Sharply 2-transitive linear groups.

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Outline

1. Basic facts and examples
2. Some history behind the problem
3. Statement of the main theorem
4. Proof
Basic facts about sharply 2-transitive groups

- A **sharply 2-transitive group** is a permutation group $\Gamma \leq \text{Sym}(\Omega)$ which acts transitively and freely on pairs of distinct points.
- Sharply 2-transitive groups contain a lot of involutions (elements of order 2), and all are conjugate.
- If an involution stabilizes a point, then the conjugation action of $\Gamma$ on $\text{Inv}(\Gamma)$ is isomorphic to it’s action on $\Omega$.
- This gives rise to the definition of the permutation characteristic of the group, $p$-char($\Gamma$):

$$p\text{-char}(\Gamma) = \begin{cases} 
2 & \Gamma_x \cap \text{Inv}(\Gamma) = \emptyset \\
p & \Gamma_x \cap \text{Inv}(\Gamma) \neq \emptyset, \quad \text{Ord}(\sigma \tau) = p \\
0 & \Gamma_x \cap \text{Inv}(\Gamma) \neq \emptyset, \quad \text{Ord}(\sigma \tau) = \infty
\end{cases}$$
The main example

Example

Given a field \( N \), the affine action \( x \mapsto ax + b \) is sharply 2-transitive.

An easy way to see this is using geometric interpretation (at least for \( N = \mathbb{R} \)). Taking \( (x, y) \) to \( (z, w) \) is equivalent to finding the unique line between \( (x, z) \) and \( (y, w) \).

Looking at this example a little closer, one can see that the same will work for a division ring or even a near-field.
Some history behind the problem

- A long standing conjecture about sharply 2-transitive groups is that every such group is the affine group of a near-field, i.e. $N^\times \rtimes N$

- In the finite case, it was completely settled by H.Zassenhaus: in his two 1936 papers he first proved this conjecture for finite groups, and later classified all finite near-fields.

- In the infinite case, much less has been done. In 1952, J.Tits proved the conjecture for locally compact connected groups. In this case all near-fields are of finite rank over $\mathbb{R}$.

- Moreover, J.Tits showed that for an infinite sharply $k$-transitive group, $k \leq 3$. 
Statement of the main theorem

**Theorem**

Let $F$ be a field and let $\Gamma \leq \GL_n(F)$ be a sharply 2-transitive group. Assume that $\text{char}(F) \neq 2$ and that $p\text{-char}(\Gamma) \neq 2$. Then $\Gamma \cong N \times N$, where $N$ is a near-field.
Proof strategy

**Theorem (Dixon and Mortimer, Permutation Groups, Theorem 7.6C)**

Let $|\Omega| \geq 2$ and let $\Gamma \leq \text{Sym}(\Omega)$ be a sharply 2-transitive group which possesses a fixed-point free normal abelian subgroup $K$. Then there exists a near-field $N$ such that $\Gamma$ is permutation isomorphic to $N^\times \rtimes N$.

Using this theorem, it suffices to prove the existence of a fixed-point free normal abelian subgroup.
Passing to algebraic groups

Let $G, H$ be the Zariski closures of $\Gamma, \Delta = \Gamma \omega$ respectively, in $GL_n(k)$ where $k = \overline{F}$.

We know that $\Gamma \bowtie \Gamma/\Delta$ sharply 2-transitively. What can we say about $G \bowtie G/H$? For that, we need to introduce the algebraic analogue of transitivity.
Passing to algebraic groups

**Definition (Generic transitivity)**

Let \( \rho : G \curvearrowright X \) be an algebraic group acting algebraically on an algebraic variety \( X \). \( \rho \) is called generically \( n \)-transitive if the action \( \rho^n \) of \( G \) on \( X^n \) admits an open dense orbit.

Idea: First, show that \( G \curvearrowright G/H \) is generically 2-transitive. If under our assumptions, there is no normal abelian subgroup then \( G \curvearrowright G/H \) can not be generically 2-transitive.
### Passing to algebraic groups

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#### Theorem (Jonathan Brundan)

Let $G$ be a connected reductive group and $H < G$ a proper reductive subgroup. Then, there is no dense $(H, H)$-double coset in $G$.

#### Theorem (Domingo Luna)

Let $H < G \leq \text{GL}_n(F)$, with $\text{char}(F) = 0$, be connected reductive groups acting on an algebraic variety $X$. Then the generic $H$ orbit is closed.

