Arithmetic groups

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The theory of arithmetic groups deals with groups of matrices whose entries are integers, or more generally, S-integers in a global field. This notion has a long history, going back to the work of Gauss on integral quadratic forms. The modern theory of arithmetic groups retains its close connection to number theory (for example, through the theory of automorphic forms) but also relies on a variety of methods from the theory of algebraic groups, particularly over local and global fields (this area is often referred to as the arithmetic theory of algebraic groups), Lie groups, algebraic geometry and various aspects of group theory (primarily, homological methods and the theory of profinite groups). At the same time, results about arithmetic groups have numerous applications in differential and hyperbolic geometry (as the fundamental groups of many important manifolds often turn out to be arithmetic), combinatorics (expander graphs), and other areas. There are also intriguing connections and parallels (which are currently not so well-understood) between arithmetic groups and other important classes of groups such as Kac-Moody groups, automorphism groups of free groups and mapping class groups.

The objective of the workshop was to survey the most significant results in the theory of arithmetic groups obtained primarily in the last five years in order to make the new concepts and methods accessible to a broader group of mathematicians whose interests are closely related to arithmetic groups. The workshop brought together 34 mathematicians, from the world's leading experts to recent PhD recipients and graduate students, working on a variety of problems involving arithmetic groups. This resulted in very active exchanges between and after the lectures. The scientific program of the workshop consisted of 3 mini-courses (two 45-min lectures each), 17 survey and research talks - 30 or 45 min - one of which (by Bertrand Rémy) was not planned in advance, a Q&A session, and an open problem session. The subjects of the mini-courses were: *Homological finiteness properties of arithmetic groups in positive characteristic* (Kevin Wortman), *Pseudo-reductive groups and their arithmetic applications* (Brian Conrad), and *Towards an arithmetic Kac-Moody theory* (Ralf Köhl). The mini-course on homological finiteness properties contained an account of a major breakthrough in the area – the proof of the Rank Conjecture. The course on the pseudo-reductive groups focused on arithmetic applications of the theory of pseudo-reductive groups, developed by Conrad, Gabber and Prasad, which include the proof of fundamental finiteness theorems (finiteness of the class number, finiteness of the Tate-Schafarevich set etc.) for *all* algebraic groups over the fields of positive characteristic, not just reductive ones. The course on Kac-Moody groups contained a series of results extending the known properties of (higher rank) arithmetic groups such as property (T) and (super)rigidity to Kac-Moody groups over rings.

While most aspects of the theory of arithmetic groups were represented at the workshop, some areas such connections between the cohomology of arithmetic groups and the theory of automorphic forms, the virtual positivity of the first Betti number of certain rank one lattices and the growth of higher Betti numbers, the analysis on homogeneous spaces modulo arithmetic groups were left out due primarily to time limitations. These topics should be included in the program of future meetings on arithmetic groups.

One of the more unusual features in the program was a Q&A session. It was meant to clarify as necessary the concepts extensively used in the lectures given in the first two days of the workshop, which was particularly useful to the junior researchers and graduate students. The session was based on the questions solicited by the organizers and submitted by the participants.

According to the official testimonials and numerous private e-mails received by the organizers, the workshop was very successful. In fact, many participants expressed the idea of holding mathematical meetings on the theory of arithmetic groups and related areas on a regular basis.

Below are detailed summaries of the mini-courses given during the workshop. Space constraints do not allow us to cover the talks uniformly in this manner. Instead, we give an overview of the topics covered at the conference. In any case, we strive to provide complete references.

Mini-Courses

Brian Conrad (Stanford University)
Pseudo-reductive groups and their arithmetic applications

A very highly-recommended survey of both the general theory and arithmetic applications is given in the Bourbaki report [63] by Bertrand Rémy. This includes a user-friendly overview of the contents of [28], indicating where main results can be found and how the logical development of the main proofs proceeds.

