Geometry and Inverse problems

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1 Overview of the Field

Inverse problems appear naturally in many areas of mathematics. The focus of this workshop was on several remarkable inverse problems which occur in Geometry broadly understood, and on employing geometric thinking as a tool in the resolution of inverse problems in PDEs, geophysics or medical imaging. There has been great activity in recent years around this circle of ideas, and the purpose of this workshop was to bring together specialists with expertise in different aspects of these interconnected problems with the hope of making substantial further progress.

2 Recent Developments and Open Problems

In order to make this discussion more concrete we now present the background for the topics that were addressed at the workshop.

2.1 Boundary and lens rigidity

An important geometric inverse problem consists in determining a Riemannian metric on a bounded set of Euclidean space (or a Riemannian manifold with boundary) from the boundary distance function, which is defined as the lengths of geodesics joining points of the boundary. Physically, these are the first arrival times of geodesics (rays) going through the domain. This is known in the geophysics literature as the *inverse kinematic problem* and in differential geometry as the *boundary rigidity problem*. The boundary distance function is unchanged under any isometry which is the identity at the boundary, so the question is whether one can determine the metric up to this obstruction. The answer is no since the boundary distance function takes into account only length minimizing geodesics. Any region of the manifold with a very large metric will not be seen from the boundary distance function so one needs some restriction on the metric. One such restriction is that the manifold is simple: given any two points they can be joined by a unique geodesic and the boundary is strictly convex. The conjecture proposed by Michel (1981) is that simple Riemannian manifolds with boundary can be determined uniquely up to an isometry from the boundary distance function.

The conjecture was proven in dimension two by Pestov and Uhlmann (2005) making a connection between the boundary distance function and the Dirichlet to Neumann map for a Riemannian metric discussed below.

In higher dimensions this conjecture is unproven and this was one of the topics that was discussed in the workshop. It was shown by Stefanov and Uhlmann (2005) that the conjecture holds locally and generically near an open and dense set of simple metrics. Burago and Ivanov (2010) extended the semiglobal result of Lassas, Sharafutdinov and Uhlmann (2003) showing that any metric close to Euclidean is boundary rigid.

For non-simple manifolds one can consider the behavior of all the geodesics going through the domain, not just the minimizing ones. This information is encoded in the scattering relation, which maps the point and direction of entrance of a geodesic to the point and direction of exit. The scattering relation was defined by Guillemin (1976) in the context of scattering theory. The natural *lens rigidity conjecture* is that for non-trapping manifolds the scattering relation determines the metric up to isometry. The lens rigidity and boundary rigidity problems are equivalent for simple manifolds.

There are very few results about this conjecture for non-simple manifolds. Vargo (2009) proved it for real-analytic metrics satisfying a mild condition. Croke (2005) has shown that if a manifold is lens rigid, a finite quotient of it is also lens rigid. He also has proved recently that the torus is lens rigid. Stefanov and Uhlmann (2009) have shown lens rigidity locally near a generic class of non-simple manifolds. Lens rigidity was one of the major topics considered in the workshop.

One can discuss analogous problems to boundary rigidity and lens rigidity for more general flows. For instance, many of the results mentioned have been generalized by Dairbekov, Paternain, Stefanov and Uhlmann to magnetic geodesics (2007).

2.2 The Calderón and Gel'fand problems

Geometric inverse problems arise already in connection with the famous inverse conductivity problem due to Calderón. This problem is the model for Electrical Impedance Tomography, an imaging modality proposed for use in medical and seismic imaging. In the case of anisotropic media in dimensions three and higher, the problem reduces to the question of determining a Riemannian manifold from the elliptic Dirichlet to Neumann map (or Cauchy data of harmonic functions) on its boundary. This is one of the major open questions related to the Calderón problem. Recently, Dos Santos Ferreira, Kenig, Salo and Uhlmann (2009) opened a new direction of research in the problem by showing that certain smooth manifolds in a fixed conformal class are determined by the Dirichlet to Neumann map. This result connects with the integral geometry questions described below, since the attenuated geodesic ray transform appears as a main ingredient in the proofs. The corresponding two-dimensional question for Schrödinger operators on Riemann surfaces was taken up by Guillarmou and Tzou (2011) based on methods introduced by Bukhgeim. This work raises the interesting possibility of extensions to complex manifolds in higher dimensions.

