)R_0(\phi, \psi, T)(1-pT)^{18}(1-p^2T)}. \tag{3}$$

3 Outcome of the Meeting and Future Plans

The beginning of the meeting was devoted to establishing a common language and presenting each person's background and area of expertise. There are still many open problems to work on and computations to refine, and the group intends to meet many more times in the next year. This meeting allowed us to begin what promises to be a long and fruitful research collaboration that spans an international community.

The mirror symmetry perspective highlights the importance of understanding the behaviour of the Zeta function for parametrized families of varieties, rather than concentrating on individual hypersurfaces. In particular, one is naturally led to investigate degenerations of hypersurfaces: how does the Zeta function change at a singular member of a family? The properties of the Zeta function for singular varieties are not well understood. (For some results on deformations of Fermat hypersurfaces in projective space or weighted projective space, see [5, 13].) As we develop computational methods, we create the tools and identify the examples necessary to investigate these properties in a controlled fashion.

A complete understanding of arithmetic mirror symmetry demands techniques for computing the Zeta functions of mirror pairs of varieties obtained via Batyrev's reflexive polytope construction. A natural extension of the current work is to extend the p-adic point-counting methods to the case of hypersurfaces in toric varieties, particularly toric varieties that are not weighted projective spaces or discrete group quotients of weighted projective spaces.

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