Leavitt path algebras of separated graphs and paradoxical decompositions

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Leavitt (1962) defined algebras $L_K(m, n)$ for $1 \leq m \leq n$ in the following way:

$L_K(m, n)$ is the $K$-algebra with generators

$$\{X_{ji}, X^*_{ji} : 1 \leq j \leq m, 1 \leq i \leq n\}$$

and defining relations:

$$XX^* = I_m, \quad X^*X = I_n,$$

where $X = (X_{ji})$. 

Separated graphs: the initial motivation

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Separated graphs

**Definition**

A *separated graph* is a pair \((E, C)\) where \(E\) is a graph, 
\[ C = \bigsqcup_{v \in E^0} C_v, \]
and \(C_v\) is a partition of \(s^{-1}(v)\) (into pairwise disjoint nonempty subsets) for every vertex \(v\):

\[
s^{-1}(v) = \bigsqcup_{X \in C_v} X.
\]

(In case \(v\) is a sink, we take \(C_v\) to be the empty family of subsets of \(s^{-1}(v)\).)

The constructions we introduce revert to existing ones in case \(C_v = \{s^{-1}(v)\}\) for each \(v \in E^0\). We refer to a *non-separated graph* in that situation.
The Leavitt path algebra of a separated graph

Definition

The Leavitt path algebra of the separated graph \((E, C)\) with coefficients in the field \(K\), is the \(K\)-algebra \(L_K(E, C)\) with generators \(\{v, e, e^* \mid v \in E^0, e \in E^1\}\), subject to the following relations:

\[(V)\quad vv' = \delta_{v,v'} v \quad \text{for all } v, v' \in E^0,
\[(E1)\quad s(e)e = er(e) = e \quad \text{for all } e \in E^1,
\[(E2)\quad r(e)e^* = e^*s(e) = e^* \quad \text{for all } e \in E^1,
\[(SCK1)\quad e^*e' = \delta_{e,e'} r(e) \quad \text{for all } e, e' \in X, X \in C, \text{ and}
\[(SCK2)\quad v = \sum_{e \in X} ee^* \quad \text{for every finite set } X \in C_v, v \in E^0.

Example

Let $1 \leq m \leq n$. Let us consider the separated graph $(E(m, n), C(m, n))$, where $E(m, n)$ is the graph consisting of two vertices $v, w$ and with

$$E(m, n)^1 = \{\alpha_1, \ldots, \alpha_n, \beta_1, \ldots, \beta_m\},$$

with $s(\alpha_i) = s(\beta_j) = v$ and $r(\alpha_i) = r(\beta_j) = w$ for all $i, j$, and $C(m, n)$ consists of two elements $X = \{\alpha_1, \ldots, \alpha_n\}$ and $Y = \{\beta_1, \ldots, \beta_m\}$. 
Figure: The separated graph \((E(2, 3), C(2, 3))\)
Lemma (E. Pardo)

There is a natural isomorphism

\[ \gamma : L_K(m, n) \rightarrow wL_K(E(m, n), C(m, n))w \]

given by

\[ \gamma(X_{ji}) = \beta_j^* \alpha_i, \quad \gamma(X_{ji}^*) = \alpha_i^* \beta_j. \]

This induces an isomorphism

\[ L_K(E(m, n), C(m, n)) \cong M_{n+1}(L_K(m, n)) \cong M_{m+1}(L_K(m, n)). \]
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$$L_K(E(m, n), C(m, n)) \cong M_{n+1}(L_K(m, n)) \cong M_{m+1}(L_K(m, n)).$$

Note that

$$\gamma\left(\sum_{i=1}^{n} X_{ji}X_{ki}^*\right) = \sum_{i=1}^{n} \beta_j^* \alpha_i \alpha_i^* \beta_k = \beta_j^* \beta_k = \delta_{jk}w$$

and similarly $$\gamma\left(\sum_{j=1}^{m} X_{ji}^*X_{jk}\right) = \delta_{ik}w$$ so $$\gamma$$ is a well-defined homomorphism, which is shown to be an isomorphism.
Definition

\((E, C)\) is \textit{finitely separated} in case \(|X| < \infty\) for all \(X \in C\).
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Definition

Let \((E, C)\) be a finitely separated graph. The \textit{monoid} of \((E, C)\) is the abelian monoid \(M(E, C)\) with generators \(\{a_v \mid v \in E^0\}\) and relations

\[a_v = \sum_{e \in X} a_{r(e)}, \quad \forall X \in C_v, \forall v \in E^0.\]
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Theorem (Goodearl-A)