The *k-unipotent radical* $\mathscr{R}_{u,k}(G)$ of a smooth connected k-group G is its largest smooth connected normal unipotent k-subgroup. The formation of

$\mathscr{R}_{u,k}(G)$ commutes with separable extension on k, including $K \to K_v$ for a global function field K and place v. We say G is *pseudo-reductive* if it is smooth connected affine and $\mathscr{R}_{u,k}(G) = 1$. This coincides with "connected reductive" when k is perfect. The most basic example is the Weil restriction $\mathrm{R}_{k'/k}(G')$ for a finite extension k'/k and a connected reductive k'-group G'. The Weil restriction is pseudo-reductive, but it is never reductive if $G' \neq 1$ and k' is not separable over k.

Observation 1. *Any smooth connected k-group G has a short exact sequence*

$$1 \to \mathscr{R}_{u,k}(G) \to G \to \bar{G} \to 1$$

with pseudo-reductive $\bar{G}$.

This observation becomes useful since there is a structure theory for pseudo-reductive groups. However, we cannot hope to understand the commutative pseudo-reductive groups, but we will aim to describe the general structure modulo that ignorance.

Pseudo-reductive groups have some nice properties. Cartan k-subgroups are always commutative, $\mathrm{R}_{k'/k}(G')$ is perfect when G' is simply connected, and there is a theory of root systems (but can be non-reduced in characteristic 2, even if $k = k_s$). Overall, pseudo-reductivity is not a particularly robust concept, so its main purpose is the role it plays in trying to prove theorems about rather general linear algebraic groups when one might not have much control (e.g. for Zariski closures, working with stabilizer schemes for a group action, etc.).

There is a procedure (called the *standard construction*) which is a kind of pushout that replaces a Weil-restricted maximal torus $\mathrm{R}_{k'/k}(T')$ from a simply connected semisimple k'-group G' with another commutative pseudo-reductive group C according to a very specific kind of procedure. The final output of this process is a central quotient presentation

$$G = (\mathrm{R}_{k'/k}(G') \rtimes C)/\mathrm{R}_{k'/k}(T')$$

of a pseudo-reductive k-group G.

Theorem 2 (Structure Theorem)**.** *Given a pseudo-reductive k-group G, then there are $G', k'/k, T'$, and C as above such that $G = (\mathrm{R}_{k'/k}(G') \rtimes C)/\mathrm{R}_{k'/k}(T')$. Moreover, the data $(G', k'/k, T', C)$ are determined up to canonical isomorphism by the choice of a maximal k-torus in G.*

A general principle for applying the structure theory of pseudo-reductive groups: if a theorem is known in the smooth connected solvable affine case over k and in the connected semisimple case over all finite extensions of k then "probably" one can use the structure theorem via standard presentations (plus extra care in characteristics 2 and 3) to prove the result for all smooth connected linear algebraic groups (and something without smoothness or connectedness, depending on the specific assertion).

This principle is successful in various cases (e.g., [27]). First: finiteness for Tate-Shafarevich sets can be reduced to the pseudo-reductive case, and the *form* of the "standard presentation" is very well-suited to then pulling up the known result for the semisimple and commutative cases, essentially using vanishing theorems for simply connected groups. Second: finiteness for Tamagawa numbers is settled in general by a different kind of argument with the "standard presentation" for pseudo-reductive groups to pull it up from the known semisimple and commutative cases. Third: the original finiteness question for the local-global principle with orbits. That indeed works out affirmatively, due to the established case for Tate-Shafarevich sets. Fourth: various formulas for the behavior of Tamagawa numbers under exact sequences that were proved in Oesterlé's paper [52] conditionally on certain unknown finiteness results are now all valid unconditionally.

Ralf Köhl (University of Gießen)

Kac-Moody groups

By theorems of Tits and Curtis, a Chevalley group over a field with at least four elements is the product of its rank two subgroups amalgamated over the rank one subgroups. The following result goes to show that in the context of Chevalley groups over local fields, the topology is also forced by the rank one subgroups:

Theorem 1 (Glöckner-Hartnick-Köhl, [32])**.** *Let $\mathbb{F}$ be a local field, and let G be a Chevalley group over $\mathbb{F}$. Then the Lie group topology on G is the finest group topology making the embeddings of the fundamental rank one subgroups (endowed with their Lie group topologies) continuous.*

Now let Δ be a 2-spherical Dynkin diagram without loops and let $\mathbb{F}$ be a field with at least four elements. For each node $\alpha \in \Delta$ let G_α be a copy of $\mathrm{SL}_2(\mathbb{F})$; and for each pair $\alpha, \beta \in \Delta$ let $G_{\alpha,\beta}$ be a simply connected Chevalley group over $\mathbb{F}$ of the type given in Δ. There are obvious inclusions $G_\alpha \rightarrowtail G_{\alpha,\beta}$. The Kac-Moody group $G_\Delta(\mathbb{F})$ can then be described as the product of the $G_{\alpha,\beta}$ amalgamated over the G_α and is uniquely determined by Δ since Δ does not contain loops. Every 2-spherical split Kac-Moody group arises this way.

