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Orbit equivalence and graph C^*-algebras

Work in progress with Nathan Brownlowe and Michael Whittaker

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Norwegian University of Science and Technology

Graph Algebras: Bridges between graph C^*-algebras and Leavitt path algebras
Banff 2013-04-23

Directed graphs

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Directed graphs

- A *directed graph* E is a quadruple (E^0, E^1, r, s) consisting of two sets E^0 and E^1 and two maps $r, s : E^1 \to E^0$.

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- If e is an edge, $s(e)$ is called the *source* of e, and $r(e)$ is called the *range* of e.

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- If $s(e) = v$ and $r(e) = w$, then we say that v *emits* e, and that w *receives* e.

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- If $s(e) = v$ and $r(e) = w$, then we say that v *emits* e, and that w *receives* e.
- If $v \in E^0$, then we let $vE^1 = \{e \in E^n : r(e) = v\}$ and $E^1 v = \{e \in E^n : s(e) = v\}$.

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Paths

Paths

- A *path of length n* in a directed graph E is a sequence $\mu = \mu_1\mu_2\ldots\mu_n$ of edges in E such that $s(\mu_i) = r(\mu_{i+1})$ for $i \in \{1, 2, \ldots, n-1\}$.

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- We denote by E^n the set of paths of length n, and let $E^* = \bigcup_{n=0}^{\infty} E^n$.

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- We denote by E^n the set of paths of length n, and let $E^* = \bigcup_{n=0}^{\infty} E^n$.
- We extend the range and source maps to E^* by setting $r(\mu) = r(\mu_1)$ and $s(\mu) = s(\mu_n)$ when $|\mu| \geq 1$, and $r(\mu) = s(\mu) = \mu$ when $\mu \in E^0$.

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- If $\mu, \nu \in E^*$ and $s(\mu) = r(\nu)$, then we write $\mu\nu$ for the path $\mu_1 \dots \mu_{|\mu|}\nu_1 \dots \nu_{|\nu|}$.

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Sinks, sources and row-finite graphs

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Sinks, sources and row-finite graphs

- A vertex $v \in E^*$ is called a *sink* if $E^1 v = \emptyset$, and a *source* if $vE^1 = \emptyset$.

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Sinks, sources and row-finite graphs

- A vertex $v \in E^*$ is called a *sink* if $E^1 v = \emptyset$, and a *source* if $vE^1 = \emptyset$.
- A directed graph is said to be *row-finite* if vE^1 is finite for all $v \in E^0$.

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Graph C^*-algebras

Graph C^*-algebras

Let E be a row-finite directed graph with no sources.

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Graph C^*-algebras

Let E be a row-finite directed graph with no sources. The C^*-algebra $C^*(E)$ of the graph E is defined as the universal C^*-algebra generated by a family $(s_e, p_v)_{e\in E^1, v\in E^0}$ consisting of partial isometries $(s_e)_{e\in E^1}$ with mutually orthogonal range projections and mutually orthogonal projections $(p_v)_{v\in E^0}$ satisfying

1. $s_e^* s_e = p_{s(e)}$ for all $e \in E^1$,
2. $p_v = \sum_{e\in vE^1} s_e s_e^*$ for all $v \in E^0$.

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The C^*-subalgebra $\mathcal{D}(E)$

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The C^*-subalgebra $\mathcal{D}(E)$

- For $\mu \in E^*$, we let $s_\mu = s_{\mu_1} \dots s_{\mu_{|\mu|}}$ when $|\mu| \geq 1$, and $s_\mu = p_\mu$ when $\mu \in E^0$.

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The C^*-subalgebra $\mathcal{D}(E)$

- For $\mu \in E^*$, we let $s_\mu = s_{\mu_1} \dots s_{\mu_{|\mu|}}$ when $|\mu| \geq 1$, and $s_\mu = p_\mu$ when $\mu \in E^0$.
- We let $\mathcal{D}(E)$ denote the C^*-subalgebra of $C^*(E)$ generated by $\{s_\mu s_\mu^* \mid \mu \in E^*\}$.

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- Let E and F be two row-finite directed graphs with no sources.

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The C^*-subalgebra $\mathcal{D}(E)$

- For $\mu \in E^*$, we let $s_\mu = s_{\mu_1} \dots s_{\mu_{|\mu|}}$ when $|\mu| \geq 1$, and $s_\mu = p_\mu$ when $\mu \in E^0$.
- We let $\mathcal{D}(E)$ denote the C^*-subalgebra of $C^*(E)$ generated by $\{s_\mu s_\mu^* \mid \mu \in E^*\}$.
- Let E and F be two row-finite directed graphs with no sources. We are interested in determining when there is an isomorphism $\psi : C^*(E) \to C^*(F)$ such that $\psi(\mathcal{D}(E)) = \mathcal{D}(F)$.