\textit{If} \((E, C)\) \textit{is a finitely separated graph then the natural map}

\[ M(E, C) \rightarrow \mathcal{V}(L_K(E, C)) \]

\textit{is an isomorphism}.
Example

For \((E, C) = (E(m, n), C(m, n))\), we have

\[ \mathcal{V}(L(E, C)) \cong M(E, C) \cong \langle a \mid ma = na \rangle. \]

a result originally due to Bergman.
Proposition

If \( M \) is any conical abelian monoid, then there exists a bipartite, finitely separated graph \((E, C)\) such that
\[
M \cong M(E, C) \cong \mathcal{V}(L_K(E, C)).
\]

\( E \) can be taken finite if \( M \) is finitely generated.
Example

In the example $M = \langle a, b \mid 2a = a + 2b \rangle$, we have two generators $a, b$ and one relation $R: 2a = a + 2b$. 
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**Figure:** $M(E, C) = \langle R, a, b \mid R = 2a, R = a + 2b \rangle \cong M$. 
We remark that, in contrast, the monoids $M_E \cong \mathcal{V}(L_K(E))$ of a Leavitt path algebra have very special properties:

- $M_E$ is **conical** if $x + y = 0 \implies x = y = 0$ (this is a general property of $\mathcal{V}(R)$ for any ring $R$)
We remark that, in contrast, the monoids $M_E \cong \mathcal{V}(L_K(E))$ of a Leavitt path algebra have very special properties:

- $M_E$ is **conical** $x + y = 0 \implies x = y = 0$ (this is a general property of $\mathcal{V}(R)$ for any ring $R$)

- $M_E$ has the **Riesz refinement property**: If $a + b = c + d$ then $\exists x, y, z, t$ such that $a = x + y$, $b = z + t$, $c = x + z$ and $d = y + t$:

\[
\begin{array}{cc}
  & c & d \\
 a & x & y \\
b & z & t
\end{array}
\]
• $M_E$ is a **separative monoid**: If $a + c = b + c$ and $c \leq na$, $c \leq mb$ for some $n, m \in \mathbb{N}$, then $a = b$.

where, for $x, y$ in an abelian monoid $M$, we write $x \leq y$ in case $y = x + z$ for some $z \in M$.

• $M_E$ is **unperforated**: $na \leq nb \implies a \leq b$.

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Even amongst the abelian monoids satisfying all these conditions, the ones of the form $M_E$ are special! (by work of A-Perera-Wehrung)
Computation of $K_0$

Let $(E, C)$ be a finitely separated graph. We denote by $1_C: \mathbb{Z}^C \to \mathbb{Z}^{E^0}$ and $A^t_{(E, C)}: \mathbb{Z}^C \to \mathbb{Z}^{E^0}$ the homomorphisms defined by

$$1_C(\delta X) = \delta_v \quad \text{if } X \in C_v$$

and

$$A^t_{(E, C)}(\delta X) = \sum_{w \in E^0} a_X(v, w)\delta_w \quad (v \in E^0, X \in C_v),$$

where $(\delta X)_{X \in C}$ denotes the canonical basis of $\mathbb{Z}^C$, $(\delta_w)$ the canonical basis of $\mathbb{Z}^{E^0}$ and, for $X \in C_v$, $a_X(v, w)$ is the number of arrows in $X$ from $v$ to $w$. 

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The next theorem follows from the computation of $\mathcal{V}(LK(E, C))$. 

**Theorem**

Let $(E, C)$ be a finitely separated graph. Then

$$K_0(LK(E, C)) \cong \text{coker}(1_C - A^t_{(E, C)}: \mathbb{Z}^{(C)} \to \mathbb{Z}^{(E^0)}).$$
**Definition**

For any separated graph \((E, C)\), the (full) graph C*-algebra of the separated graph \((E, C)\) is the universal C*-algebra with generators \(\{v, e \mid v \in E^0, \ e \in E^1\}\), subject to the following relations:

1. \((V)\) \(vw = \delta_{v,w} v\) and \(v = v^*\) for all \(v, w \in E^0\),
2. \((E)\) \(s(e)e = er(e) = e\) for all \(e \in E^1\),
3. \((SCK1)\) \(e^*f = \delta_{\epsilon,f} r(e)\) for all \(e, f \in X, X \in C\), and
4. \((SCK2)\) \(v = \sum_{e \in X} ee^*\) for every finite set \(X \in C_v, v \in E^0\).
Definition

