Classification of Leavitt path algebras:
How to use tools from the classification of C*-algebras in the Algebra setting

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April 22, 2013
Hey, have you read my new paper on the orbifold quantum cohomology of...

The Landscape of Modern Mathematics
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Today, I’m going to talk about classification up to Morita equivalence.

Morita equivalence in the category of $C^*$-algebras is often called “strong Morita equivalence” to distinguish it from Morita equivalence of rings. Also, strong Morita equivalence for $C^*$-algebras is the same as being stably isomorphic.

\[ A \sim_{SME} B \iff A \otimes K \cong B \otimes K \]
There are many important results from the classification program, but two major accomplishments are:

**Theorem (Elliott’s Theorem)**

If $A$ and $B$ are $C^*$-algebras that are AF (i.e., direct limits of finite-dimensional algebras), then $A \sim_{SME} B$ if and only if

$$(K_0^{top}(A), K_0^{top, +}(A)) \cong (K_0^{top}(B), K_0^{top, +}(B)).$$
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**Theorem (Kirchberg-Phillips Classification Theorem)**

If $A$ and $B$ are purely infinite, simple, separable, nuclear $C^*$-algebras that are in the bootstrap class to which the UCT applies, then $A \sim_{SME} B$ if and only if

$$K_{0}^{\text{top}}(A) \cong K_{0}^{\text{top}}(B) \text{ and } K_{1}^{\text{top}}(A) \cong K_{1}^{\text{top}}(B).$$

Note: Many purely infinite, simple $C^*$-algebras fall into this class.
Can a similar classification be done for algebras?

The proof of Elliott’s Theorem works for ultramatricial algebras over a field $K$ (i.e., algebraic direct limits of finite-dimensional $K$-algebras), and can be used to show the ordered $K_0$-group is a complete Morita equivalence invariant for ultramatricial algebras. (Indeed, Elliott showed this in his original paper.)
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What about the Kirchberg-Phillips Classification Theorem? Can a similar result be obtained for purely infinite algebras? Can we use algebraic $K$-theory in place of topological $K$-theory? (We may need the higher algebraic $K$-groups . . . these may be harder to compute . . .)

Definition: A ring $R$ is *purely infinite* if every left ideal of $R$ contains an infinite idempotent.
ALL THE BEST WORK HAS BEEN DONE OVER HERE!
We cannot hope to mimic or adapt the proof of the Kirchberg-Phillips theorem. We will need to use different techniques. What characteristics should we look for in our class of algebras?

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- A “nice” class of algebras that are similar to $C^*$-algebras. Ideally, we would like this class to be similar to a subclass of purely infinite $C^*$-algebras that can be classified by other, more direct, methods.

Unital. (At least as a starting point.) The $K$-groups should be computable.

Answer: Cuntz-Krieger $C^*$-algebras are simple, purely infinite, and were originally classified by techniques different from those used in the Kirchberg-Phillips theorem. Also, the $K$-groups are easy to compute.

We will consider algebras that are analogues of the Cuntz-Krieger algebras. Cuntz-Krieger algebras were originally associated to finite square matrices, but the modern approach is to formulate them in terms of graphs.
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Cuntz-Krieger algebras were originally associated to finite square matrices, but the modern approach is to formulate them in terms of graphs.
A (directed) graph $E = (E^0, E^1, r, s)$ consists of a set of vertices $E^0$, a set of edges $E^1$, and maps $r : E^1 \to E^0$ and $s : E^1 \to E^0$ identifying the range and source of each edge. (We’ll allow infinite graphs, but assume the vertex set and edge set are countable.)
Definition (Graph $C^*$-algebras)

If $E$ is a graph, the graph $C^*$-algebra $C^* (E)$ is the universal $C^*$-algebra generated by a Cuntz-Krieger $E$-family, which consists of mutually orthogonal projections $\{p_v : v \in E^0\}$ and partial isometries with mutually orthogonal ranges $\{s_e : e \in E^1\}$ satisfying:

1. $s^*_e s_e = p_{r(e)}$ for all $e \in E^1$
2. $p_v = \sum_{e \in E^1 : s(e) = v} s_e s^*_e$ for all $v \in E^0$ with $0 < |s^{-1}(v)| < \infty$
3. $s_e s^*_e \leq p_{s(e)}$ for all $e \in E^1$.

