Report on Whitney Problems Workshop
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1 Introduction

1.1 Whitney Problems

Motivated by boundary value problems for partial differential equations, classical trace and extension theorems characterize traces of spaces of generalized smoothness (e.g., Sobolev, Besov, etc.) to smooth submanifolds of a Euclidean space. But in many cases one needs similar results for subsets of a more complicated geometric structure (for instance, after the change of variables initial data may be situated on a Lipschitz surface). The subject is originated from Hassler Whitney seminal papers of 1934 which, in particular, deal with the following problem: given a real function on an arbitrary subset of a Euclidean space to determine whether it is extendible to a function of a prescribed smoothness on the whole of space?

Whitney developed important analytic and geometric techniques which allowed him to solve this problem for functions defined on subsets of the real line to be extended to m-times continuously differentiable functions on it. He also formulated and solved similar problems related to jets of functions defined on a subset of a Euclidean space (thought of as the family of Taylor polynomials of a function at points of this subset) in any dimension. Another important result of Whitney (later generalized by G. Glaeser) asserts that a Whitney-regular domain (also known as a quasi-convex domain) allows an extension of functions with bounded derivatives of a given order to functions of the same smoothness on the whole space.

In the decades since Whitney's seminal work, fundamental progress to the problem was made by Georges Glaeser, Yuri Brudnyi and Pavel Shvartsman and Edward Bierstone, Pierre Milman, Len Bos and Wiesław Pawłucki. Also, very important results related to a geometric description of Sobolev extension domains were obtained by Nahum Zobin and Pavel Shvartsman.

In 1958 Georges Glaeser gave a geometric description of traces to subsets of a Euclidean space of 1-times continuously differentiable functions based on his notion of the "iterated paratangent space".

Later on Yuri Brudnyi and Pavel Shvartsman conjectured a "finiteness principle" for traces of m-times continuously differentiable functions with the m^{th} derivative having modulus of continuity ω (a similar principle for traces of smooth functions on the real line was established by H. Whitney) and proved this conjecture for $m = 1$ by establishing a deep Lipschitz selection theorem, an analog of the classical Helly theorem on convex sets.

For subanalytic sets X with $C^\infty(X) = \cap_{m\geq 0} C^m(X)$, called 'tame', Edward Bierstone and Pierre Milman proved in 90's existence of C^∞ extensions and, for arbitrary subanalytic sets in 2003 with Wiesław Pawłucki, a variant of C^m extensions based on their C^m analogue of Glaeser's C^1 iterated paratangent spaces, i.e., a variant of the Whitney problem for traces of C^m functions to subanalytic sets. For sets on which the traces of the C^∞ functions determine their infinite jets Len Bos and Pierre Milman in 1995 found geometric and analytic characterizations of the existence of C^∞ extensions majorating the sup norm by the sup norm. Curiously the latter property for an arbitrary subanalytic set implies that it is 'tame' (but is not known to hold even for a union of a parabola with its tangent line).

Building on this work, in a series of recent papers Charles Fefferman solved the original problem of Whitney in full generality. His methods led to a number of very important developments in the field, including new analytic and geometric methods in the study of Lipschitz structures on finite sets. Also, Charles Fefferman, partially in collaboration with Bo'az Klartag, developed powerful methods of computation of extensions.

It is natural also to consider similar extension and trace problems for functions in Sobolev spaces. These results are at a much earlier stage. A substantial progress in this direction for functions on subsets of a Euclidean space was recently obtained by the "Princeton group" (Charles Fefferman, Arie Israel, Garving Luli) and, by very different methods and in very different terms, by Pavel Shvartsman.

There is a substantial progress in the old problem of description of Sobolev extension domains. The case of Sobolev L_∞ norms was completely resolved by Nahum Zobin in the 1990s; Pavel Shvartsman has recently made a serious progress in determining good sufficient conditions for a domain to be a Sobolev extension domain.

1.2 Workshop

The Whitney Problems Workshop at the Banff International Research Station in April of 2013 was the sixth meeting devoted to these problems since 2008 (Williamsburg, VA 2008, 2009, 2011, Palo Alto, CA, 2010, Toronto, Canada, 2012). It has brought together an international group of researchers actively involved in this field as well as several newcomers who got interested in the topic to discuss recent progress and open problems in the area of Whitney type problems and thus foster interaction and collaboration between researchers in the field.