The geometric Calderón problem is also connected to another major question in geometric inverse problems, the *Gel'fand problem* which reduces to determining a Riemannian manifold from the hyperbolic Dirichlet to Neumann map. This problem has been thoroughly studied in a number of situations by Kurylev, Lassas and others using the Boundary Control method introduced by Belishev, and recent work of Kurylev, Lassas and Yamaguchi (2010) extends uniqueness results for this problem to the case of partly collapsed manifolds. The relation to the Calderón problem arises since the special form of manifolds treated in the elliptic case formally corresponds to the Lorentz metrics considered in the Boundary Control method. There is ongoing joint work by Dos Santos Ferreira, Kurylev, Lassas and Salo which exploits this relation, and this emerging connection between two major strands in the theory of inverse problems was discussed in the workshop. We mention that geodesic ray transforms are naturally encountered also in hyperbolic inverse problems on manifolds.

2.3 Tensor tomography

The geodesic ray transform, where one integrates a function or a tensor field along geodesics of a Riemannian metric, is closely related to the boundary rigidity problem. The integration of a function along geodesics is the linearization of the boundary rigidity problem in a fixed conformal class. The standard X-ray transform, where one integrates a function along straight lines, corresponds to the case of the Euclidean metric and is the basis of medical imaging techniques such as CT and PET. The case of integration along more general geodesics arises in geophysical imaging in determining the inner structure of the Earth since the speed of elastic waves generally increases with depth, thus curving the rays back to the Earth surface. It also arises in ultrasound imaging, where the Riemannian metric models the anisotropic index of refraction. In *tensor* *tomography problems* one would like to determine a symmetric tensor field up to natural obstruction from its integrals over geodesics.

The case of integrating tensors of order one corresponds to the geodesic Doppler transform in which one integrates a vector field along geodesics. This transform appears in ultrasound tomography to detect tumors using blood flow measurements and also in non-invasive industrial measurements for reconstructing the velocity of a moving fluid. The integration of tensors of order two along geodesics, also known as *deformation boundary rigidity*, arises as the linearization of the boundary rigidity problem. The case of tensor fields of rank four describes the perturbation of travel times of compressional waves propagating in slightly anisotropic elastic media.

There are recent advances on tensor tomography on manifolds due to Dairbekov (2006), Sharafutdinov (2007) and Stefanov and Uhlmann (2009). Recently Paternain, Salo and Uhlmann (2011) settled the tensor tomography problem for simple surfaces by proving that the ray transform is injective on symmetric tensors of any order up to the natural obstruction. The proof introduces new methods and makes a connection to the attenuated ray transforms described below and also to methods in Complex Geometry such as the Kodaira Vanishing Theorem. The main open problem here is to extend this result to higher dimensional simple Riemannian manifolds, and there are related open problems on closed manifolds.

2.4 Attenuated ray transforms

Recently there has been lots activity in the study of these transforms. Besides their importance in imaging technology, they arise naturally in various inverse problems in geometry as explained above.

In the case of Euclidean space with the Euclidean metric the attenuated ray transform is the basis of the medical imaging technology of SPECT and has been extensively studied. There are two natural directions in which this transform can be extended: one is to replace Euclidean space by a Riemannian manifold, and the other is to consider the case of systems where the attenuation is given for example by a unitary connection.

If $a \in C^\infty(M)$ is the attenuation coefficient, the attenuated ray transform of a function $f \in C^\infty(M)$ is

$$I_a f(x,v) = \int_0^{\tau(x,v)} f(\gamma(t,x,v)) \exp\Big[\int_0^t a(\gamma(s,x,v))\,ds\Big]\,dt.$$

Here $x \in \partial M$, v is a unit tangent vector at x pointing inside M and $\gamma(t,x,v)$ is the geodesic determined by (x,v).

There has been remarkable progress in the understanding of injectivity properties of these transforms. Injectivity in the Euclidean case was proved by Arbuzov, Bukhgeim and Kazantsev (1998) and an inversion formula was provided by Novikov (2000). Recently, Salo and Uhlmann (2010) proved that the attenuated ray transform is injective for simple two dimensional manifolds. Moreover, stability estimates and a reconstruction procedure of the function from the attenuated transform were given. In the case of systems, one considers instead of a scalar function a on M, an attenuation given by a connection A and a Higgs field Φ on the trivial bundle $M \times \mathbb{C}^n$. The pairs (A, Φ) often appear in the so-called Yang-Mills-Higgs theories. A good example of this is the Bogomolny equation in Minkowski $(2+1)$-space which appears as a reduction of the self-dual Yang-Mills equation in $(2+2)$-space and has been object of intense study in the literature of Solitons and Integrable Systems. There is a remarkable connection between the Bogomolny equation and scattering data which deserves further attention.