Definition 2. *The* Kac-Peterson *topology on $G_\Delta(\mathbb{F})$ is the finest group topology that makes the canonical embeddings $G_\alpha \rightarrowtail G_\Delta(\mathbb{F})$ continuous.*

Theorem 3 (Hartnick-Köhl-Mars, [38])**.** *The group $G_\Delta(\mathbb{F})$ with the Kac-Peterson topology is Hausdorff and a k_ω-space, i.e., it is the direct limit of an ascending sequence of compact Hausdorff subspaces. If Δ is not spherical, then the Kac-Peterson topology is neither locally compact nor metrizable.*

In particular, the existence of a Haar measure is not guaranteed for non-spherical Kac-Moody groups with the Kac-Peterson topology.

Theorem 4 (Hartnick-Köhl, [37])**.** *Let $\mathbb{F}$ be a local field and let G_Δ be an irreducible (i.e., Δ is connected), 2-spherical split Kac-Moody group. Then $G_\Delta(\mathbb{F})$ with the Kac-Peterson topology has Kazhdan's property (T).*

The subgroup $G_\Delta(\mathbb{Z})$ is discrete and finitely generated. One would like to think of $G_\Delta(\mathbb{Z})$ in analogy to arithmetic lattices. It is, however, an open question whether it inherits property (T) from $G_\Delta(\mathbb{R})$.

The following follows easily from a theorem of Caprace and Monod on Chevalley groups acting on CAT(0) polyhedral complexes applied to the Davis realization of the twin building for $G_\Delta(\mathbb{R})$:

Proposition 5. *Let L be an irreducible Chevalley group of rank at least two, let $G_\Delta(\mathbb{R})$ be a Kac-Moody group, and let $\varphi\colon L(\mathbb{Z}) \to G_\Delta(\mathbb{R})$ be a group homomorphism. Then the image $\varphi(L(\mathbb{Z}))$ is a bounded subgroup, i.e., it lies in the intersection of two parabolic subgroups of opposite sign. In particular, $\varphi(L(\mathbb{Z}))$ is contained in an algebraic subgroup of $G_\Delta(\mathbb{R})$.*

This can be extended to yield an analogue of Margulis superrigidity:

Theorem 6 (Farahmand-Horn-Köhl)**.** *Let $G_\Delta(\mathbb{R})$ and $G_{\Delta'}(\mathbb{R})$ be irreducible 2-spherical Kac-Moody groups and let $\varphi\colon G_\Delta(\mathbb{Z}) \to G_{\Delta'}(\mathbb{R})$ be a group homomorphism with Zariski dense image. Then there exists $n \in \mathbb{N}$ such that the restriction of φ to $G_\Delta(n\mathbb{Z})$ extends uniquely to a continuous homomorphism $G_\Delta(\mathbb{R}) \to G_{\Delta'}(\mathbb{R})$ with respect to the Kac-Peterson topologies.*

The proof proceeds by first dealing with the case that G_Δ is a Chevalley group, where the Kac-Peterson topology is the Lie group topology. For general G_Δ, the statement is reduced to the rank two subgroups, which are Chevalley groups. Using the the presentations of G_Δ and $G_{\Delta'}$ as products of their respective rank two subgroups amalgamated along their rank one subgroups, one constructs the extension of φ. Since the Kac-Peterson topology is universal with respect to the Lie topologies on the rank one subgroups, it follows that φ is continuous.

Kevin Wortman (University of Utah)

Finiteness Properties of Arithmetic Groups over Function Fields

Recall that a group Γ has finiteness length $\leq m$ if it has a classifying space whose m-skeleton is finite. In this case, the cohomology of Γ is clearly finitely generated in dimensions $\leq m$.

