Imprimitive irreducible modules for finite quasisimple groups

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4 Harish-Chandra induction
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**Project**

Classify the pairs \((G, G \to \text{SL}(V))\) such that

1. \(G\) is a finite quasisimple group,
2. \(V\) a finite dimensional vector space over some field \(K\),
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1. *G is quasisimple, if* \(G = G'\) *and* \(G/Z(G)\) *is simple.*
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1. \(G\) is quasisimple, if \(G = G'\) and \(G/Z(G)\) is simple.
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We call \(H := \text{Stab}_G(V_1)\) a **block stabilizer**.

We have \(V \cong \text{Ind}_H^G(V_1) := KG \otimes_{KH} V_1\) as \(KG\)-modules.
Motivation I: Maximal Subgroups

Let \( K \) be a finite field and \( V \) a f.d. \( K \)-vector space.
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Is $N_X(G)$ a maximal subgroup of $X$?
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What about $\varphi : M \to \text{SO}_{196882}^-(2)$? \quad ($M$: Monster)
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A table look-up in our lists might help to answer this question.
Let $K$ be algebraically closed. All irreducible, imprimitive $KG$-modules are known for

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A sample of results

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   - $G$ a Suzuki or Ree group, $G = G_2(q)$, or $G$ a Steinberg triality group (Seitz, H.-Husen-Magaard).
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THE ALTERNATING GROUPS; $K = \mathbb{C}$

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**Theorem (Dragomir Djoković, Jerry Malzan, 1976)**

Suppose that $G = A_n$, $n \geq 10$, and let $\chi \in \text{Irr}(G)$ be imprimitive. Then one of the following holds.
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   Also, \( \chi = \text{Ind}_{NG(s_m \times s_m)}^{G}(\chi_1) \) with \( \chi_1(1) = 1 \).
Again we take $K = \mathbb{C}$.

**Theorem (Daniel Nett, Felix Noeske, 2009)**

*Suppose that $G = 2.A_n$, $n \geq 10$, is the covering group of $A_n$, and let $\psi \in \text{Irr}(G)$ be imprimitive.*
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Suppose that $G = 2.A_n$, $n \geq 10$, is the covering group of $A_n$, and let $\psi \in \text{Irr}(G)$ be imprimitive.

Then $n = 1 + m(m + 1)/2$, and $\psi = \text{Res}_{2.S_n}^G(\sigma^\lambda)$ with $\lambda = (m + 1, m - 1, m - 2, \ldots, 1)$. 
Again we take $K = \mathbb{C}$.

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*Also, $\psi = \text{Ind}^{G}_{2.A_{n-1}}(\psi_1)$ with $\psi_1$ a constituent of $\text{Res}^{2.S_{n-1}}_{2.A_{n-1}}(\sigma^\mu)$ with $\mu = (m, m - 1, \ldots, 1)$.***
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Such a pair $(L, P)$ gives rise to a parabolic subgroup $P = P^F$ of $G$ with Levi complement $L = L^F$. 
THEOREM (Gary Seitz, 1988)

Let $G$ be a finite reductive, quasisimple group of characteristic $p$. Suppose that $V$ is an irreducible, imprimitive $FG$-module. Then $G$ is one of $\text{SL}_2(5)$, $\text{SL}_2(7)$, $\text{SL}_3(2)$, $\text{Sp}_4(3)$, and $V$ is the Steinberg module. Thus it remains to study finite reductive groups in non-defining characteristic.
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SOME RESULTS
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REDUCTIVE GROUPS IN DEFINING CHARACTERISTICS

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THE MAIN REDUCTION THEOREM

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Let $G$ and $K$ be as above. Let $H \leq G$ be a maximal subgroup. Suppose that $\text{Ind}^G_H(V_1)$ is irreducible for some $K$-module $V_1$. Then $H = P$ is a parabolic subgroup of $G$. 
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Let $G$ be a finite group, $H \leq G$, and $K$ a field.
Some easy characteristic-free criteria

Let $G$ be a finite group, $H \leq G$, and $K$ a field.
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4. Suppose that $H = C_G(a)$ for some $a \in G$. Then $t \not\in \langle ^t a, a \rangle$ for all $t \in G \setminus H$. 