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Infinite paths

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Infinite paths

- An *infinite path* in a directed graph E is an infinite sequence $x = x_1 x_2 \dots$ of edges in E such that $s(x_i) = r(x_{i+1})$ for $i \in \{1, 2, \dots\}$.

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- If $\mu \in E^*$, $x \in E^\infty$ and $s(\mu) = r(x)$, then we write μx for the path $\mu_1 \dots \mu_{|\mu|}x_1x_2\dots$ (if $\mu \in E^0$, then $\mu x = x$).

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The infinite path space

- We equip E^∞ with the topology generated by $\{Z(u) \mid u \in E^*\}$.

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- We equip E^∞ with the topology generated by $\{Z(u) \mid u \in E^*\}$.
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- $Z(\mu)$ is open and compact for all $\mu \in E^*$.

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- E^∞ is compact if and only if E^0 is finite.
- There is a $*$-isomorphism from $\mathcal{D}(E)$ to $C_0(E^\infty)$ which, for every $\mu \in E^*$, maps $s_\mu s_\mu^*$ to the characteristic function of $Z(\mu)$.

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- There is a $*$-isomorphism from $\mathcal{D}(E)$ to $C_0(E^\infty)$ which, for every $\mu \in E^*$, maps $s_\mu s_\mu^*$ to the characteristic function of $Z(\mu)$.
- We let $\sigma_E : E^\infty \to E^\infty$ denote the map

$$x_1 x_2 x_3 \ldots \mapsto x_2 x_3 \ldots .$$

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Continuously orbit equivalence

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Let E and F be two row-finite directed graphs with no sources. We say the infinite path spaces E^∞ and F^∞ are *continuously orbit equivalent*

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Continuously orbit equivalence

Let E and F be two row-finite directed graphs with no sources. We say the infinite path spaces E^∞ and F^∞ are *continuously orbit equivalent* if there exists a homeomorphism $h : E^\infty \to F^\infty$ and continuous functions $k_1, l_1 : E^\infty \to \mathbb{N}$ and $k_2, l_2 : F^\infty \to \mathbb{N}$ such that

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$$\sigma_F^{k_1(x)} \circ h \circ \sigma_E(x) = \sigma_F^{l_1(x)} \circ h(x) \text{ and}$$
$$\sigma_E^{k_2(y)} \circ h^{-1} \circ \sigma_F(y) = \sigma_E^{l_2(y)} \circ h^{-1}(y),$$

for all $x \in E^\infty, y \in F^\infty$.

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Cycles

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Cycles

- A *cycle* is a path $\mu \in E^*$ for which $\mu \geq 1$ and $s(\mu) = r(\mu)$.
- An *entry* for a cycle μ is an edge $e \in E^1$ such that $r(e) = r(\mu_i)$ and $e \neq \mu_i$ for some $i \in \{1, 2, \ldots, |\mu|\}$.

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The main theorem

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Suppose E and F are row-finite directed graphs with no sources and in which every cycle has an entry.

The main theorem

Suppose E and F are row-finite directed graphs with no sources and in which every cycle has an entry. Then the following are equivalent:

1. There is an isomorphism $\psi : C^*(E) \to C^*(F)$ such that $\psi(\mathcal{D}(E)) = \mathcal{D}(F)$;
2. E^∞ and F^∞ are continuously orbit equivalent.

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Examples

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1. Let E be the graph $\bullet$

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Then $E^\infty = \{\star\} = F^\infty$, so E^∞ and F^∞ are continuously orbit equivalent, but $C^*(E) \cong \mathbb{C} \ncong C(\mathbb{T}) \cong C^*(F)$.