In recent work, Paternain, Salo and Uhlmann (2011) proved that the attenuated ray transform is injective for unitary pairs (A, Φ) and simple surfaces. There are two outstanding open problems here: prove injectivity for simple Riemannian manifolds of dimension greater than or equal to three and remove the condition that the pair is unitary, i.e. consider other structure groups, like $GL(n, C)$.

Injectivity properties of attenuated ray transforms have several applications. One of them implemented by Paternain, Salo and Uhlmann (2011) for arbitrary simple surfaces is to recover a unitary connection from the scattering data given by parallel transport along geodesics. Previous results in this direction for domains in Euclidean space were obtained by Finch and Uhlmann (2001), Novikov (2002) and Eskin (2004). Results for general simple Riemannian manifolds but for small connections are due to Sharafutdinov (2000). The problem of determining a unitary connection from its parallel transport on a simple Riemannnian manifold of dimension ≥ 3 remains a major challenge.

3 Presentation Highlights

The scientific program of the workshop consisted of four days of presentations from Monday September 16 to Thursday September 19. The program was designed so that there would be plenty of time for free exchange of ideas and collaboration between participants, with six presentations on Monday, Tuesday and Thursday and only three on Wednesday.

The content of the presentations together with related discussions and joint work are described below.

3.1 Monday 16 September

- **Mikko Salo** (University of Jyväskylä), as the first speaker, gave an overview talk on several geometric inverse problems pointing out relations and connections between them. The underlying theme was the geodesic ray transform, which was shown to arise in many contexts: integral geometry problems (applications in medical imaging), inverse problems for hyperbolic PDE (Gel'fand's inverse problem), inverse problems for elliptic PDE (Calderón's inverse problem), rigidity for manifolds with boundary (boundary/lens rigidity), and spectral rigidity (Can you hear the shape of a drum?). The talk presented also open questions, including invertibility of the geodesic ray transform on nontrapping or tensor situations, the boundary rigidity problem on simple manifolds for $n \geq 3$, and the anisotropic Calderón problem for $n \geq 3$. A particular question was raised related to finding precise conditions (simple/no conjugate points/nontrapping) under which various geometric problems can be solved. All these topics were covered in more detail in later talks.

- **Chris Croke** (University of Pennsylvania) gave a presentation on recent results due to him, Herreros and Wen on scattering and lens rigidity. In particular this addressed the question raised in the talk of Salo on finding precise conditions for scattering or lens rigidity to hold. A standard condition that has been considered is nontrapping, but the results of Croke gave the first examples of manifolds where scattering or lens rigidity hold even if there are trapped rays (but not too many). The main issue was to show that the set of vectors that are tangent to trapped rays have measure zero. Also, the results indicated that scattering and lens data are almost equivalent in some situations.

- **András Vasy** (Stanford University) presented his recent work with Uhlmann and also with Stefanov on invertibility of the geodesic ray transform. This is based on a new microlocal method for studying the ray transform which has certain remarkable features: it allows to prove local invertibility results under a convexity assumption (earlier results required real-analyticity) and global invertibility results under a condition which is different from simplicity, and it introduces to this problem the scattering pseudodifferential calculus due to Melrose and others. The result are so far restricted to 0-tensors and dimensions $n \geq 3$, and it is interesting to see if these restrictions can be removed.

- **Yang Yang** (University of Washington) discussed an inverse problem for the transport equation, where measurements given by the albedo operator are used to recover the attenuation and scattering coefficients. The result extended earlier work in the Euclidean and simple Riemannian manifold case to the case of magnetic geodesics, which is a dynamical system generalizing the usual geodesic flow and was discussed in the background section. The main point was an expansion of the kernel of the albedo operator in terms of its singularities.

- **Alexandru Tamasan** (University of Central Florida) gave a talk on the attenuated Radon transform, a particular case of the ray transforms mentioned in the talk of Salo, in the Euclidean plane. This work gave new explicit types of range characterizations for this transform. The method of proof was based on the theory of A-analytic functions introduced by Bukhgeim and coauthors for studying transforms of this type. This theory seems potentially useful for other applications related to ray transforms as well.