For any separated graph \((E, C)\), the (full) graph \(C^\ast\)-algebra of the separated graph \((E, C)\) is the universal \(C^\ast\)-algebra with generators \(\{v, e \mid v \in E^0, \ e \in E^1\}\), subject to the following relations:

(V) \(vw = \delta_{v,w}v\) and \(v = v^\ast\) for all \(v, w \in E^0\),

(E) \(s(e)e = er(e) = e\) for all \(e \in E^1\),

(SCK1) \(e^\ast f = \delta_{e,f} r(e)\) for all \(e, f \in X, \ X \in C\), and

(SCK2) \(v = \sum_{e \in X} ee^\ast\) for every finite set \(X \in C_v, \ v \in E^0\).

In case \((E, C)\) is trivially separated, \(C^\ast(E, C)\) is just the classical graph \(C^\ast\)-algebra \(C^\ast(E)\).
Graph C*-algebras and dynamics

It is well-known that graph C*-algebras (of ordinary graphs) are closely related to dynamics. This was first discovered by Cuntz and Krieger for $\mathcal{O}_n$ and related C*-algebras $\mathcal{O}_A$, nowadays known as Cuntz-Krieger C*-algebras.

In particular $\mathcal{O}_n$ is related to the shift on $X = \{1, \ldots, n\}^\mathbb{N}$. 
Graph C*-algebras and dynamics

It is well-known that graph C*-algebras (of ordinary graphs) are closely related to dynamics. This was first discovered by Cuntz and Krieger for $O_n$ and related C*-algebras $O_A$, nowadays known as Cuntz-Krieger C*-algebras.

In particular $O_n$ is related to the shift on $X = \{1, \ldots, n\}^\mathbb{N}$.

Note that $X = \bigsqcup_{i=1}^n H_i$, with $X \cong H_i$ for all $i$.

($H_i = \{(i, x_2, x_3, \ldots, )\}.$)

We extend this to the case $(m, n)$, as follows:
Dynamical systems of type \((m,n)\)

We study pairs of compact Hausdorff topological spaces \((X, Y)\) such that

\[
X = \bigcup_{i=1}^{n} H_i = \bigcup_{j=1}^{m} V_j,
\]

where the \(H_i\) are pairwise disjoint clopen subsets of \(X\), each of which is homeomorphic to \(Y\) via given homeomorphisms \(h_i : Y \to H_i\). Likewise we will assume that the \(V_i\) are pairwise disjoint clopen subsets of \(X\), each of which is homeomorphic to \(Y\) via given homeomorphisms \(v_i : Y \to V_i\).
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**Definition**

We will refer to the quadruple \((X, Y, \{h_i\}_{i=1}^{n}, \{v_j\}_{j=1}^{m})\) as an \((m, n)\)-dynamical system.
\[ \begin{array}{cccc}
V_1 & V_2 & \cdots & V_m \\
\vdots & \ddots & & \vdots \\
H_1 & & & H_n \\
\vdots & & & \vdots \\
v_1 & & & v_m \\
\end{array} \]
Definition

An \((m, n)\)-dynamical system \((X^u, Y^u, \{h_i^u\}_{i=1}^n, \{v_j^u\}_{j=1}^m)\) is \textit{universal} if it satisfies the following condition: given any \((m, n)\)-dynamical system

\[(X, Y, \{h_i\}_{i=1}^n, \{v_j\}_{j=1}^m),\]

there exists a unique continuous map

\[\gamma: \Omega = X \bigsqcup Y \rightarrow \Omega^u = X^u \bigsqcup Y^u,\]

such that

1. \(\gamma(Y) \subseteq Y^u,\)
2. \(\gamma(X) \subseteq X^u,\)
3. \(\gamma \circ h_i = h_i^u \circ \gamma,\)
4. \(\gamma \circ v_j = v_j^u \circ \gamma.\)
Example