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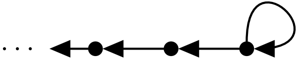
   and let F be the graph 

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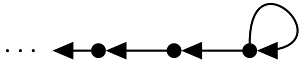
   and let F be the graph 
   Then $E^\infty = \mathbb{N} = F^\infty$,

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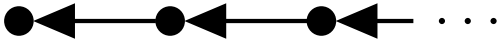
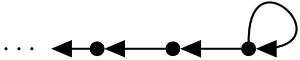
   Then $E^\infty = \{\star\} = F^\infty$, so E^∞ and F^∞ are continuously orbit equivalent, but $C^*(E) \cong \mathbb{C} \ncong C(\mathbb{T}) \cong C^*(F)$.
2. Let E be the graph 

   and let F be the graph 
   Then $E^\infty = \mathbb{N} = F^\infty$, so E^∞ and F^∞ are continuously orbit equivalent,

Examples

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   Then $E^\infty = \{\star\} = F^\infty$, so E^∞ and F^∞ are continuously orbit equivalent, but $C^*(E) \cong \mathbb{C} \not\cong C(\mathbb{T}) \cong C^*(F)$.
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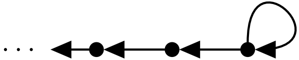
   and let F be the graph 
   Then $E^\infty = \mathbb{N} = F^\infty$, so E^∞ and F^∞ are continuously orbit equivalent, but $C^*(E) \cong \mathcal{K} \not\cong \mathcal{K} \otimes C(\mathbb{T}) \cong C^*(F)$.

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The full inverse semigroup

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The full inverse semigroup

Let E be a row-finite directed graph with no sources and in which every cycle has an entry.

The full inverse semigroup

Let E be a row-finite directed graph with no sources and in which every cycle has an entry. We denote by $\mathcal{S}(E^\infty)$ the set of all partial homeomorphisms of E^∞ whose domain and range are compact open sets, and such that there exist continuous functions $k_\tau, l_\tau : \mathrm{Dom}(\tau) \to \mathbb{N}$ satisfying

$$\sigma_E^{k_\tau(x)}(\tau(x)) = \sigma_E^{l_\tau(x)}(x).$$

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The full inverse semigroup

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$$\sigma_E^{k_\tau(x)}(\tau(x)) = \sigma_E^{l_\tau(x)}(x).$$

If $h : E^\infty \to F^\infty$ is a homeomorphism, we denote by $h \circ \mathcal{S}(E^\infty) \circ h^{-1}$ the set

$$\{h \circ \tau \circ h^{-1}|_{h(\mathrm{Dom}(\tau))} : \tau \in \mathcal{S}(E^\infty)\}.$$

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The main theorem

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Suppose E and F are row-finite directed graphs with no sources and in which every cycle has an entry. Then the following are equivalent:

1. There is an isomorphism $\psi : C^*(E) \to C^*(F)$ such that $\psi(\mathcal{D}(E)) = \mathcal{D}(F)$;
2. E^∞ and F^∞ are continuously orbit equivalent;
3. there is a homeomorphism $h : E^\infty \to F^\infty$ such that $h \circ \mathcal{S}(E^\infty) \circ h^{-1} = \mathcal{S}(F^\infty)$.

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The groupoid of the full inverse semigroup

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The groupoid of the full inverse semigroup

- When E is a row-finite directed graph with no sinks in which every cycle has an entry, then we let $\mathcal{G}_{\mathcal{S}(E^\infty)}$ be the groupoid

$$\{(x,\tau) \mid \tau \in \mathcal{S}(E^\infty),\ x \in \mathrm{Dom}(\tau)\}/\sim$$

where $(x_1,\tau_1) \sim (x_2,\tau_2)$ if $x_1 = x_2$ and there is a a compact open neighbourhood $U \subseteq \mathrm{Dom}(\tau_1) \cap \mathrm{Dom}(\tau_2)$ of x_1 such that τ_1 and τ_2 are equal on U.

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- $[x,\tau]^{-1} = [\tau(x),\tau^{-1}]$.

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  where $(x_1,\tau_1) \sim (x_2,\tau_2)$ if $x_1 = x_2$ and there is a a compact open neighbourhood $U \subseteq \mathrm{Dom}(\tau_1) \cap \mathrm{Dom}(\tau_2)$ of x_1 such that τ_1 and τ_2 are equal on U.
- $[x,\tau]^{-1} = [\tau(x),\tau^{-1}]$.
- $[x_1,\tau_1]$ and $[x_2,\tau_2]$ are composable if $x_1 = \tau_2(x_2)$ in which case $[x_1,\tau_1][x_2,\tau_2] = [x_2,\tau_1 \circ \tau_2]$.

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The groupoid of the full inverse semigroup

- When $\tau \in \mathcal{S}(E^\infty)$ and U is an open subset of $\mathrm{Dom}(\tau)$, then we let $Z(U,\tau) = \{[x,\tau] \mid x \in U\}$.