- **Hanming Zhou** (University of Washington) presented new results for the boundary rigidity problem on simple manifolds, where the usual geodesic flow is replaced by a magnetic flow with a potential (MP-systems). This work extends several basic results on the usual boundary rigidity problem to this new setting. There is a difference to the geodesic flow case since knowing the boundary measurements for only one energy level is not sufficient to determine the coefficients, so more measurements are required.

3.2 Tuesday 17 September

- **Matti Lassas** (University of Helsinki) described recent work with Yaroslav Kurylev and Gunther Uhlmann on inverse problems for Einstein equations and other non-linear hyperbolic equations. The talk considered inverse problems for the Einstein equation with a time-depending metric on a 4-dimensional globally hyperbolic Lorentzian manifold. The inverse problem discussed is the question, do the observations of the solutions of the coupled system in an open subset of the space-time with the sources supported in this open set determine the properties of the metric in a larger domain. To study this problem the concept of light observation sets was defined and was shown that these sets determine the conformal class of the metric.

- **Spyros Alexakis** (University of Toronto) considered some problems on unique continuation motivated also by questions in general relativity. He presented joint work with Volker Sclue and Arick Shao on uniqueness results from null infinity for linear waves on asymptotically flat space times. If the solution vanishes to infinite order on suitable parts of future and past null infinities the solution must vanish in an open set of the interior. The method of proof consists in deriving appropriate new Carleman estimates for wave operators on Minkowski, Scharzschild and Kerr space time and certain perturbations of these.

- **Katya Krupchyk** (University of Helsinki) discussed joint work with Gunther Uhlmann on L^p resolvent estimates for higher order elliptic self-adjoint operators on compact manifolds without boundary, generalizing the result of David Dos Santos Ferreira, Carlos Kenig and Mikko Salo for second order elliptic operators. Krupchyk also show that the spectral regions in the resolvent estimates are optimal in general.

- **Raluca Felea** (Rochester Institute of Technology) discussed microlocal analysis of Synthetic Aperture Radar (SAR) with moving objects. She considered four particular cases and in each case she analyzed the forward operator F which maps the image to the data and the normal operator F^*F which is used to recover the image. In general, by applying the backprojection operator F^* to the data, artifacts appear in the reconstructed image. She described these artifacts in terms of several Lagrangian manifolds intersecting and show how to microlocally reduce their strength to obtain a better image.

- **Francois Monard** (University of Washington) reviewed recent progress made by the author on some inverse problems involving geodesic X-ray transforms on Riemannian surfaces with boundary. He explained how to reconstruct sections of certain bundles (k-differentials for k an integer), which in some cases coincide with solenoidal tensor fields, from knowledge of their ray transform. Such reconstruction formulas take the form of Fredholm equations when the metric is simple. Furthermore, the error is proved to be a contraction when the gaussian curvature is small in C^1 norm, in which case the unknowns can be reconstructed via Neumann series. He also presented numerical implementation of these formulas even in the non-simple case where these formulas are not known to be valid.

- **Kiril Datchev** (MIT) discussed the quantitative limiting absorption principle in the semiclassical limit He described his elementary proof of Burq's resolvent bounds for long range semiclassical Schrodinger operators. Globally, the resolvent norm grows exponentially in the inverse semiclassical parameter, and near infinity it grows linearly. His proof also weaken the regularity assumptions on the potential.

3.3 Wednesday 18 September

- **Michael Eastwood** (Australian National University) described results identifying the kernel of the geodesic ray transform acting on symmetric tensors on projective spaces. For the case of 2-tensors, this could be seen as an infinitesimal rigidity result. The techniques used are from representation theory and they are much more powerful than the more traditional approaches using some standard Fourier analysis.

- **Gerard Besson** (Université Grenoble, CNRS) gave a talk on 3-manifolds. After commenting on the recent spectacular results on geometrization of closed 3-manifolds, Besson described some very fundamental open questions on the geometry of open 3-manifolds. The idea was to still use Perelman's techniques on the Ricci flow with surgery to approach these issues (which he called "Post-Perelman questions"). A good part of the talk was motivated by the discussion of the Withehead manifolds. These manifolds are contractible but not connected at infinity (and hence they are not diffeomorphic to $\mathbb{R}^3$). What is the "best" metric on them? Complements of Cantor sets on the sphere were also discussed at some length, including the Yamabe problem for them.

