Permanence properties for graph algebras

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Graph Algebras: Bridges between graph $C^*$-algebras and Leavitt path algebras
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Definition

A **Cuntz-Krieger algebra** is a graph $C^*$-algebra $C^*(E)$ arising from a finite graph $E$ with no sinks and sources. $C^*(E) = O_A^{\text{top}}$. 
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A $C^*$-algebra $A$ **looks like a Cuntz-Krieger algebra** if
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Definition

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A $C^*$-algebra $A$ **looks like a Cuntz-Krieger algebra** if

- $A$ is unital, purely infinite, nuclear, separable, and of real rank zero;
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A $C^*$-algebra $\mathfrak{A}$ **looks like a Cuntz-Krieger algebra** if

- $\mathfrak{A}$ is unital, purely infinite, nuclear, separable, and of real rank zero;
- $\mathfrak{A}$ has finitely many ideals;
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**Definition**

A $C^*$-algebra $A$ **looks like a Cuntz-Krieger algebra** if

- $A$ is unital, purely infinite, nuclear, separable, and of real rank zero;
- $A$ has finitely many ideals;
- for all $I_1 \subseteq I_2 \subseteq A$, the group $K_0(I_2/I_1)$ is finitely generated and the group $K_1(I_2/I_1)$ is finitely generated and free, and $\text{rank}(K_0(I_2/I_1)) = \text{rank}(K_1(I_2/I_1))$;
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A $C^*$-algebra $\mathcal{A}$ \textit{looks like a Cuntz-Krieger algebra} if

- $\mathcal{A}$ is unital, purely infinite, nuclear, separable, and of real rank zero;
- $\mathcal{A}$ has finitely many ideals;
- for all $\mathcal{I}_1 \subseteq \mathcal{I}_2 \subseteq \mathcal{A}$, the group $K_0(\mathcal{I}_2/\mathcal{I}_1)$ is finitely generated and the group $K_1(\mathcal{I}_2/\mathcal{I}_1)$ is finitely generated and free, and $\text{rank}(K_0(\mathcal{I}_2/\mathcal{I}_1)) = \text{rank}(K_1(\mathcal{I}_2/\mathcal{I}_1))$; and
- the simple sub-quotients of $\mathcal{A}$ are in the bootstrap class.
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- the simple sub-quotients of $A$ are in the bootstrap class.

**Definition**

A $C^*$-algebra $A$ is a *phantom Cuntz-Krieger algebra* if $A$ looks like a Cuntz-Krieger algebra but $A$ is not isomorphic to a Cuntz-Krieger algebra.
Question of George Elliott (2012)

Can a phantom Cuntz-Krieger algebra be SME to a Cuntz-Krieger algebra?

\[ A \otimes K \cong O \text{top} \]

But

\[ A \not\cong O \text{top} \]

A

\[ A \otimes K \cong O \text{top} \iff A \cong p(\text{Otop} \otimes K) \]
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Question

Can a phantom Cuntz-Krieger algebra be SME to a Cuntz-Krieger algebra?

\[ \mathcal{A} \otimes K \cong O^\text{top}_{A} \otimes K \quad \text{but} \quad \mathcal{A} \not\cong O^\text{top}_{A'} \]
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Reformulation

Can a phantom Cuntz-Krieger algebra be isomorphic to a unital full hereditary sub-algebra of a stabilized Cuntz-Krieger algebra?

\[ \mathbb{A} \otimes K \cong \mathcal{O}_{A}^{\text{top}} \otimes K \iff \mathbb{A} \cong p (\mathcal{O}_{A}^{\text{top}} \otimes K) p \]
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Reformulation
Can a phantom Cuntz-Krieger algebra be isomorphic to a unital full hereditary sub-algebra of a stablized Cuntz-Krieger algebra?