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- When $\tau \in \mathcal{S}(E^\infty)$ and U is an open subset of $\mathrm{Dom}(\tau)$, then we let $Z(U,\tau) = \{[x,\tau] \mid x \in U\}$.
- We equip $\mathcal{G}_{\mathcal{S}(E^\infty)}$ with the topology generated by $\{Z(U,\tau) \mid \tau \in \mathcal{S}(E^\infty),\ U \text{ is an open subset of } \mathrm{Dom}(\tau)\}$.

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- We equip $\mathcal{G}_{\mathcal{S}(E^\infty)}$ with the topology generated by $\{Z(U,\tau) \mid \tau \in \mathcal{S}(E^\infty),\ U \text{ is an open subset of } \mathrm{Dom}(\tau)\}$.
- Then $\mathcal{G}_{\mathcal{S}(E^\infty)}$ becomes a locally compact, Hausdorff, étale topological groupoid and $\mathcal{G}^0_{\mathcal{S}(E^\infty)}$ is homeomorphic to E^∞.

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The Cuntz-Krieger uniqueness theorem

Let E be a row-finite directed graph with no sources and in which every cycle has an entry.

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The Cuntz-Krieger uniqueness theorem

Let E be a row-finite directed graph with no sources and in which every cycle has an entry. Let ϕ be a $*$-homomorphism defined on $C^*(E)$.

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Let E be a row-finite directed graph with no sources and in which every cycle has an entry. Let ϕ be a $*$-homomorphism defined on $C^*(E)$. Then ϕ is injective if and only if $\phi(p_v) \neq 0$ for all $v \in E^0$.

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1. $\phi(p_v) = \chi_{Z(Z(v),\mathrm{Id}_{Z(v)})}$ for $v \in E^0$,

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3. $\phi(\mathcal{D}(E)) = C_0(\mathcal{G}^0_{\mathcal{S}(E^\infty)})$.

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Proof:

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Proof:

For $v \in E^0$ let q_v denote the characteristic function of $Z(Z(v), \mathrm{Id}_{Z(v)})$, and for $e \in E^1$ let t_e denote the characteristic function of $Z(Z(e), (\sigma_E)_{|Z(e)})$.

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1. $t_e^* t_e = q_{s(e)}$ for all $e \in E^1$,
2. $q_v = \sum_{e \in vE^1} t_e t_e^*$ for all $v \in E^0$.

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2. $q_v = \sum_{e \in vE^1} t_e t_e^*$ for all $v \in E^0$.

It then follows from the universal property of $C^*(E)$ that there exists a $*$-homomorphism $\phi : C^*(E) \to C^*(\mathcal{G}_{\mathcal{S}(E^\infty)})$ such that $\phi(p_v) = q_v$ and $\phi(s_e) = t_e$.

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Proof:

For $v \in E^0$ let q_v denote the characteristic function of $Z(Z(v), \mathrm{Id}_{Z(v)})$, and for $e \in E^1$ let t_e denote the characteristic function of $Z(Z(e), (\sigma_E)_{|Z(e)})$. It is not difficult to check that $(t_e, q_v)_{e \in E^1, v \in E^0}$ is a family consisting of partial isometries $(t_e)_{e \in E^1}$ with mutually orthogonal range projections and mutually orthogonal projections $(q_v)_{v \in E^0}$ satisfying

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Proof:

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It then follows from the universal property of $C^*(E)$ that there exists a $*$-homomorphism $\phi : C^*(E) \to C^*(\mathcal{G}_{\mathcal{S}(E^\infty)})$ such that $\phi(p_v) = q_v$ and $\phi(s_e) = t_e$. ϕ is surjective since $C^*(\mathcal{G}_{\mathcal{S}(E^\infty)})$ is generated by $(t_e, q_v)_{e \in E^1, v \in E^0}$. It is easy to check that $\phi(\mathcal{D}(E)) = C_0(\mathcal{G}^0_{\mathcal{S}(E^\infty)})$ and that ϕ restricted to $\mathcal{D}(E)$ is injective.

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It then follows from the universal property of $C^*(E)$ that there exists a $*$-homomorphism $\phi : C^*(E) \to C^*(\mathcal{G}_{\mathcal{S}(E^\infty)})$ such that $\phi(p_v) = q_v$ and $\phi(s_e) = t_e$. ϕ is surjective since $C^*(\mathcal{G}_{\mathcal{S}(E^\infty)})$ is generated by $(t_e, q_v)_{e \in E^1, v \in E^0}$. It is easy to check that $\phi(\mathcal{D}(E)) = C_0(\mathcal{G}^0_{\mathcal{S}(E^\infty)})$ and that ϕ restricted to $\mathcal{D}(E)$ is injective. It then follows from the Cuntz-Krieger uniqueness theorem that ϕ is injective.