Let K be a global function field of characteristic $p > 0$, and let $\mathcal{G}$ be a connected noncommutative absolutely almost simple K-isotropic K-group. Let $d := \sum_{p \in S} \mathrm{rk}_{K_p}(\mathcal{G})$ denote the sum of the local ranks of $\mathcal{G}$. With this notation fixed, the two main results are:

Theorem 1 (Bux-Köhl-Witzel [18])**.** *The finiteness length $\phi(\Gamma)$ of the S-arithmetic subgroup $\Gamma = \mathcal{G}(\mathcal{O}_S)$ is $d-1$. (Rank-Theorem)*

Theorem 2 (Wortman)**.** *The cohomology $\mathrm{H}^d(\Gamma', \mathbb{F}_p)$ is not finitely generated for some subgroup Γ' of finite index in Γ and where $\mathbb{F}_p$ is the finite field with p elements. (At the time of the conference, a mild restriction on the K-type of $\mathcal{G}$ was needed. Meanwhile, Wortman was able to remove this restriction.)*

Results on finiteness properties of arithmetic groups have a long history. The euclidean algorithm shows that $\mathrm{SL}_n(\mathbf{Z})$ is finitely generated. Finite presentability of these groups is a classical application of Siegel domains. Raghunathan [56] proved that arithmetic groups in characteristic 0 enjoy all finiteness properties. In fact, he showed that they have a torsion free subgroup of finite index which is the fundamental group of a compact aspherical manifold with boundary. Borel-Serre [16] have shown that S-arithmetic subgroups of reductive groups in characteristic 0 also enjoy all finiteness properties.

The picture in positive characteristic is different. Nagao [49] showed that $\mathrm{SL}_2(\mathbf{F}_q[t])$ is not even finitely generated. Behr [10] proved that Γ as in the Rank Theorem is finitely generated if and only if $d > 1$. Stuhler [67] showed that $\mathrm{SL}_2(\mathcal{O}_S)$ has finiteness length $|S| - 1 = d - 1$. Abels [1] and Abramenko [2] independently showed that $\mathrm{SL}_n(\mathbf{F}_q[t])$ has finiteness length $n - 2 = d - 1$ provided q is large enough. Sometime during the 1980s, the pattern became transparent. Behr turned it into a serious conjecture when he proved in [11] that the S-arithmetic subgroup Γ of the Rank Theorem is finitely presented if and only if $d > 2$.

A significant step toward the Rank Theorem was the proof of its "negative half" by Bux and Wortman in [19], where they showed that $\phi(\Gamma) < d$. In 2008 (published in [20]), Bux-Wortman also settled the Rank Theorem in full for groups of global rank 1. The major improvement was a geometric filtration of the Bruhat-Tits building for Γ defined by Busemann functions. The relative links of this filtration are larger than those occuring in combinatorially defined filtrations used previously: the new relative links are hemi-sphere complexes in spherical buildings, whose connectivity properties have been established by Schulz [64]. The proof of the Rank Theorem for arbitrary groups follows this line of thought. Here, Behr-Harder reduction theory is the source of the Busemann functions and the associate filtration.

The second theorem above strengthens the negative half of the Rank Theorem considerably. For $\mathrm{SL}_2(\mathcal{O}_S)$, Stuhler had obtained infinite generation of homology in the critical dimension (here $d = |S|$) by means of a spectral sequence argument. In the other works cited above, the finiteness length was deduced by combinatorial or geometrical means that do not detect homology in the critical dimension.

The main difficulty is that the action of Γ on its associated Bruhat-Tits building X is not free; in fact, the order of point stabilizers is not bounded. Wortman uses the height function from the Rank Theorem to pass to a cocompact subspace $X(0)$, which is $(d-2)-$*conected.* Gluing in free Γ-orbits of cells of dimensions d and $d+1$, he obtains a d-connected space Y on which Γ acts with stabilizers of uniformly bounded order, and he obtains a Γ-equivariant map $Y \to X$. He can now pass to a finite index subgroup $\Gamma' \leq \Gamma$ that acts freely on Y. He then constructs an infinite family of cocycles on Y and a "dual" family of cycles on X paired via the comparison map $Y \to X$. The supports of the cycles in X increase in height and "low" cocycles evaluate trivially on higher cycles. On the other hand, each cocycle evaluates non-trivially on the corresponding cycle. This shows that the cocycles are non-trivial and linearly independent.