#### Corollary

If $G, H$ are both reductive then $G \curvearrowright G/H$ can not be generically 2-transitive.
Proof of the corollary

Assume that $H$ admits an open orbit in its action on $G/H$ then at least one of the $H^0$ orbits is open as well, since any $H$-orbit is a finite union of $H^0$-orbits. So we have $\overline{O} = H^0gH$ open.

$\overline{O}$ is connected and so is contained in the connected component $\overline{X} = G^0gH$.

The natural map $\varphi : G/H^0 \to G/H$ restricted to $X = G^0gH^0$ is a covering map (since $[H : H^0]$ is finite), so $O = \varphi^{-1}(\overline{O}) \cap X = H^0gH^0$ is open (and hence dense) in $X$. This contradicts Brundan’s theorem.

If $\text{char}(F) = 0$, we can use Luna’s theorem instead: the generic $H^0$-orbit in $G^0/H^0$ is closed. But we have just seen that there exists an open orbit. Hence the action $H^0 \acts G^0/H^0$ has to be transitive.
Proving splitting

Theorem

Let $\Gamma \leq \text{GL}_n(F)$ be a sharply 2-transitive group. Assume that $\text{char}(F) \neq 2$ and that $p\text{-char}(\Gamma) \neq 2$. Then there exist a non-trivial abelian normal subgroup $N \triangleleft \Gamma$.

Proposition (1)

Let $\Gamma$ be as in the assumptions of the theorem. If the conclusion of the theorem fails, then there exists an algebraically closed field $k$ and a faithful representation $\rho : \Gamma \to \text{GL}_n(k)$ such that $G = \overline{\rho(\Gamma)^Z}$ is reductive.
Proof of Proposition 1

- Take a faithful representation $\rho_0 : \Gamma \to \text{GL}_n(k)$ for $k = \overline{F}$.
- Let $G_0 = \rho(\Gamma)^Z$. Let $G_u$ be the unipotent radical of $G_0$ and $N = \rho(\Gamma) \cap G_u$.
- Since $N$ is nilpotent, its penultimate element of the lower central series is a characteristic subgroup of $N$ and hence is a normal abelian subgroup of $\rho(\Gamma)$ - a contradiction.
- So we can divide by $G_u$ and obtain the required representation $\rho$. 
**Generic 2-transitivity**

**Proposition (2)**

Let $\Gamma, \Delta = \Gamma_\omega$ be as before and denote $G = \Gamma^Z, H = \Delta^Z$. Let $\sigma \in \Delta$ be the unique involution. Then $\sigma$ is semi-simple, $H \leq C_G(\sigma)$ and $G$ acts generically 2-transitively on $G/H$.

- Fix the unique involution $\sigma \in \Delta$. Since $\Delta$ centralizes $\sigma$, so does $H$.
- Take any $\gamma \in \Gamma$ not in $\Delta$. $\Delta$ acts transitively on $\Gamma/\Delta \setminus \{\Delta\}$, so $\Gamma/\Delta = \Delta \sqcup \Delta \gamma \Delta$.
- The set $H \sqcup H\gamma H \subseteq G/H$ is dense, since it contains the dense set $\Gamma H = H \sqcup \Delta \gamma H$ and locally closed, hence open.
- It follows that the orbit of $(H, \gamma H)$ is open in $G/H \times G/H$.

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Generic 2-transitivity

- Take the natural map \( \pi : G/H \times G/H \to G/H \).
- The intersection \( G(H, \gamma H) \cap \pi^{-1}(H) \) is open and dense in the fiber.
- Since the action is transitive, this is true for any fiber.
- Hence the \( G \)-orbit \( G(H, \gamma H) \) is dense. It is also locally closed, as an orbit of an algebraic action - and thus open.
Existence of a normal abelian subgroup

By the previous results we see that if there is no such subgroup, then $G, C = C_G(\sigma)$ are reductive and $G \curvearrowright G/H$ is generically 2-transitive.

Since $H \leq C$ and $G \curvearrowright G/H$ is generically 2-transitive, then so is $G \curvearrowright G/C$.

But $G, C$ are both reductive, which contradicts the Corollary.
Let $N \triangleleft \Gamma$ be the normal abelian subgroup we just obtained.

For any $\omega \in \Omega$, $[N_\omega, N] = \langle e \rangle$ and hence $N_\omega = \langle e \rangle$.

So $N$ is a fixed-point free normal abelian subgroup.
Thank You

Thank You!