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The main theorem

Suppose E and F are row-finite directed graphs with no sources and in which every cycle has an entry.

The main theorem

Suppose E and F are row-finite directed graphs with no sources and in which every cycle has an entry. Then the following are equivalent:

1. There is an isomorphism $\psi : C^*(E) \to C^*(F)$ such that $\psi(\mathcal{D}(E)) = \mathcal{D}(F)$;
2. E^∞ and F^∞ are continuously orbit equivalent;
3. there is a homeomorphism $h : E^\infty \to F^\infty$ such that $h \circ \mathcal{S}(E^\infty) \circ h^{-1} = \mathcal{S}(F^\infty)$;
4. the groupoids $\mathcal{G}_{\mathcal{S}(E^\infty)}$ and $\mathcal{G}_{\mathcal{S}(F^\infty)}$ are isomorphic (as topological groupoids with Haar systems).

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Remark

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Remark

The main theorem, and its proof, is inspired by the results in Kengo Matsumoto's two papers

1. *Orbit equivalence of topological Markov shifts and Cuntz-Krieger algebras*,
2. *Orbit equivalence of one-sided subshifts and the associated C^*-algebras*.

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On the proof of the main theorem

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On the proof of the main theorem

2 $\iff$ 3:

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On the proof of the main theorem

(2) $\iff$ (3): Let $h : E^\infty \to F^\infty$ be a homeomorphism.

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On the proof of the main theorem

2 $\iff$ 3: Let $h : E^\infty \to F^\infty$ be a homeomorphism. It is straight forward to check that there exist continuous functions $k_1, l_1 : E^\infty \to \mathbb{N}$ and $k_2, l_2 : F^\infty \to \mathbb{N}$ such that

$$\sigma_F^{k_1(x)} \circ h \circ \sigma_E(x) = \sigma_F^{l_1(x)} \circ h(x) \text{ and}$$
$$\sigma_E^{k_2(y)} \circ h^{-1} \circ \sigma_F(y) = \sigma_E^{l_2(y)} \circ h^{-1}(y),$$

for all $x \in E^\infty, y \in F^\infty$,

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On the proof of the main theorem

2 $\iff$ 3: Let $h : E^\infty \to F^\infty$ be a homeomorphism. It is straight forward to check that there exist continuous functions $k_1, l_1 : E^\infty \to \mathbb{N}$ and $k_2, l_2 : F^\infty \to \mathbb{N}$ such that

$$\sigma_F^{k_1(x)} \circ h \circ \sigma_E(x) = \sigma_F^{l_1(x)} \circ h(x) \text{ and}$$
$$\sigma_E^{k_2(y)} \circ h^{-1} \circ \sigma_F(y) = \sigma_E^{l_2(y)} \circ h^{-1}(y),$$

for all $x \in E^\infty, y \in F^\infty$, if and only if $h \circ \mathcal{S}(E^\infty) \circ h^{-1} = \mathcal{S}(F^\infty)$.

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On the proof of the main theorem

3 $\Longrightarrow$ 4:

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On the proof of the main theorem

③ $\implies$ ④: It is also easy to check that if $h : E^\infty \to F^\infty$ is a homeomorphism such that $h \circ \mathcal{S}(E^\infty) \circ h^{-1} = \mathcal{S}(F^\infty)$,

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On the proof of the main theorem

3 $\implies$ 4: It is also easy to check that if $h : E^\infty \to F^\infty$ is a homeomorphism such that $h \circ \mathcal{S}(E^\infty) \circ h^{-1} = \mathcal{S}(F^\infty)$, then $[x, \tau] \mapsto [h(x), h \circ \tau \circ h^{-1}]$ is an isomorphism between $\mathcal{G}_{\mathcal{S}(E^\infty)}$ and $\mathcal{G}_{\mathcal{S}(F^\infty)}$.