When $m = 1$, the universal $(1, n)$ dynamical system consists of $X^u = \{1, \ldots, n\}^\mathbb{N}$, $Y^u = \{1', \ldots, n'\}^\mathbb{N}$, a disjoint copy of $X^u$, $X^u = \bigcup_{i=1}^n H_i$, where

$$H_i = \{(i, x_2, x_3, \ldots) : x_n \in \{1, \ldots, n\}\},$$

$h_i : Y^u \to X^u$ sends $(x_1', x_2', \ldots)$ to $(i, x_1, x_2, \ldots)$, and $\nu : Y^u \to X^u$ sends $(x_1', x_2', \ldots)$ to $(x_1, x_2, \ldots)$. 
In general, the universal $(m, n)$ dynamical system is related to the graph C*-algebra $A_{m,n} := C^*(E(m, n), C(m, n))$, as follows:

**Definition**

Let $U$ be the subset of partial isometries in $A_{m,n}$ given by

$$U = \{\alpha_1, \ldots, \alpha_n, \beta_1, \ldots, \beta_m\}.$$  

We will let $O_{m,n}$ be the quotient of $A_{m,n}$ by the closed two-sided ideal generated by all elements of the form

$$xx^* x - x,$$

as $x$ runs in $\langle U \cup U^* \rangle$. 
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It is worth to mention that $A_{1,n} = O_{1,n} \cong M_2(O_n)$, because $\alpha_1, \ldots, \alpha_n, \beta_1$ is a tame set of partial isometries when $m = 1$. 
Note that there is a partial action $\theta$ of $\mathbb{F}_{n+m}$, the free group on $\{a_1, \ldots, a_n, b_1, \ldots, b_m\}$ on $\Omega^u = X^u \sqcup Y^u$, obtained by sending $a_i$ to $h_i$ and $b_j$ to $v_j$.

**Theorem**

*There is a natural isomorphism*

$$\mathcal{O}_{m,n} \cong C(\Omega^u) \rtimes_{\theta^*} \mathbb{F}_{n+m},$$

where $C(\Omega^u) \rtimes_{\theta^*} \mathbb{F}_{n+m}$ denotes the crossed product of the C*-algebra $C(\Omega^u)$ by the induced partial action $\theta^*$ of $\mathbb{F}_{n+m}$. 
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All the above can be generalized to any finite bipartite separated graph $(E, C)$, obtaining C*-algebras $\mathcal{O}(E, C)$ which are suitable full crossed products of commutative C*-algebras by partial actions of free groups.
The algebra $L_{K}^{ab}(E, C)$

The theory is very similar in the purely algebraic case. Let $(E, C)$ be as before. We look at the construction in some detail:
The algebra $L^\text{ab}_K(E, C)$

The theory is very similar in the purely algebraic case. Let $(E, C)$ be as before. We look at the construction in some detail:

Set $U = \langle E^1 \cup (E^1)^* \rangle$, the multiplicative semigroup of $L_K(E, C)$ generated by $E^1 \cup (E^1)^*$. For $u \in U$ set $e(u) = uu^*$ (not an idempotent in general). Write

$$L^\text{ab}_K(E, C) = L_K(E, C)/\langle [e(u), e(u')] : u, u' \in U \rangle.$$ 

It can be shown that $\{e(u) : u \in U\}$ is a family of commuting idempotents in $L^\text{ab}_K(E, C)$. 
Let $\mathcal{B}$ be the commutative subalgebra of $L_K^{ab}(E, C)$ generated by the idempotents $e(u)$, for $u \in U$.

There exists a totally disconnected, metrizable, compact space $\Omega(E, C)$ such that

$$\mathcal{B} = C_K(\Omega(E, C)),$$

where $C_K(\Omega)$ denotes the algebra of locally constant functions $\Omega \to K$. 
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where $C_K(\Omega)$ denotes the algebra of locally constant functions $\Omega \rightarrow K$.

Moreover, there is a partial action $\alpha$ of $F = F\langle E_1 \rangle$ on $\mathcal{B}$ (given essentially by conjugation) which induces a partial action $\alpha^*$ by homeomorphisms of $F$ on $\Omega(E, C)$. Moreover, we show:

**Theorem**

$$L_K^{ab}(E, C) \cong C_K(\Omega(E, C)) \rtimes_{\alpha} F.$$
We can compute precisely the structure of the monoid $\mathcal{V}(L^\text{ab}(E, C))$ thanks to the following approximation result:

**Theorem (A-Exel)**

There exists a sequence of separated graphs $\{(E_n, C^n)\}$ canonically associated to $(E, C)$ such that $(E_0, C^0) = (E, C)$ and

$$L^\text{ab}_K(E, C) \cong \lim_{\to} L_K(E_n, C^n).$$

Moreover all the connecting maps $L_K(E_n, C^n) \to L_K(E_{n+1}, C^{n+1})$ are surjective.
We can compute precisely the structure of the monoid $\mathcal{V}(L_{ab}^K(E, C))$ thanks to the following approximation result:

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Moreover all the connecting maps $L_K(E_n, C^n) \to L_K(E_{n+1}, C^{n+1})$ are surjective.