The workshop was organized as follows: each day we had two hour lectures in the morning and two hour lectures in the afternoon, followed by informal discussions after 5 pm every day. Because one of the planned lecturers, Professor Goldshtein of the Ben Gurion University in Israel, was unable to come due to a pilots' strike in Israel, we had to somewhat change the initial schedule – Professor Goldshtein was asked by the organizers to prepare a special three lectures course on subjects closely related to the Whitney Problems but not well known to other participants.

By the end of the Workshop we felt that we have not set aside sufficient time for the participants to work together in small groups, which was obviously very much needed, and therefore we decided to use three one hour slots on Thursday and Friday for such collaborative work and discussions in small groups, which we feel were extremely useful.

2 Main topics

The topics most discussed at the Workshop were the following

(i) Computational aspects of Whitney extension problems in Sobolev spaces (C. Fefferman, A. Israel, K. Luli, P. Shvartsman).

(ii) Finiteness principle (C. Fefferman).

(iii) Sobolev extension domains (P. Shvartsman, N. Zobin).

(iv) Spaces of functions of bounded mean oscillation and problems of their extension (P. Shvartsman).

The discussions in (i) and (ii) were based on the papers [1]–[22]; in (iii) on the papers [23]–[27]; in (iv) on the papers [28]–[37].

There were also lectures devoted to various problems arising in Denjoy-Carleman classes (A. Nicoara), to nonlinear methods of fast reconstruction of piecewise smooth functions from their Fourier coefficients (Y. Yomdin), Sobolev-Poincare inequalities in complicated domains (R. Hurri-Syrjanen), exact constants for Whitney-type extension operators (E. Le Gruyer), problems of geometric theory of quasiconformal mappings (I. Uriarte-Tuero). In all of these lectures there was a clear feeling of their connections to Whitney Problems, and they were offering insights which might prove to be very useful for Whitney Problems.

Participants were invited to suggest open problems and questions during the workshop. These included specific problems on which there was hope of making some progress during the program, as well as more ambitious problems which may influence the future activity of the field. Most of the suggested problems were discussed in small groups after 5 pm every day.

3 Main results reported

C. Fefferman, A. Israel and K. Luli reported on their results (partially joint with B. Klartag) regarding algorithmic questions of Sobolev-Whitney extension problems, which show that they are very close to the same deep understanding in this field as was previously achieved by C. Fefferman for spaces $C^m(\mathbb{R}^n)$. This is a very important progress since these problems for Sobolev spaces are very important for applications, and only several years ago they seemed to be quite inaccessible. Now there are two quite different approaches to these problems, one developed by K. Luli, A. Israel and C. Fefferman, and a quite different one developed by P. Shvartsman, which essentially resolve, in a very computable way, the main difficulties in this field. There were very intensive discussions of these topics during the Workshop, and the participants feel that these discussions were very instrumental in clarifying a number of serious issues.

C. Fefferman, in a series of three lectures, explained main ideas of his work on Whitney extension problems for spaces $C^m(\mathbb{R}^n)$, the key tool in which play a special Calderon-Zygmund decomposition of finite sets. These results are very technical and difficult to master, and the participants feel that after these talks a number of fundamental ideas has been thoroughly explained. A discussion of these ideas was very intensive, and the participants suggested some new ways to utilize them in other problems.

P. Shvartsman and N. Zobin reported on their work in progress aimed at a full geometric description of finitely connected planar Sobolev extension domains. A powerful sufficient geometric condition was established by P. Shvartsman several years ago, and now it seems that this condition is also necessary. A number of new geometric constructions were presented, which will hopefully lead to a proof of the desired result. These constructions are far reaching generalizations of constructions which allowed N. Zobin to resolve this problem in the case of Sobolev spaces $W^m_\infty(\mathbb{R}^2)$ back in the 90s. During the Workshop this problem was also very intensively discussed and these discussions have led to a much better understanding of many involved issues.