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On the proof of the main theorem

4 $\implies$ 1:

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On the proof of the main theorem

- ❹ $\Longrightarrow$ ❶: If $\mathcal{G}_{\mathcal{S}(E^\infty)}$ and $\mathcal{G}_{\mathcal{S}(F^\infty)}$ are isomorphic,

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On the proof of the main theorem

4 $\Longrightarrow$ 1: If $\mathcal{G}_{\mathcal{S}(E^\infty)}$ and $\mathcal{G}_{\mathcal{S}(F^\infty)}$ are isomorphic, then there is an isomorphism between $C^*(\mathcal{G}_{\mathcal{S}(E^\infty)})$ and $C^*(\mathcal{G}_{\mathcal{S}(F^\infty)})$ which maps $C_0(\mathcal{G}^0_{\mathcal{S}(E^\infty)})$ onto $C_0(\mathcal{G}^0_{\mathcal{S}(F^\infty)})$,

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On the proof of the main theorem

4 $\implies$ 1: If $\mathcal{G}_{\mathcal{S}(E^\infty)}$ and $\mathcal{G}_{\mathcal{S}(F^\infty)}$ are isomorphic, then there is an isomorphism between $C^*(\mathcal{G}_{\mathcal{S}(E^\infty)})$ and $C^*(\mathcal{G}_{\mathcal{S}(F^\infty)})$ which maps $C_0(\mathcal{G}^0_{\mathcal{S}(E^\infty)})$ onto $C_0(\mathcal{G}^0_{\mathcal{S}(F^\infty)})$, and since there is an isomorphism between $C^*(E)$ and $C^*(\mathcal{G}_{\mathcal{S}(E^\infty)})$ which maps $\mathcal{D}(E)$ onto $C_0(\mathcal{G}^0_{\mathcal{S}(E^\infty)})$, and an isomorphism between $C^*(F)$ and $C^*(\mathcal{G}_{\mathcal{S}(F^\infty)})$ which maps $\mathcal{D}(F)$ onto $C_0(\mathcal{G}^0_{\mathcal{S}(F^\infty)})$,

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On the proof of the main theorem

(4) $\implies$ (1): If $\mathcal{G}_{\mathcal{S}(E^\infty)}$ and $\mathcal{G}_{\mathcal{S}(F^\infty)}$ are isomorphic, then there is an isomorphism between $C^*(\mathcal{G}_{\mathcal{S}(E^\infty)})$ and $C^*(\mathcal{G}_{\mathcal{S}(F^\infty)})$ which maps $C_0(\mathcal{G}^0_{\mathcal{S}(E^\infty)})$ onto $C_0(\mathcal{G}^0_{\mathcal{S}(F^\infty)})$, and since there is an isomorphism between $C^*(E)$ and $C^*(\mathcal{G}_{\mathcal{S}(E^\infty)})$ which maps $\mathcal{D}(E)$ onto $C_0(\mathcal{G}^0_{\mathcal{S}(E^\infty)})$, and an isomorphism between $C^*(F)$ and $C^*(\mathcal{G}_{\mathcal{S}(F^\infty)})$ which maps $\mathcal{D}(F)$ onto $C_0(\mathcal{G}^0_{\mathcal{S}(F^\infty)})$, it follows that there is an isomorphism between $C^*(E)$ and $C^*(F)$ which maps $\mathcal{D}(E)$ onto $\mathcal{D}(F)$.

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On the proof of the main theorem

1 $\implies$ 3:

On the proof of the main theorem

1 $\implies$ 3: Let

$$N_E = \{u \in C^*(E) : u \text{ is a partial isometry}, \\ u\mathcal{D}(E)u^* \subseteq \mathcal{D}(E), u^*\mathcal{D}(E)u \subseteq \mathcal{D}(E)\}.$$

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On the proof of the main theorem

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$$N_E = \{u \in C^*(E) : u \text{ is a partial isometry}, \\ u\mathcal{D}(E)u^* \subseteq \mathcal{D}(E), u^*\mathcal{D}(E)u \subseteq \mathcal{D}(E)\}.$$

If $u \in N_E$, then uu^* and u^*u belong to $\mathcal{D}(E)$ which we will identify with $C_0(E^\infty)$.

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On the proof of the main theorem

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If $u \in N_E$, then uu^* and u^*u belong to $\mathcal{D}(E)$ which we will identify with $C_0(E^\infty)$. There is for each $u \in N_E$ a unique $\tau_u \in \mathcal{S}(E^\infty)$ satisfying $ufu^* = f \circ \tau_u$ and $u^*fu = f \circ \tau_u^{-1}$ for all $f \in C_0(E^\infty)$.