\[ \mathcal{A} \otimes \mathbb{K} \cong \mathcal{O}_{\mathcal{A}}^{\text{top}} \otimes \mathbb{K} \iff \mathcal{A} \cong p (\mathcal{O}_{\mathcal{A}}^{\text{top}} \otimes \mathbb{K}) p \]
Bad permanence properties

\[ E : \quad \bullet^{v_1} \quad \xrightarrow{\quad} \quad \bullet^{v_2} \quad \xrightarrow{\quad} \quad \bullet^{v_3} \ldots \]
Bad permanence properties

\[ E : v_1 \xleftrightarrow{} v_2 \xleftrightarrow{} v_3 \ldots \]

\[ p_{v_1} \left( C^* (E) \otimes \mathbb{K} \right) p_{v_1} \sim_{SME} M_{2^\infty} \]
Bad permanence properties

$E: \quad \bullet^v_1 \xleftrightarrow{\cdots} \bullet^v_2 \xleftrightarrow{\cdots} \bullet^v_3 \cdots$

$p_{v_1}(C^*(E) \otimes \mathbb{K})p_{v_1} \sim_{SME} M_{2^\infty}$

$p_{v_1}(C^*(E) \otimes \mathbb{K})p_{v_1}$ is not isomorphic to a graph $C^*$-algebra
Good permanence properties

\[ E : \quad \ldots \bullet v_2 \longrightarrow \bullet v_1 \longrightarrow \bullet v_0 \]

\[ F : \quad \bullet v_1 \longrightarrow \bullet \quad \bullet v_n \]

\[ p \left( C^* (E) \otimes K \right) \sim = M_{n+1} (C(T)) \sim = C^* (F) \]
Good permanence properties

\[ E : \quad \ldots \rightarrow v_2 \rightarrow v_1 \rightarrow v_0 \quad \circlearrowleft \]

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Motivation

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Good permanence properties

\[ E : \quad \ldots \rightarrow v_2 \rightarrow v_1 \rightarrow v_0 \rightarrow \quad \]

\[ p(C^*(E) \otimes \mathbb{K})p \cong M_{n+1}(C(T)) \cong C^*(F) \]

\[ F : \quad \ldots \rightarrow v_n \rightarrow v_1 \rightarrow v_0 \rightarrow \quad \]

\[ F : \quad v_1 \rightarrow v_2 \rightarrow v_3 \rightarrow v_4 \rightarrow v_1 \rightarrow v_n \rightarrow v_1 \]

\[ e_1 \rightarrow e_2 \rightarrow e_3 \rightarrow e_n \rightarrow \]
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**E**: $v_1 \rightarrow v_2 \rightarrow v_3 \ldots$

$C^*(E)$ is not SME to a unital graph $C^*$-algebra

**E**: $\ldots \rightarrow v_2 \rightarrow v_1 \rightarrow v_0$

$C^*(E)$ is SME to a unital graph $C^*$-algebra

Question

Let $E$ be a graph with finitely many vertices.

1. Is every unital hereditary sub-algebra of $C^*(E) \otimes K$ isomorphic to a graph $C^*$-algebra?

2. Is every hereditary sub-algebra of $C^*(E) \otimes K$ with an approximate identity consisting of projections isomorphic to a graph $C^*$-algebra?
Motivation

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\[
E : \bullet_{v_1} \rightarrow \bullet_{v_2} \rightarrow \bullet_{v_3} \rightarrow \cdots
\]

\[
C^*(E) \text{ is not SME to a unital graph } C^*\text{-algebra}
\]

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E : \cdots \bullet_{v_2} \rightarrow \bullet_{v_1} \rightarrow \bullet_{v_0}
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Question

Let \( E \) be a graph with finitely many vertices.