P. Shvartsman, in a series of two lectures, presented his results on extension of BMO functions. These results were based on a beautiful construction of an exotic nonlinear extension operator for these spaces. There has been a lively discussion (C. Fefferman, I. Uriarte-Tuero, P. Shvartsman, N. Zobin) on whether one can construct a linear extension operator, which led to a number of quite unexpected developments.

A. Nicoara reported on her recent results (joint with F. Aquistapace and F. Broglia) related to analogs of the Hilbert Nullstellensatz for Denjoy-Carleman quasi-analytic classes. These classes were proven to lack a number of algebraic properties usually employed to prove Nullstellensatz: they are non-Noetherian (Kessler-Nicoara), they do not satisfy the Weierstarss Division Theorem (Childress), they do not satisfy the Weierstrass Preparation Theorem (Aquistapace-Broglia-Bronshtein-Nicoara-Zobin). It has been shown that a resolution of the problem boils down to an analytic construction of a DC function with very special properties. Possible approaches to this problems have been discussed by the participants, hopefully one of them could lead to a solution of the problem.

Y. Yomdin reported on his results (joint with D. Batenkov) regarding the fast reconstruction of a piecewise smooth function from its Fourier coefficients. He has reported a solution of an old Eckloff problem who has conjectured that a nonlinear reconstruction can be done as fast as a linear reconstruction of a smooth function without discontinuities. It is worth mentioning that one of crucial ideas of this solution comes from discussions on the previous Whitney Problems Workshop in Toronto in August 2012.

In recent years there has been a growing interest in relations between the Whitney Problems and the theory of quasiconformal mappings. In particular, the Whitney partition of unity, which is of such importance in Whitney Problems (and in many problems of Harmonic Analysis) is essentially quasiconformally invariant. Moreover, theory of capacities, which has been developed for quite different reasons is now understood as essentially a Whitney type problem. We hoped that the planned lectures by Goldshtein, who is a leading authority in this field, would further clarify these connections. As it has been mentioned he was unable to come. Nevertheless, I. Uriarte-Tuero gave a wonderful lecture devoted to other aspects of this beautiful theory, and helped all participants to better understand its ideas.

R. Hurri-Syrjanen reported on her results (joint with A. Vakahangas) regarding the validity of the Poincare inequality and its relations to geometric properties of planar domains. These ideas have a lot in common with Shvartsman-Zobin investigation of geometry of Sobolev extension domains.

E. Le Gruyer reported on his results (joint with M. Hirn) related to attempts to compute the norm of the minimal extension, which generalize his previous work on minimal C^1 extensions. This is a very difficult problem and the main thrust is on attempts to guess the natural norm for which one can obtain the minimal extension.

Also, there has been a fruitful discussion (L. Bos, A. Brudnyi, Y. Yomdin) related to Bernstein-Markov type inequalities for polynomials and their use in the extension and trace problems for some classes of Morrey-Campanato spaces defined on certain fractal type subsets of $\mathbb{R}^n$ and for ϵ-nets in them, based on similar results presented in [19].

More specific details of the described presentations can be found in the Abstracts of Talks below.

4 Some lessons from the Workshop

During the Workshop it became clear that the program was overloaded with lectures. After four intensive lectures the participants were very exhausted and this prevented them from getting more actively involved in discussions. We have also seen that it is better to have discussions in smaller groups so that everyone is indeed interested in the discussion. In our future meetings we shall set aside more time for such informal discussions, even if this means a reduction in time devoted to lectures.

Preparing for our previous meetings we used to post problems proposed by the participants before the meeting. This helped the participants to find common interests and get involved in joint projects. This time we did not do this ahead, which is definitely a drawback.

5 Organization of the Workshop

The staff of the BIRS did a wonderful job in making the Workshop very productive and enjoyable. All our requests were met, everything was going very smoothly. The only problem we encountered was a poor quality of Wi-Fi connection at the Corbett Hall, and food was too good to resist.

6 Abstracts of Talks

Charles Fefferman. Finiteness Principle I, II, III.