On the proof of the main theorem

1 $\implies$ 3: Let

$$N_E = \{u \in C^*(E) : u \text{ is a partial isometry}, \\ u\mathcal{D}(E)u^* \subseteq \mathcal{D}(E), u^*\mathcal{D}(E)u \subseteq \mathcal{D}(E)\}.$$

If $u \in N_E$, then uu^* and u^*u belong to $\mathcal{D}(E)$ which we will identify with $C_0(E^\infty)$. There is for each $u \in N_E$ a unique $\tau_u \in \mathcal{S}(E^\infty)$ satisfying $ufu^* = f \circ \tau_u$ and $u^*fu = f \circ \tau_u^{-1}$ for all $f \in C_0(E^\infty)$. The map $u \mapsto \tau_u$ is a surjective map from N_E to $\mathcal{S}(E^\infty)$,

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On the proof of the main theorem

1 $\Longrightarrow$ 3: Let

$$N_E = \{u \in C^*(E) : u \text{ is a partial isometry}, \quad u\mathcal{D}(E)u^* \subseteq \mathcal{D}(E), u^*\mathcal{D}(E)u \subseteq \mathcal{D}(E)\}.$$

If $u \in N_E$, then uu^* and u^*u belong to $\mathcal{D}(E)$ which we will identify with $C_0(E^\infty)$. There is for each $u \in N_E$ a unique $\tau_u \in \mathcal{S}(E^\infty)$ satisfying $ufu^* = f \circ \tau_u$ and $u^*fu = f \circ \tau_u^{-1}$ for all $f \in C_0(E^\infty)$. The map $u \mapsto \tau_u$ is a surjective map from N_E to $\mathcal{S}(E^\infty)$, and $\tau_{u_1} = \tau_{u_2}$ iff $u_1u_1^* = u_2u_2^*$, $u_1^*u_1 = u_2^*u_2$, and $u_1u_2^*$ and $u_1^*u_2$ both belong to $\mathcal{D}(E)$.

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On the proof of the main theorem

❶ $\implies$ ❸: Let

$$N_E = \{u \in C^*(E) : u \text{ is a partial isometry}, \quad u\mathcal{D}(E)u^* \subseteq \mathcal{D}(E), u^*\mathcal{D}(E)u \subseteq \mathcal{D}(E)\}.$$

If $u \in N_E$, then uu^* and u^*u belong to $\mathcal{D}(E)$ which we will identify with $C_0(E^\infty)$. There is for each $u \in N_E$ a unique $\tau_u \in \mathcal{S}(E^\infty)$ satisfying $ufu^* = f \circ \tau_u$ and $u^*fu = f \circ \tau_u^{-1}$ for all $f \in C_0(E^\infty)$. The map $u \mapsto \tau_u$ is a surjective map from N_E to $\mathcal{S}(E^\infty)$, and $\tau_{u_1} = \tau_{u_2}$ iff $u_1u_1^* = u_2u_2^*$, $u_1^*u_1 = u_2^*u_2$, and $u_1u_2^*$ and $u_1^*u_2$ both belong to $\mathcal{D}(E)$. It follows that if there is an isomorphism between $C^*(E)$ and $C^*(F)$ which maps $\mathcal{D}(E)$ onto $\mathcal{D}(F)$, then there is a homeomorphism $h : E^\infty \to F^\infty$ such that $h \circ \mathcal{S}(E^\infty) \circ h^{-1} = \mathcal{S}(F^\infty)$.

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Some remarks about the main theorem

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Some remarks about the main theorem

1. We believe that the assumptions that E and F are row-finite with no sources can be dropped without too much problems.

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Some remarks about the main theorem

1. We believe that the assumptions that E and F are row-finite with no sources can be dropped without too much problems.
2. We also believe that the theorem (and the proof) holds if E and F are replaced by higher rank graphs.

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Examples

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Examples

1. If (E^∞, σ_E) and (F^∞, σ_F) are conjugate, then E^∞ and F^∞ are continuously orbit equivalent.

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Examples

1. If (E^∞, σ_E) and (F^∞, σ_F) are conjugate, then E^∞ and F^∞ are continuously orbit equivalent. It follows that if F is an in-split of E, then there is an isomorphism between $C^*(E)$ and $C^*(F)$ which maps $\mathcal{D}(E)$ onto $\mathcal{D}(F)$ (this is a small improvement of a result by Bates and Pask).