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Unital graph algebras

Approximate identity consisting of projections is necessary

\[ E : \quad \ldots \rightarrow v_2 \rightarrow v_1 \rightarrow v_0 \]
Approximate identity consisting of projections is necessary

Set $\mathcal{A} = \left\{ f \in C(S^1, M_2) : f(1) \in \begin{bmatrix} \mathbb{C} & 0 \\ 0 & 0 \end{bmatrix} \right\} \subseteq C(\mathbb{T}) \otimes \mathbb{K} \cong C^*(E)$. 
Approximate identity consisting of projections is necessary

\[ E : \quad \ldots \xrightarrow{v_2} \xrightarrow{v_1} \xrightarrow{v_0} \]

Set \( \mathcal{A} = \left\{ f \in C(S^1, M_2) : f(1) \in \begin{bmatrix} \mathbb{C} & 0 \\ 0 & 0 \end{bmatrix} \right\} \subseteq C(\mathbb{T}) \otimes \mathbb{K} \cong C^*(E). \)

- \( \mathcal{A} \) is a full hereditary sub-algebra of \( C(\mathbb{T}) \otimes \mathbb{K} \)
Motivation

Cuntz-Krieger algebras

Unital graph algebras

Approximate identity consisting of projections is necessary

\[ E : \quad \cdots \quad \bullet^2 \rightarrow \bullet^1 \rightarrow \bullet^0 \]

Set \( \mathcal{A} = \left\{ f \in C(S^1, M_2) : f(1) \in \begin{bmatrix} \mathbb{C} & 0 \\ 0 & 0 \end{bmatrix} \right\} \subseteq C(T) \otimes \mathbb{K} \cong C^*(E) \).

- \( \mathcal{A} \) is a full hereditary sub-algebra of \( C(T) \otimes \mathbb{K} \)
- every projection in \( \mathcal{A} \) has rank 1
Approximate identity consisting of projections is necessary

Set $\mathcal{A} = \left\{ f \in C(S^1, M_2) : f(1) \in \begin{bmatrix} \mathbb{C} & 0 \\ 0 & 0 \end{bmatrix} \right\} \subseteq C(\mathbb{T}) \otimes K \cong C^*(E)$.

- $\mathcal{A}$ is a full hereditary sub-algebra of $C(\mathbb{T}) \otimes K$
- every projection in $\mathcal{A}$ has rank 1

Therefore, $\mathcal{A}$ is not isomorphic to a graph $C^*$-algebra.
Theorem (Arklint-R)

Let $\mathcal{B}$ be a unital hereditary sub-algebra of $O_{A}^{\text{top}} \otimes K$. Then $\mathcal{B} \cong O_{A'}^{\text{top}}$. 
Theorem (Arkling-R)
Let $\mathcal{B}$ be a unital hereditary sub-algebra of $O_{\mathcal{A}}^{\text{top}} \otimes K$. Then $\mathcal{B} \cong O_{\mathcal{A}'}^{\text{top}}$.

Theorem
Let $\mathcal{B}$ be a hereditary sub-algebra of $O_{\mathcal{A}}^{\text{top}} \otimes K$. Then the following are equivalent.

(a) $\mathcal{B}$ is isomorphic to a graph $C^*$-algebra.
(b) $\mathcal{B}$ has an approximate identity consisting of projections.
Definition

An **algebraic Cuntz-Krieger algebra** is $L_K(E)$ arising from a finite graph $E$ with no sinks and sources. $L_K(E) = \mathcal{O}_A^{\text{alg}}$. 
Definition

An *algebraic Cuntz-Krieger algebra* is $L_K(E)$ arising from a finite graph $E$ with no sinks and sources. $L_K(E) = O^\text{alg}_A$.

Definition

A sub-ring $S$ of $R$ is *hereditary* if $S = pRp$ for some idempotent $p$ in the multiplier ring $\mathcal{M}(R)$.
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An algebraic Cuntz-Krieger algebra is $L_K(E)$ arising from a finite graph $E$ with no sinks and sources. $L_K(E) = \mathcal{O}_A^{\text{alg}}$.

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A sub-ring $S$ of $R$ is hereditary if $S = pRp$ for some idempotent $p$ in the multiplier ring $\mathcal{M}(R)$.

Theorem
Let $S$ be a hereditary sub-ring of $M_\infty(\mathcal{O}_A^{\text{alg}})$. Then the following are equivalent.