- **Maciej Dunajski** (Cambridge University) discussed the following problem: given a set of curves in the plane so that one has a curve through each point in each direction, when does there exist a metric so that these curves are unparametrized geodesics of the metric? This inverse problem was addressed locally and it lead to a system of PDEs and from those some obstructions were derived. By using some techniques inspired by twistor theory, Brian, Dunajski and Eastwood were able to solve the problem completely. Dunajski presented also some progress in the analogous 3D question.

3.4 Thursday 19 September

- **Adrian Nachman** (University of Toronto) talked about recovering an anisotropic conductivity in a known conformal class from knowledge of one current. Nachman also indicated how to obtain the necessary data from Magnetic Resonance Imagers. The main tools for the results came from geometric measure theory and an important point was that that the corresponding electric potential is the unique solution of a constrained minimization problem with respect to a weighted total variation functional defined entirely in terms of the physical data. The talk also emphasized the fact that the associated equipotential surfaces are area minimizing with respect to a Riemannian metric obtained from the data.

- **Lionel Mason** (Oxford University) gave a talk on his joint work with LeBrun characterizing de Sitter $2 + 1$ Einstein-Weyl spaces. The characterization comes in the form of a 1-1 correspondence with orientation reversing diffeomorphisms of the 2-sphere. The correspondence is established by considering the space of holomorphic discs with boundary on a special subset of the 2-quadric $\mathbb{C}P^1 \times \mathbb{C}P^1$. Similar ideas lead also to the classification of Zoll projective structures on the 2-sphere and projective space.

- **Maarten de Hoop** (Purdue University) presented his recent results with coauthors (of whom Lassas and Holman were also in the audience), which give a rigorous and very general mathematical basis for Dix's method that has been used in seismic imaging. In this method, seismic scattering data are interpreted mathematically as "sphere data" (shape operators of wave fronts generated by many point scatterers), and it is remarkable that the recovery of a sound speed from this data can be done also in the presence of caustics. Stated in geometric terms, it is interesting that this method recovers the universal cover of the Riemannian manifolds instead of the manifold itself.

- **Alex Strohmaier** (University of Loughborough) gave a talk on analytic properties of resolvents and scattering matrices on manifolds. The main point was to exploit the notion of Hahn holomorphic/meromorphic function. The set of Hahn holomorphic functions form an integral domain and this allows for a good notion of Hahn meromorphic function. The key result was that with these definitions in place, resolvents and scattering matrices end up being Hahn meromorphic and from that one can derive for instance that the expansion coefficients determine the scattering matrix near the bottom of the essential spectrum as well as several other consequences.

- **Lauri Oksanen** (University College London) presented his work with Lassas and Helin on recovering a nontrapping Riemannian metric (or sound speed) from boundary measurements for the wave equation with a single boundary source. If one allows all possible boundary sources, this inverse problem can be solved for arbitrary metrics by the boundary control method, but it turns out that one can obtain much information from only one well chosen source. Following ideas in engineering, a white noise source should be sufficient, and this result makes the idea mathematically precise.

- **Rafe Mazzeo** (Stanford University) talked about spectral geometry of the moduli space of Riemann surfaces. The lecture started with a substantial amount of background material including a description of the Weil-Petersson metric and its expansion as one approaches the singularities given by the divisors of the Deligne-Mumford compactification. The existence of an asymptotic expansion of the metric to all orders was highlighted. The lecture concluded with a discussion of the Laplacian of the Weil-Petersson metric including a result asserting that the spectrum is discrete and satisfies a Weyl law. The first steps in spectral geometry of moduli may allow in the future the consideration of quantities like the determinant of the Laplacian of the Weil-Petersson metric.

4 Outcome of the Meeting

We believe the workshop was extremely beneficial for all of us. Besides the talks, there were many groups meeting independently to discuss various open problems. One instance of this was a lenghty meeting between Eastwood, Graham, Paternain, Salo and Alexakis in which various geometric inverse problems related with conformal geometry were discussed. One such problem was the existence of conformal Killing tensors on Anosov manifolds or the possible absence of them for a generic metric. Another problem was the existence of certain special n-harmonic coordinates leading to regularity results for conformal mappings.

The research program of the organizers also benefited a great deal form the meeting: we made progress in the tensor tomography problem in higher dimensions and we identified an important analogue of the Beurling transform for manifolds of non-positive curvature. The contraction properties of this transform has consequences for the transport equation such as existence of certain invariant distributions for the geodesic flow. A preprint is being drafted as we finish this report.

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