Abstract: These expository talks motivate the proof of the Brudnyi-Shvartsman finiteness principle for $C^m(\mathbb{R}^n)$ starting with the simplest nontrivial case, $C^2(\mathbb{R}^2)$, and then passing to the general case.

Charles Fefferman. Whitney problems for Sobolev functions.

Abstract: Extension of functions from finite subsets of $\mathbb{R}^n$ to functions in a Sobolev space. Theorems (proven and checked), algorithms (being written up, probably correct), open problems.

Ritva Hurri-Syrjanen. On fractional Poincare and Hardy inequalities.

Abstract: Fractional Poincare inequalities and fractional Hardy inequalities come from the classical Poincare inequality and the classical Hardy inequality respectively, where the pth power of the absolute value of the function's gradient has been replaced by a fractional integral. In this talk we will review some known results on these inequalities. We will also discuss some recent developments of fractional inequalities in bounded irregular domains. Joint work with Antti V. Vahakangas.

Arie Israel. Computational Aspects of the Sobolev Extension Problem: Part 2.

Abstract: In this talk we describe some algorithms for manipulating dyadic decompositions of Euclidean space. These procedures are important components in a current work on near-linear time algorithms for computing Sobolev extensions. This is joint work with Charles Fefferman, Boaz Klartag, and Garving Luli.

Erwan Le Gruyer. Some results on minimal Lipschitz extensions for $m-$jets from $\mathbb{R}^D$ to $\mathbb{R}^n$ under restrictive assumptions.

Abstract: The purpose of the talk is to present some results involving the Lipschitz constant for $m-$jets defined on a non-empty subset of $\mathbb{R}^D$ with values in $\mathbb{R}^n$ under restrictive assumptions. In particular we restrict ourselves to functions

$F \in C^{m,1}(\mathbb{R}^D, \mathbb{R}^n)$ of the form

$$F = \sum_{k=1}^{D} F_k(x_k), F_k : \mathbb{R} \longmapsto \mathbb{R}^n, \ \forall k.$$

We produce minimal extensions and associated formulas which generalize Glaesers formula when ($D = 1$, $n = 1$) and also generalize a recently derived formula when ($D = 1$, n is arbitrary) (with Matthew Hirn).

Kevin Luli. Computational Aspects of the Sobolev Extension Problem: Part 1.

Abstract: Given a real-valued function f defined on a finite subset E of a Euclidean space $\mathbb{R}^n$, how can one find a near-optimal extension of f belonging to the Sobolev space $L^{m,p}(\mathbb{R}^n)$? How small can one take the Sobolev norm of an extension?

These questions are an instance of the Whitney extension problem. In previous work of Fefferman-Israel-Luli it was shown that there exists a bounded linear extension operator, and a formula was given for the near-optimal norm. The bounded linear extension operator and the approximate formula for the optimal norm were interesting theoretically, but the proof provided no means to construct these objects.

In this talk, we will review the recent results in the Fefferman-Israel-Luli paper. We will describe in broad strokes some of the key ingredients in the proofs and explain how the procedure can be modified to yield an efficient algorithm.

Andreea Nicoara. A Nullstellensatz for Lojasiewicz Ideals

Abstract: I will talk about recent progress on the Bochnak Nullstellensatz Conjecture, namely that a finitely generated ideal of smooth functions on an n-dimensional manifold equals the ideal of functions vanishing on its zero set iff the ideal is closed in the Whitney topology and real. Time permitting, I will indicate a possible way of completely settling this conjecture. (Joint work with Francesca Acquistapace and Fabrizio Broglia.)

Pavel Shvartsman. Extensions of BMO-functions and fixed points of contractive mappings in L_2 I, II.

Abstract: Let E be a closed subset of $\mathbb{R}^n$ of positive Lebesgue measure. We discuss a constructive algorithm which to every function f defined on E assigns its almost optimal extension to a function $F(f) \in BMO(\mathbb{R}^n)$. We obtain the extension $F(f)$ as a fixed point of a certain contractive mapping $T_f : L_2(\mathbb{R}^n) \to L_2(\mathbb{R}^n)$.