1. $S$ has an approximate identity consisting of idempotents.
2. $S \cong L_K(F)$. 

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Definition

A sub-ring $S$ of $R$ is **hereditary** if $S = pRp$ for some idempotent $p$ in the multiplier ring $\mathcal{M}(R)$.

Theorem

Let $S$ be a hereditary sub-ring of $M_\infty(\mathcal{O}_A^\text{alg})$. Then the following are equivalent.

1. $S$ has an approximate identity consisting of idempotents.
2. $S \simeq L_K(F)$.

Moreover, if $S$ is unital, then $S \simeq \mathcal{O}_A^\text{alg}$.
Consequences
Consequences

(1) Every hereditary sub-algebra of $\mathcal{O}_A^{\text{top}}$ with an approximate identity consisting of projections is isomorphic to a graph $C^*$-algebra.

(1) Every hereditary sub-algebra of $\mathcal{O}_A^{\text{alg}}$ with an approximate identity consisting of idempotents is isomorphic to a Leavitt path algebra.
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Consequences

\( C^* \)-algebras

(1) Every hereditary sub-algebra of \( \mathcal{O}_A^{\text{top}} \) with an approximate identity consisting of projections is a isomorphic to a graph \( C^* \)-algebra.

(2) If \( \mathcal{A} \) has an approximate identity consisting of projections and \( \mathcal{A} \sim_{\text{SME}} \mathcal{O}_A^{\text{top}} \), then \( \mathcal{A} \cong C^*(E) \).

Rings

(1) Every hereditary sub-algebra of \( \mathcal{O}_A^{\text{alg}} \) with an approximate identity consisting of idempotents is a isomorphic to a Leavitt path algebra.

(2) If \( \mathcal{R} \) has an approximate identity consisting of idempotents and \( \mathcal{R} \sim_{\text{ME}} \mathcal{O}_A^{\text{alg}} \), then \( \mathcal{R} \cong L_K(E) \).
Consequences

**C*-algebras**

1. Every hereditary sub-algebra of $O^\text{top}_A$ with an approximate identity consisting of projections is isomorphic to a graph $C^*$-algebra.

2. If $\mathcal{A}$ has an approximate identity consisting of projections and $\mathcal{A} \sim_{SME} O^\text{top}_A$, then $\mathcal{A} \cong C^*(E)$.

3. If $\mathcal{A}$ is unital and $\mathcal{A} \sim_{SME} O^\text{top}_A$, then $\mathcal{A} \cong O^\text{top}_{A'}$.

**Rings**

1. Every hereditary sub-algebra of $O^\text{alg}_A$ with an approximate identity consisting of idempotents is isomorphic to a Leavitt path algebra.

2. If $R$ has an approximate identity consisting of idempotents and $R \sim_{ME} O^\text{alg}_A$, then $\mathcal{A} \cong L_K(E)$.

3. If $R$ is unital and $R \sim_{ME} O^\text{alg}_A$, then $R \cong O^\text{alg}_{A'}$. 
Consequences

**C*-algebras**

(4) If $C^*(E)$ is a unital graph $C^*$-algebra, then $C^*(E) \cong O^\text{top}_A$ if and only if

$$\text{rank}(K_0(C^*(E))) = \text{rank}(K_1(C^*(E))).$$

**Rings**

(4)
**Consequences**

**$C^*$-algebras**

(4) If $C^*(E)$ is a unital graph $C^*$-algebra, then $C^*(E) \cong O_A^{\text{top}}$ if and only if

$$\text{rank}(K_0(C^*(E))) = \text{rank}(K_1(C^*(E))).$$

**Rings**

(4)

Example

If

then

$$K_0(L_Q(F)) = \mathbb{Z} \quad \text{and} \quad K_1(L_Q(F)) = \mathbb{Z}_2 \oplus \mathbb{Z} \oplus \mathbb{Z} \oplus \cdots.$$
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Proof