**Theorem**

$$\mathcal{V}(L_{ab}^K(E, C)) \cong \lim \to M(E_n, C^n).$$

Moreover the map $M(E, C) = \mathcal{V}(L_K(E, C)) \to \mathcal{V}(L_{ab}^K(E, C))$ is an order-embedding.
Paradoxical decompositions

Let $G$ be a group acting on a set $X$. $E, E' \subseteq X$ are **equidecomposable** if

$$E = A_1 \sqcup A_2 \sqcup \cdots \sqcup A_n, \quad E' = B_1 \sqcup B_2 \sqcup \cdots \sqcup B_n$$

and there exist $g_1, g_2, \ldots, g_n \in G$ such that $B_i = g_i A_i$ for all $i = 1, \ldots, n$. 
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and there exist $g_1, g_2, \ldots, g_n \in G$ such that $B_i = g_iA_i$ for all $i = 1, \ldots, n$.

The *type semigroup* $S(X, G)$ is defined by using this relation. Elements of $S(X, G)$ are finite sums of equidecomposability classes $[E]$, for $E \subseteq X$. 
A subset $E \subseteq X$ is called **paradoxical** if $E_1 \sqcup E_2 \subseteq E$ with $E_1 \sim_G E$ and $E_2 \sim_G E$. 
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Note that $E \subseteq X$ is paradoxical $\iff 2[E] \leq [E]$ in $S(X, G)$. 
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The Banach-Tarski Theorem (or Paradox) asserts that the unit ball $\mathbb{B}^1$ is $G$-paradoxical, where $G$ is the group of all the isometries of $\mathbb{R}^3$. 

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The study of this concept led to the notion of **amenable group**: A discrete group $\Gamma$ is **amenable** if $\Gamma\Gamma$ is not paradoxical.
Tarski’s Theorem

**Theorem (Tarski)**

Let $G$ be a group acting on a set $X$. Then the following conditions are equivalent:

1. $E$ is not $G$-paradoxical, i.e. $2[E] \not\subseteq [E]$
2. There exists a finitely additive $G$-invariant measure $\mu : \mathcal{P}(X) \to [0, +\infty]$ such that $\mu(E) = 1$.

This result gives the transition from the paradoxical decompositions characterization of amenable groups to other characterizations, notably the one involving invariant means.

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About the proof

The proof of Tarski’s Theorem is based on the purely semigroup theoretic result:

**Theorem**

Let $(S, +)$ be an abelian semigroup and $e \in S$. Then the following are equivalent:

1. (a) There exists a semigroup homomorphism $\mu : S \to [0, \infty]$ such that $\mu(e) = 1$.
2. (b) For all $n \in \mathbb{N}$, we have $(n + 1)e \not\leq ne$. 
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Let \((S, +)\) be an abelian semigroup and \(e \in S\). Then the following are equivalent:

(a) There exists a semigroup homomorphism \(\mu: S \to [0, \infty]\) such that \(\mu(e) = 1\).

(b) For all \(n \in \mathbb{N}\), we have \((n + 1)e \not\preceq ne\).

and the following properties of \(S(X, G)\):

**Schröder-Bernstein axiom:** \(a \leq b\) and \(b \leq a \implies a = b\).

**Cancellation law:** \(\forall n \in \mathbb{N}, \; na = nb \implies a = b\).
In fact, with these conditions at hand we can easily show that condition (b) in the Theorem is equivalent to $2e \not\leq e$, or equivalently

$$2e \leq e \iff (n + 1)e \leq ne \text{ for some } n.$$  

If $(n + 1)e \leq ne$ then $(n + 1)e = ne$ by Schröder-Bernstein, and then

$$(n + 1)e = ne \implies n(2e) = ne \implies 2e = e \text{ by the cancellation law.}$$
There has been recent interest in trying to extend Tarski’s theorem to a more general context:

Assume that $G$ acts on a set $X$ and let $\mathcal{D}$ be a $G$-invariant subalgebra of sets of $X$. Then one can restrict the $G$-equidecomposability relation to elements of $\mathcal{D}$, and obtain a type semigroup $S(X, G, \mathcal{D})$. 
In recent papers by Rørdam–Sierakowski and Kerr–Nowak, the following particular case has been considered:

$G$ acts by homeomorphisms on a totally disconnected compact Hausdorff space $X$ (e.g. the Cantor set) and $\mathbb{D}$ is the subalgebra $\mathbb{K}$ of clopen subsets of $X$.