**Proof of 1:** Clear, since $\dim(V) = [G : H] \dim(V_1)$.

**Proof of 2:** $[G : H]^2 \leq \dim(V)^2 \leq |G|$.

**Proof of 3:** This is a consequence of Mackey's theorem.

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**SOME EASY CHARACTERISTIC-FREE CRITERIA**
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Non-parabolic block stabilizers

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**Example**

Let $G = \text{Sp}_{2m}(q)$ with $m$ even and $q > 3$ odd, and let $H = \langle H_0, s \rangle$ with $H_0 = \text{Sp}_m(q) \times \text{Sp}_m(q)$ and $s = \begin{bmatrix} 0 & I_m \\ I_m & 0 \end{bmatrix}$. 
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Put \( t := \begin{bmatrix} I_m & N \\ N & I_m \end{bmatrix} \) with \( N := \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix} \).
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Then \( t \in \langle t a, a \rangle \), hence \( t \) centralizes \( t H_0 \cap H_0 \).
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Let $G = \text{Sp}_{2m}(q)$ with $m$ even and $q > 3$ odd, and let $H = \langle H_0, s \rangle$ with $H_0 = \text{Sp}_m(q) \times \text{Sp}_m(q)$ and $s = \begin{bmatrix} 0 & l_m \\ l_m & 0 \end{bmatrix}$.

Then $H_0 = C_G(a)$ with $a = \begin{bmatrix} \alpha l_m & 0 \\ 0 & \alpha^{-1} l_m \end{bmatrix}$, where $\langle \alpha \rangle = \mathbb{F}^*$. 

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Then $t \in \langle t^a, a \rangle$, hence $t$ centralizes $tH_0 \cap H_0$.

Finally, $t \in C_G(s)$ and $tH_0 \cap sH_0 = \emptyset$, thus $t \in C_G(tH \cap H)$. 
Let $G$ be a finite reductive, quasisimple group of characteristic $p$, and let $K$ be an algebraically closed field with $\text{char}(K) \neq p$. According to our main reduction theorem, we may restrict our investigation to parabolic subgroups. Proposition (H.-H.USEN-MAGAARD, 2013) Let $P$ be a parabolic subgroup of $G$ with unipotent radical $U$. Let $V_1$ be a $KP$-module such that $\text{Ind}_{G}^{P}(V_1)$ is irreducible. Then $U$ is in the kernel of $V_1$. In other words, $\text{Ind}_{G}^{P}(V_1)$ is Harish-Chandra induced. This allows to apply Harish-Chandra theory to our classification problem, reducing certain aspects to Weyl groups.
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**Proposition (H.-Husen-Magaard, 2013)**

Let $P$ be a parabolic subgroup of $G$ with unipotent radical $U$. 
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Let $P$ be a parabolic subgroup of $G$ with unipotent radical $U$. Let $V_1$ be a $K_P$-module such that $\text{Ind}_P^G(V_1)$ is irreducible.
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### Sketch Proof of Proposition

**Proposition**

Let $P$ be a parabolic subgroup of $G$ with unipotent radical $U$. Let $V_1$ be a $KP$-module such that $\text{Ind}_P^G(V_1)$ is irreducible. Then $U$ is in the kernel of $V_1$. 

**Proof:** (Sketch) Let $L$ be a Levi complement of $U$ in $P$. Choose a head composition factor $V_2$ of $\text{Res}_L^P(V_1)$. Let $Q$ be the opposite parabolic subgroup of $P$, so $P \cap Q = L$. Mackey's theorem yields a non-trivial homomorphism $\text{Ind}_P^G(V_1) \rightarrow \text{Ind}_Q^G(\tilde{V}_2)$, where $\tilde{V}_2 = \text{Infl}_Q^L(V_2)$. As $\text{Ind}_P^G(V_1)$ is simple, and $\dim(\text{Ind}_Q^G(\tilde{V}_2)) \leq \dim(\text{Ind}_P^G(V_1))$, this implies that $\text{Ind}_P^G(V_1) \sim \text{Ind}_Q^G(\tilde{V}_2)$. It follows that $\dim(V_1) = \dim(V_2)$. 


**PROPOSITION**

Let $P$ be a parabolic subgroup of $G$ with unipotent radical $U$. Let $V_1$ be a KP-module such that $\text{Ind}_G^P (V_1)$ is irreducible. Then $U$ is in the kernel of $V_1$.