Definition

A graph $E$ with finitely many vertices is in standard form if

(1) every regular vertex of $E$ is a base point of a loop and

(2) for every infinite emitter $v \in E^0$ and $e \in s^{-1}(v)$, we have that $|s^{-1}(v) \cap r^{-1}(r(e))| = \infty$. 
Definition

A graph $E$ with finitely many vertices is in standard form if

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Theorem (Sørensen)

If $E$ is a graph with finitely many vertices, then there exists a graph $F$ in standard form such that $C^*(E) \otimes K \cong C^*(F) \otimes K$. 
Theorem (Ara-Moreno-Pardo)

Let $E$ be a finite graph in standard form such that $E$ has no sinks and sources. If $p$ is a non-zero projection (idempotent) in $C^*\left(\mathbb{K}(L_\mathbb{K}(E))\right)$, then $p \sim \sum_{v \in H} m_v p v$ with $m_v > 0$ where $H$ is the hereditary subset of $E_0$ such that $I_H = \text{Ideal}(p)$.

$p(C^*\left(\mathbb{K}(L_\mathbb{K}(E))\right)p) \sim = C^*\left(F\right)$ where $F$ is the graph obtained from $(H, r - 1(H), r, s)$ by adding a head of length $m_v - 1$ to each vertex $v$ in $H$. 
Unital Case

Theorem (Ara-Moreno-Pardo)

Let $E$ be a finite graph in standard form such that $E$ has no sinks and sources. If $p$ is a non-zero projection (idempotent) in $C^*(E) \otimes \mathbb{K}(\mathcal{M}_\infty(\mathbb{K}(E)))$, then

$$p \sim \sum_{v \in H} m_v p_v$$

with $m_v > 0$ where $H$ is the hereditary subset of $E^0$ such that $I_H = \text{Ideal}(p)$. 
Theorem (Ara-Moreno-Pardo)

Let $E$ be a finite graph in standard form such that $E$ has no sinks and sources. If $p$ is a non-zero projection (idempotent) in $C^*(E) \otimes K (M_\infty(L_K(E)))$, then

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$$p(C^*(E) \otimes K)p \cong C^*(F)$$

where $F$ is the graph obtained from $(H, r^{-1}(H), r, s)$ by adding a head of length $m_v - 1$ to each vertex $v$ in $H$. 
$E : \cdot$

$p(C^*(E) \otimes K)p$
Motivation

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\[ p(C^*(E) \otimes \mathbb{K})p \cong C^*(F_1) \]
\[ p(C^*(E) \otimes \mathbb{K})p \cong C^*(F_1) \cong C^*(F_2) \]
Non-unital case

Take \( \{ p_n \} \) be a sequence of non-zero mutually orthogonal projections such that \( \{ \sum_{k=1}^{n} p_k \} \) is an approximate identity consisting of projections for \( B \sim p(\mathcal{C}^*(E) \otimes \mathcal{K}) \subseteq \mathcal{C}^*(E) \otimes \mathcal{K} \), for some projection \( p \) in the multiplier algebra \( \mathcal{M}(\mathcal{C}^*(E) \otimes \mathcal{K}) \).
Take \( \{p_n\}_{n=1}^{\infty} \) be a sequence of non-zero mutually orthogonal projections such that \( \{\sum_{k=1}^{n} p_k\}_{n=1}^{\infty} \) is an approximate identity consisting of projections for

\[
\mathfrak{B} \cong p(C^*(E) \otimes K)p \subseteq C^*(E) \otimes K,
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for some projection \( p \) in the multiplier algebra \( \mathcal{M}(C^*(E) \otimes K) \).
Non-unital case

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for some projection \( p \) in the multiplier algebra \( \mathcal{M}(C^*(E) \otimes \mathbb{K}) \).