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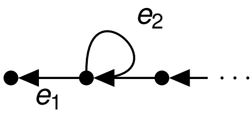
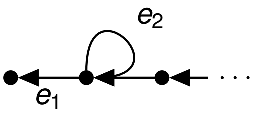
Examples

2. Let E be the graph

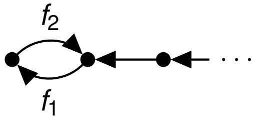


Examples

2. Let E be the graph



and let F be the graph



Examples

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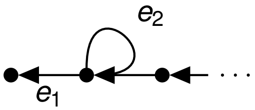
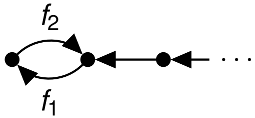


and let F be the graph



Then $e_1 e_2^n x \mapsto f_1(f_2 f_1)^n x$, $e_2^n x \mapsto (f_2 f_1)^n x$, $x \mapsto x$ give raise to a continuously orbit equivalence between E^∞ and F^∞.

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Examples

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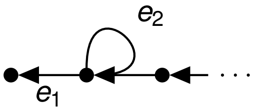
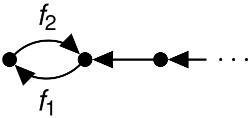


and let F be the graph



Then $e_1 e_2^n x \mapsto f_1(f_2 f_1)^n x$, $e_2^n x \mapsto (f_2 f_1)^n x$, $x \mapsto x$ give raise to a continuously orbit equivalence between E^∞ and F^∞. So there is an isomorphism between $C^*(E)$ and $C^*(F)$ which maps $\mathcal{D}(E)$ onto $\mathcal{D}(F)$.

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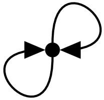
Examples

3. Let E be the graph



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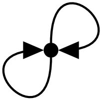
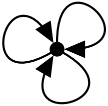
Examples

3. Let E be the graph



and let F be the graph



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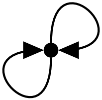
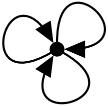
Examples

3. Let E be the graph



and let F be the graph



Then E^∞ and F^∞ are both homeomorphic to the Cantor set,

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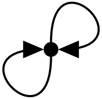
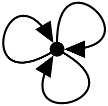
Examples

3. Let E be the graph



and let F be the graph



Then E^∞ and F^∞ are both homeomorphic to the Cantor set, but $C^*(E) \cong \mathcal{O}_2 \not\cong \mathcal{O}_3 \cong C^*(F)$,

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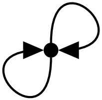
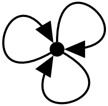
Examples

3. Let E be the graph



and let F be the graph



Then E^∞ and F^∞ are both homeomorphic to the Cantor set, but $C^*(E) \cong \mathcal{O}_2 \ncong \mathcal{O}_3 \cong C^*(F)$, so E^∞ and F^∞ are not continuously orbit equivalent.

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Questions

Questions

1. Can any of you find graphs E and F such that $C^*(E) \cong C^*(F)$, and E^∞ is not homeomorphic to F^∞?

Questions

1. Can any of you find graphs E and F such that $C^*(E) \cong C^*(F)$, and E^∞ is not homeomorphic to F^∞?
2. Can any of you find graphs E and F such that $C^*(E) \cong C^*(F)$, E^∞ is homeomorphic to F^∞, but E^∞ and F^∞ are not continuously orbit equivalent?

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Questions

1. Can any of you find graphs E and F such that $C^*(E) \cong C^*(F)$, and E^∞ is not homeomorphic to F^∞?
2. Can any of you find graphs E and F such that $C^*(E) \cong C^*(F)$, E^∞ is homeomorphic to F^∞, but E^∞ and F^∞ are not continuously orbit equivalent?
3. Can any of you find graphs E and F such that $C^*(E) \cong C^*(F)$, E^∞ is homeomorphic to F^∞, the diagram

$$\begin{array}{ccc} K_0(\mathcal{D}(E)) & \longrightarrow & K_0(C^*(E)) \\ \cong \big\updownarrow & & \big\updownarrow \cong \\ K_0(\mathcal{D}(F)) & \longrightarrow & K_0(C^*(F)) \end{array}$$

commutes, but E^∞ and F^∞ are not continuously orbit equivalent?

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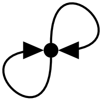
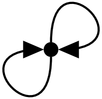
Questions

4. Let E be the graph

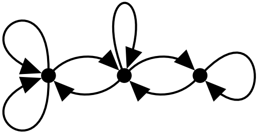


Questions

4. Let E be the graph

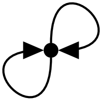


and let F be the graph

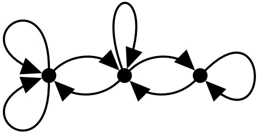


Questions

4. Let E be the graph



and let F be the graph



Are E^∞ and F^∞ continuously orbit equivalent?