**Proof:** (Sketch) Let $L$ be a Levi complement of $U$ in $P$. 

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**Proof:** (Sketch) Let $L$ be a Levi complement of $U$ in $P$. Chose a head composition factor $V_2$ of $\text{Res}^P_L(V_1)$. Let $Q$ be the opposite parabolic subgroup of $P$, so $P \cap Q = L$. Mackey's theorem yields a non-trivial homomorphism $\text{Ind}_P^G(V_1) \to \text{Ind}_Q^G(\tilde{V}_2)$, where $\tilde{V}_2 = \text{Infl}_L^Q(V_2)$. 


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As $\text{Ind}_P^G(V_1)$ is simple, and $\dim(\text{Ind}_Q^G(\tilde{V}_2)) \leq \dim(\text{Ind}_P^G(V_1))$, this implies that

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**Proposition**

Let $P$ be a parabolic subgroup of $G$ with unipotent radical $U$. Let $V_1$ be a KP-module such that $\text{Ind}_P^G(V_1)$ is irreducible. Then $U$ is in the kernel of $V_1$.

**Proof:** (Sketch) Let $L$ be a Levi complement of $U$ in $P$. Chose a head composition factor $V_2$ of $\text{Res}_L^P(V_1)$. Let $Q$ be the opposite parabolic subgroup of $P$, so $P \cap Q = L$. Mackey’s theorem yields a non-trivial homomorphism $\text{Ind}_P^G(V_1) \to \text{Ind}_Q^G(\tilde{V}_2)$, where $\tilde{V}_2 = \text{Infl}_L^Q(V_2)$.

As $\text{Ind}_P^G(V_1)$ is simple, and $\dim(\text{Ind}_Q^G(\tilde{V}_2)) \leq \dim(\text{Ind}_P^G(V_1))$, this implies that

$$\text{Ind}_P^G(V_1) \cong \text{Ind}_Q^G(\tilde{V}_2).$$

It follows that $\dim(V_1) = \dim(V_2)$. 
A consequence for maximal subgroups

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$G = \text{GL}_n(q)$, $L = \text{GL}_m(q) \times \text{GL}_{n-m}(q)$ with $m \neq n - m$. 
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Analogous results hold for the other classical groups.
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In particular, the elements of $\mathcal{E}(G, [1])$ are HC-primitive.
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Notice that the minimal polynomial of $s$ is irreducible if and only if $C_G(s) \cong \text{GL}_m(q^d)$ for integers $m, d$ with $md = n$. 
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**Example (Cédric Bonnafé)**

Suppose that $q$ is odd, let $G = \text{GL}_4(q)$ and $P$ a parabolic subgroup with Levi complement $L = \text{GL}_2(q) \times \text{GL}_2(q)$. 
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*Suppose that $q$ is odd, let $G = GL_4(q)$ and $P$ a parabolic subgroup with Levi complement $L = GL_2(q) \times GL_2(q)$. Let $1$ denote the trivial character and $1^-$ the unique linear character of $GL_2(q)$ of order 2.*
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**Theorem (H.-Husen-Magaard, 2013)**

Let $\chi \in \text{Irr}(GL_n(q))$ be Harish-Chandra primitive.

Then $\text{Res}_{SL_n(q)}^{GL_n(q)}(\chi)$ is irreducible and Harish-Chandra primitive.
Let $G = \text{SL}_n(q)$, $s \in G^* = \text{PGL}_n(q)$ semisimple.
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$$\text{Irr}(W(s)^F) \to \mathcal{E}(G, [s]), \quad \eta \mapsto \chi_\eta,$$

where $W(s)$ is the “Weyl group” of $C_{G^*}(s)$ (Bonnafé).
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Suppose that $\mathcal{E}(G, [s])$ contains Harish-Chandra primitive and imprimitive characters.

Then $W(s)^F = S: \langle \gamma \rangle$, with $S = S_m \times \cdots \times S_m$, and $\gamma$ permuting the $e$ factors $S_m$ of $S$ transitively, and $em \mid n$. 
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**Theorem (H.-Magaard)**

$\chi_\eta \in \mathcal{E}(G, [s])$ is primitive, if and only if $\text{Res}_{S : \langle \gamma \rangle}^S(\eta)$ is irreducible.
Thank you for listening!