- Let \( H \) be the hereditary subset \( E^0 \) such that
  \[
l_H = \text{Ideal}\{p_n : n \in \mathbb{N}\}
  \]

- \( p_n \sim \sum_{v \in H} m(v, n)p_v \) where \( m(v, n) \geq 0 \) and

\[
\bigcup_{n=1}^{\infty} \{v \in H : m(v, n) > 0\} = H.
\]
Non-unital case

Take \( \{p_n\}_{n=1}^\infty \) be a sequence of non-zero mutually orthogonal projections such that \( \{\sum_{k=1}^{n} p_k\}_{n=1}^\infty \) is an approximate identity consisting of projections for

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  \]

Then \( \mathfrak{B} \cong C^*(F) \) where \( F \) is obtained from \( (H, r^{-1}(H), r, s) \) by adding a head of length \(-1 + \sum_{n=1}^{\infty} m(v, n)\) to each vertex \( v \in H \).
Motivation

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\[ p_1 \sim n_1 p_v + m_1 p_w, \quad p_2 \sim n_2 p_v + m_2 p_w, \quad p_3 \sim m_3 p_w, \quad p_4 \sim m_4 p_w, \quad \ldots \]
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Cuntz-Krieger algebras

Unital graph algebras

\[ p_1 \sim n_1 p_v + m_1 p_w, \quad p_2 \sim n_2 p_v + m_2 p_w, \quad p_3 \sim m_3 p_w, \quad p_4 \sim m_4 p_w, \]
\[ \ldots \]

\[ \mathcal{B} \cong C^*(F) \]
The unitization of a graph algebra
The unitization of a graph algebra

**Theorem**

If $C^*(E)$ is a non-unital $C^*$-algebra and $C^*(E) \sim_{SME} \mathcal{O}_A^{top}$, then $C^*(E)^\dagger \simeq C^*(F)$. 

The diagram illustrates the relationship between $C^*(E)$ and $C^*(F)$ after unitization.
The unitization of a graph algebra

**Theorem**

If $C^*(E)$ is a non-unital $C^*$-algebra and $C^*(E) \sim_{SME} \mathcal{O}_A^{\text{top}}$, then $C^*(E)^\dagger \cong C^*(F)$. 

\[ \mathcal{B} \cong C^*(F) \quad \text{and} \quad \mathcal{B}^\dagger \cong C^*(F)^\dagger \]
Unital graph algebras

Theorem
Let $E$ be a graph with finitely many vertices.

(1) Every unital hereditary sub-algebra of $C^*(E) \otimes K$ is isomorphic to a graph $C^*$-algebra.

(2) Every unital hereditary sub-algebra of $M_\infty(L_K(E))$ is isomorphic to a Leavitt path algebra.
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Theorem

Let $E$ be a graph and $p$ be a projection (idempotent) in $C^*(E) \otimes K (M_\infty(L_K(E)))$. Then

$$p \sim \sum_{v \in S} m_v \left( p_v - \sum_{e \in T_v} s_e s_e^* \right)$$

$T_v \subseteq_{\text{fin}} s^{-1}(v)$ and $T_v = \emptyset$ when $|s^{-1}_v(v)| < \infty$. 
Unital graph algebras

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$T_v \subseteq \text{fin} \ s^{-1}(v)$ and $T_v = \emptyset$ when $|s_E^{-1}(v)| < \infty$.

Change graph

$$C^*(E) \otimes K \cong C^*(F) \otimes K$$
$$L_K(E) \otimes K \cong L_K(F) \otimes K$$

$$p \mapsto q \sim \sum_{v \in S} m_v q_v$$
Theorem (work in progress)

Let $E$ be a graph with finitely many vertices, $\mathcal{A} \subseteq_{\text{her}} C^*(E) \otimes \mathbb{K}$, and $A \subseteq_{\text{her}} M_\infty(L_K(E))$.

1. $\mathcal{A}$ has an approximate identity consisting of projections if and only if $\mathcal{A}$ is isomorphic to a graph $C^*$-algebra.

2. $A$ has an approximate identity consisting of idempotents if and only if $A$ is isomorphic to a Leavitt path algebras.
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Theorem (work in progress)

Let $E$ be an infinite graph.