Topics represented at the workshop

(a) Structural and homological properties

In addition to the mini-course on homological finiteness properties for S-arithmetic subgroups of semi-simple algebraic groups in positive characteristic which contained an exposition of the work of Bux-Köhl-Witzel-Wortman on the rank conjecture, there was a talk by Stefan Witzel on the finiteness properties for proper actions of arithmetic groups. He presented examples of arithmetic groups showing that classical finiteness properties and finiteness length for proper actions vary independently [31, 71].

The talk of Ted Chinburg demonstrated how to use the Lefschetz Theorem from algebraic geometry to show that in certain situations a "large" arithmetic group can be generated by smaller arithmetic subgroups [26].

In his talk, Vincent Emery showed how to bound torsion in the homology of non-uniform arithmetic lattices in characteristic zero [30].

(b) Profinite techniques for arithmetic groups

The method of analyzing arithmetic groups via the study of their finite quotients has a long history. One aspect of this approach, known as the congruence subgroup problem, is focused on understanding the difference between the profinite and congruence completions, which is measured by the congruence kernel. In his talk, Andrei Rapinchuk gave a survey of the concepts and results pertaining to the congruence subgroup problem [53, 55, 57, 58].

In their talks, which together virtually constituted another mini-course, Benjamin Klopsch and Christopher Voll presented their new results on the representation growth of S-arithmetic groups satisfying the congruence subgroup property (i.e. for which the congruence kernel is finite). These results are formulated in terms of the representation zeta function of the S-arithmetic group, and give, in particular, the precise value of the abscissa of convergence that depends only on the root system [6–9].

The talk of Pavel Zalesskii was devoted to the general question of when the profinite topology on an arithmetic group should be considered to be "strong." The cohomological aspect of this question boils down to the notion of "goodness" introduced by Serre. The central conjecture here asserts that if an S-arithmetic subgroup in characteristic zero fails the congruence subgroup property then it should be good, and the talk contained an account of the results confirming this conjecture [36, 70].

(c) Connections with Kac-Moody, automorphism groups of free groups and mapping class groups

As we already mentioned, Ralf Köhl gave a mini-course on Kac-Moody groups, in which he described how Kac-Moody groups are obtained from Chevalley groups by amalgamation, defined the Kac-Peterson topology on them, and established their important properties including property (T) and a variant of superrigidity. Due to a significant interest to Kac-Moody groups, expressed particularly during the Q&A session, Bertrand Rémy kindly agreed to give a survey talk on the subject. He discussed various approaches to Kac-Moody groups and stressed the utility of buildings in their analysis [3, 21–23, 29, 42, 62, 68, 69].

In his talk, Alan Reid showed how one can use the existence of homomorphisms of the mapping class group onto certain arithmetic groups to prove that every finite group can be obtained as a quotient of a suitable finite index subgroup of the mapping class group [47].

Lizhen Ji reported on the construction of compete geodesic metrics on the outer space X_n that are invariant under $Out(F_n)$ where F_n is the free group of rank n [41].

(d) Applications to geometry, topology and beyond

Mikhail Belolipetsky gave a survey of the long line of research aimed at understanding arithmetic groups of isometries of the hyperbolic space that are generated by reflections. One of the central results here is the finiteness of the number of conjugacy classes of maximal arithmetic hyperbolic reflection groups. This opens the possibility of classifying such groups [4, 5, 12–15, 44, 46, 48, 50, 51].

Matthew Stover reported on his results that restrict the number of ends (cusps) an arithmetically defined hyperbolic manifold/orbifold or a locally symmetric space can possibly have. In particular, a one-end arithmetically defined hyperbolic n-orbifold cannot exist for any $n > 31$. The question about the existence of one-ended nonarithmetic finite volume hyperbolic manifolds remains wide open [66].