The extension operator $f \to F(f)$ is non-linear, and in general it is not known whether there exists a continuous linear extension operator

$$BMO(\mathbb{R}^n)|_E \to BMO(\mathbb{R}^n)$$

for an arbitrary set E.

In these talk we present a rather wide family of sets for which such extension operators exist. In particular, this family contains closures of domains with arbitrary internal and external cusps. The proof of this result is based on a solution to a similar problem for spaces of Lipschitz functions defined on subsets of a hyperbolic space.

Ignacio Uriarte-tuero. Two conjectures of Astala on distortion of sets under quasiconformal maps and related removability problems.

Abstract: Quasiconformal maps are a certain generalization of analytic maps that have nice distortion properties. They appear in elasticity, inverse problems, geometry (e.g. Mostow's rigidity theorem)... among other places.

In his celebrated paper on area distortion under planar quasiconformal mappings (Acta 1994), Astala proved that if E is a compact set of Hausdorff dimension d and f is K-quasiconformal, then fE has Hausdorff dimension at most $d' = \frac{2Kd}{2+(K-1)d}$, and that this result is sharp. He conjectured (Question 4.4) that if the Hausdorff measure $\mathcal{H}^d(E) = 0$, then $\mathcal{H}^{d'}(fE) = 0$.

First it was shown that Astala's conjecture is sharp in the class of all Hausdorff gauge functions (UT, IMRN, 2008).

Lacey, Sawyer and UT jointly proved completely Astala's conjecture in all dimensions (Acta, 2010). The proof uses Astala's 1994 approach, geometric measure theory, and new weighted norm inequalities for Calderón-Zygmund singular integral operators which cannot be deduced from the classical Muckenhoupt A_p theory.

These results are related to removability problems for various classes of quasiregular maps. I will mention sharp removability results for bounded K-quasiregular maps (i.e. the quasiconformal analogue of the classical Painleve problem) recently obtained jointly by Tolsa and UT.

I will further mention recent results related to another conjecture of Astala on Hausdorff dimension of quasicircles obtained jointly by Prause, Tolsa and UT.

The talk will be self-contained.

Yosef Yomdin. Fourier Sampling of Piecewise-Smooth Functions, Johnson– Lindenstrauss lemma, and Turan-Nazarov Inequality (with D. Batenkov).

Abstract: A periodic C^d smooth function f can be reconstructed from its first N Fourier coefficients with an error of order $1/N^d$. However, for f only piecewise C^d smooth the classical Fourier approximation has an error of order $1/N$, no matter how large d is. There is a long-standing open problem (Eckhoff Conjecture) concerning a possibility to gain the smooth accuracy rate $1/N^d$ via a non-linear manipulations with the first N Fourier coefficients of any piecewise C^d smooth function f. This problem was recently solved by D. Batenkov, via Algebraic Sampling approach. The key point was a proper choice of the samples among the first N Fourier coefficients of the function f. On the other hand, the problem of estimating robustness of the Fourier sampling on a given sampling set S is addressed by a discrete version of the well-known Turan-Nazarov inequality for exponential polynomials. It turns out to give rather accurate estimates and challenging predictions. There is another interesting connection between the Eckhoff problem and its versions with the general bounds on sampling accuracy of functions in given functional classes. It turns out that such bounds can be provided by a combination of Kolmogorovs ϵ-entropy and Johnson-Lindenstrauss dimensionality reduction. These bounds are accurate enough to imply a non-effective solution of the Eckhoff Conjecture, as well as many similar results for piecewise-analytic and other natural classes of non-regular functions. On the other hand, this approach raises basic problems related to the role of randomness

in the dimensionality reduction algorithms, as well as the required accuracy of the measurements.

We plan to discuss these and some other results and related open questions.

Nahum Zobin. Sobolev extension domains, I, II.

Abstract: We announce a solution of an old problem of description of planar finitely connected bounded Sobolev extension domains for $p > 2$, and for any smoothness.We present several new tools which allow to prove that the natural condition on subhyperbolic metric, which has been proven by Pavel Shvartsman to be sufficient for such a domain to be a Sobolev extension domain, is actually also necessary. The heart of the proof is an explicit construction of almost fats growing functions in such a domain (with Pavel Shvartsman).

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