1. $C^*(E)^\dagger$ is isomorphic to a graph $C^*$-algebra if and only if $C^*(E)$ is SME to a unital graph $C^*$-algebra.
2. $L_K(E)^\dagger$ is isomorphic to a Leavitt path algebra if and only if $L_K(E)$ is ME to a unital Leavitt path algebra.
Questions

(1) Can we determine exactly when \( C^*(E) \) is SME to a Cuntz-Krieger algebra?
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Necessary conditions ($K$-theory of gauge invariant quotients)

(1) $K^*_p(I_2/I_1)$ is finitely generated

(2) $\text{rank}(K_1(I_2/I_1)) \leq \text{rank}(K_0(I_2/I_1))$

(3) If $I_2/I_1$ is "gauge simple" and $K_0(I_2/I_1) + \neq K_0(I_2/I_1)$, then $K_0(I_2/I_1) \cong \mathbb{Z}$ and $K_0(I_2/I_1) + \geq \mathbb{Z}$. 
Questions

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Necessary conditions (K-theory of gauge invariant quotients)

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Questions

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Necessary conditions ($K$-theory of gauge invariant quotients)

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(2) $\text{rank}(K_1(\mathcal{I}_2/\mathcal{I}_1)) \leq \text{rank}(K_0(\mathcal{I}_2/\mathcal{I}_1))$ (equality for Cuntz-Krieger algebras)
Questions

(1) Can we determine exactly when $\mathcal{C}^*(E)$ is SME to a Cuntz-Krieger algebra?

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Theorem (Arklint-Bentmann-Katsura)

Let $C^*(E)$ purely infinite graph $C^*$-algebra with finitely many ideals. If

1. $K_*(I_2/I_1)$ is finite generated and
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then there exists a unital graph $C^*$-algebra $C^*(F)$ such that

$$\text{FK}_R(C^*(E)) \cong \text{FK}_R(C^*(E)).$$
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Moreover, if $\text{rank}(K_1(I_2/I_1)) = \text{rank}(K_0(I_2/I_1))$, then $C^*(F)$ can be chosen to be a Cuntz-Krieger algebra.
Theorem (Arklint-Bentmann-Katsura)

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$$FK_{\mathcal{R}}(C^*(E)) \cong FK_{\mathcal{R}}(C^*(E)).$$

Moreover, if $\text{rank}(K_1(\mathcal{I}_2/\mathcal{I}_1)) = \text{rank}(K_0(\mathcal{I}_2/\mathcal{I}_1))$, then $C^*(F)$ can be chosen to be a Cuntz-Krieger algebra.

If $X$ is an accordion space, then

$$C^*(E) \otimes K \cong C^*(F) \otimes K.$$
If $C^*(E) \sim_{SME} O_A^{\text{top}}$, then every unital hereditary sub-algebra of $C^*(E)$ is a Cuntz-Krieger algebra.
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If $C^*(E) \sim_{SME} C^*(F)$ with $|F^0| < \infty$, then every unital hereditary sub-algebra of $C^*(E)$ is a graph $C^*$-algebra.
If \( C^*(E) \sim_{SME} O^\text{top}_A \), then every unital hereditary sub-algebra of \( C^*(E) \) is a Cuntz-Krieger algebra.

If \( C^*(E) \sim_{SME} C^*(F) \) with \( |F^0| < \infty \), then every unital hereditary sub-algebra of \( C^*(E) \) is a graph \( C^* \)-algebra.

**Reformulation**

Suppose \( C^*(E) \) is a non-unital graph \( C^* \)-algebra with finitely many ideals and “\( K \)-theory” as a Cuntz-Krieger algebra (unital graph \( C^* \)-algebra). Is every unital hereditary sub-algebra of \( C^*(E) \) isomorphic to a Cuntz-Krieger algebra (unital graph \( C^* \)-algebra).
Motivation

Cuntz-Krieger algebras

Unital graph algebras

Unital graph algebras

\[ K_0(C^*(E)) = \mathbb{Z} \quad K_1(C^*(E)) = 0 \]
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