T.N. Venkataramana spoke about the monodromy groups associated with hypergeometric functions. One of the central questions is when the monodromy group is a finite index subgroup of the corresponding integral symplectic group. A criterion was described for this to be the case [65].

(e) Rigidity

Generally speaking, a rigidity statement is a result asserting that in certain situations any abstract homomorphism of a special subgroup (e.g, arithmetic subgroup or a lattice) of the group of rational points of an algebraic group (resp., a Lie group, or a Kac-Moody group) can be extended to an algebraic (resp., analytic or continuous) homomorphism of the ambient group. The most general result in this direction is Margulis's Superridity Theorem for irreducible lattices in higher rank semi-simple Lie groups. As was pointed out by Bass, Milnor and Serre, rigidity statements can also be proved for S-arithmetic groups if the corresponding congruence kernel is finite. The mini-course of Ralf Köhl falls within this topic.

In his talk, Igor Rapinchuk discussed his rigidity results for the finite-dimensional representations of elementary subgroups of Chevalley group of rank > 1 over arbitrary commutative rings. These results settle a conjecture of Borel and Tits about abstract homomorphisms for split algebraic groups of rank > 1 over fields of characteristic $\neq 2, 3$, and also have applications to character varieties of some finitely generated groups [59–61].

(f) Weakly commensurable groups and connections to algebraic groups

The notion of weak commensurability for Zariski-dense subgroups of the group of rational points of semi-simple algebraic groups over a field of characteristic zero was introduced by G. Prasad and A. Rapinchuk [54]. They were able to provide an almost complete answer to the question of when two S-arithmetic subgroups of absolutely almost simple algebraic groups are weakly commensurable. Using Schanuel's conjecture from transcendental number theory, they connected this work to the analysis of length-commensurable and isospectral locally symmetric spaces, and in fact obtained new important results about isospectral spaces. In his talk, Rajan reported on his work which is based on a new notion of representation equivalence of lattices. While the condition of representation equivalence is generally stronger than isospectrality, it enables one to obtain results about representation equivalent locally symmetric spaces without using Schanuel's conjecture (which is still unproven).

The work of Prasad-Rapinchuk also attracted attention to a wide range of questions in the theory of algebraic groups asking about a possible relationship between two absolutely almost simple algebraic groups G_1 and

G_2 over the same field K given the fact that they have the same isomorphism/isogeny classes of maximal K-tori. In his talk, Vladimir Chernousov reported on the recent results on a related problem of characterizing finite-dimensional central division algebras over the same field K that have the same isomorphism classes of maximal subfields. He also formulated a conjecture that would generalize these results to arbitrary absolutely almost simple groups, and indicated that the results on division algebras enable one to prove this conjecture for inner forms of type A_n [24, 25].

(g) Applications to combinatorics

In his talk, Alireza Salehi Golsefidy discussed applications of arithmetic groups and their Zariski-dense subgroups to the construction of highly connected but sparse graphs known as *expanders*. It was pointed out by Margulis that families of expanders can be constructed from a discrete Kazhdan group Γ by fixing a finite system S of generators of Γ and considering the Cayley graphs $C(\Gamma/N, S)$ of the finite quotients of Γ with respect to this generating system. Later, using deep number-theoretic results, Lubotzky, Phillips and Sarnak showed that one also obtains a family of expanders from the non-Kazhdan group $SL_2(\mathbb{Z})$ by fixing a system S of generators of the latter and considering the Cayley graphs $C(SL_2(\mathbb{Z}/d\mathbb{Z}), S)$ for the congruence quotients.

Lubotzky raised the question of whether one gets a family of expanders if one takes, for example, an arbitrary finitely generated Zariski-dense subgroup $\Gamma \subset SL_2(\mathbb{Z})$ and considers the Cayley graphs of the congruence quotients $SL_2(\mathbb{Z}/d\mathbb{Z})$ with respect to a fixed finite generating set of the subgroup (it is known that Γ will map surjectively onto $SL_2(\mathbb{Z}/d\mathbb{Z})$ for all d prime to some d_0 depending on Γ).

Golsefidy surveyed the important progress [17, 33, 39, 40, 43, 45] on this question in the context of general arithmetic groups which began with the work of Bourgain and Gamburd, and reported on his recent results with Varjú [34, 35].

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