These authors have raised the question of whether the analogue of Tarski’s Theorem holds in this context. More precisely:

Is it true that, for $E \in \mathbb{K}$, one has that the following are equivalent?

1. $2[E] \not\sim [E]$ in $S(X, G, \mathbb{K})$,
2. There exists a semigroup homomorphism $\mu : S(X, G, \mathbb{K}) \to [0, \infty]$ such that $\mu([E]) = 1$. 

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One may ask:

What are the general properties of $S(X, G, \mathbb{K})$? It is easy to show that $S(X, G, \mathbb{K})$ has the following properties:

- It is **conical** $x + y = 0 \implies x = y = 0$
- It has the **Riesz refinement property**: If $a + b = c + d$ then $\exists x, y, z, t$ such that $a = x + y$, $b = z + t$, $c = x + z$ and $d = y + t$:
We prove that these are the only general properties of $S(X, G, K)$:

**Theorem**

*Let $M$ be an arbitrary f.g. conical abelian monoid. Then there exists a totally disconnected, metrizable compact space $X$ and an action of a finitely generated free group $F$ on it such that there is an order-embedding $M \hookrightarrow S(X, F, K)$.***
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For instance, taking $M = \langle a \mid na = ma \rangle$ for $1 < m < n$ one obtains that there is a clopen subset $E \subseteq X$ such that $2[E] \nsubseteq [E]$ in $S(X, F, K)$, but there is no $\mu: S(X, F, K) \to [0, \infty]$ such that $\mu([E]) = 1$. 
In the general setting of a partial action $\theta$ of a group $\Gamma$ on a totally disconnected compact space $X$, we always have a monoid homomorphism:

$$S(X, \Gamma, K) \rightarrow \mathcal{V}(C_K(X) \rtimes_{\theta^*} \Gamma)$$

$$[Y] \mapsto \chi_Y \cdot \delta_e$$

If $X = \Omega(E, C)$ for a finite bipartite separated graph $(E, C)$, we are able to show:

**Theorem**

*The natural homomorphism*

$$S(\Omega(E, C), F, K) \rightarrow \mathcal{V}(C_K(\Omega(E, C)) \rtimes_{\alpha} F)$$

*is an isomorphism*
Now, starting with a finitely generated conical abelian monoid $M$, we choose a finite bipartite separated graph $(E, C)$ such that $M \cong M(E, C)$, and so we get a totally disconnected metrizable compact space $\Omega(E, C)$ with a partial action $\alpha^*$ of $\mathbb{F} = \mathbb{F}\langle E^1 \rangle$ such that there is an order-embedding

$$M \hookrightarrow \mathcal{V}(L^{ab}(E, C)) \cong S(\Omega(E, C), \mathbb{F}, \mathbb{K}).$$
Finally, using globalization techniques due to Abadie, we can reach the same conclusion, but with *total actions* instead of *partial actions*, obtaining:

**Theorem**

*Let $M$ be an arbitrary f.g. conical abelian monoid. Then there exist a totally disconnected, metrizable compact space $X$ and an action of a finitely generated free group $\mathbb{F}$ on it such that there is an order-embedding $M \hookrightarrow S(X, \mathbb{F}, K)$.***
Finally, using globalization techniques due to Abadie, we can reach the same conclusion, but with *total actions* instead of *partial actions*, obtaining:

**Theorem**

Let $M$ be an arbitrary f.g. conical abelian monoid. Then there exist a totally disconnected, metrizable compact space $X$ and an action of a finitely generated free group $F$ on it such that there is an order-embedding $M \hookrightarrow S(X, F, K)$.

**Corollary**

There exist a global action of a finitely generated free group $F$ on a totally disconnected metrizable compact space $Z$, and a non-$F$-paradoxical (with respect to $K$) clopen subset $A$ of $Z$ such that $\mu(A) = \infty$ for every finitely additive $F$-invariant measure $\mu: K \to [0, \infty]$ such that $\mu(A) > 0$. 

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