Fact: $C^* (E)$ is unital if and only if $E$ has a finite number of vertices.

Fact: If $E$ is finite with no sinks or sources, then $C^* (E)$ is simple if and only if $E$ is strongly connected and not a single cycle. (In this case $C^* (E)$ is purely infinite and unital.)
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Gene Abrams and Gonzalo Aranda-Pino introduced Leavitt path algebras.

**Definition (Leavitt path algebras)**

Given a graph $E = (E^0, E^1, r, s)$ and a field $K$, the Leavitt path algebra $L_K(E)$ is the universal $K$-algebra generated by a set $\{v : v \in E^0\}$ of pairwise orthogonal idempotents, together with a set $\{e, e^* : e \in E^1\}$ of elements such that the $e$'s and $e^*$'s satisfy the relations of partial isometries with mutually orthogonal ranges, and

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Let's start by considering the invariant. How can we calculate the topological K-groups?

Let $A \in \mathbb{E}$ be the square matrix indexed by the vertices, and with $A_{v,w} = \text{number of edges from } v \text{ to } w$.

If $\mathbb{E}$ has no sinks and no infinite emitters, then $K_{\text{top}}^0(\mathbb{C}^*(\mathbb{E})) \cong \text{coker}(I - A^t: \bigoplus_{v \in \mathbb{E}} \mathbb{Z} \rightarrow \bigoplus_{v \in \mathbb{E}} \mathbb{Z})$ and $K_{\text{top}}^1(\mathbb{C}^*(\mathbb{E})) \cong \ker(I - A^t: \bigoplus_{v \in \mathbb{E}} \mathbb{Z} \rightarrow \bigoplus_{v \in \mathbb{E}} \mathbb{Z})$.
The classification of simple Cuntz-Krieger algebras

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$$K_{1}^{\text{top}}(C^*(E)) \cong \text{ker} \left( I - A^t_E : \bigoplus_{E^0} \mathbb{Z} \to \bigoplus_{E^0} \mathbb{Z} \right)$$
If $E$ is a finite graph, put $I - A^t_E$ into Smith Normal Form.

$$I - A^t_E \leftrightarrow \begin{pmatrix} d_1 & \cdots & 0 \\ \vdots & d_k & \cdots \\ 0 & \cdots & 0 \end{pmatrix}$$

Then $K_{\text{top}}^0(\mathbb{C}^*(E)) \cong \text{coker}(I - A^t_E) \cong \mathbb{Z}^{d_1} \oplus \cdots \oplus \mathbb{Z}^{d_k} \oplus \mathbb{Z}^{m}$

$K_{\text{top}}^1(\mathbb{C}^*(E)) \cong \ker(I - A^t_E) \cong \mathbb{Z}^{m}$.

Note: $K_{\text{top}}^1(\mathbb{C}^*(E))$ is the free part of $K_{\text{top}}^0(\mathbb{C}^*(E))$. 

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If $E$ is a finite graph, put $I - A_E^t$ into Smith Normal Form.

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Note: $K_1^{\text{top}}(C^*(E))$ is the free part of $K_0^{\text{top}}(C^*(E))$. 
We expect that if \( E \) is a finite graph with no sinks or sources (and not a single cycle), then \( C^*(E) \) is determined up to strong Morita equivalence by
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K_{0}^{\text{top}}(C^*(E)) \cong \text{coker}(I - A_E^t).
\]
This was proved by Cuntz and Krieger (and also relied on some work of Eilliott and of Rørdam) almost two decades before the Kirchberg-Phillips classification theorem. How was this accomplished?

A large component of the proof used Symbolic Dynamics.
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If $E$ is a finite graph, the (two-sided) shift space $X_E$ is the set

$$ X_E := \{ \ldots e_{-2} e_{-1} e_0 e_1 e_2 \ldots \mid e_i \in E^1 \text{ and } r(e_i) = s(e_{i+1}) \text{ for all } i \in \mathbb{Z} \} $$

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We give the finite set of edges $E^1$ the discrete topology, so the infinite product

$$\prod_{\mathbb{Z}} E^1 = \ldots E^1 \times E^1 \times E^1 \times \ldots$$

is compact by Tychonoff’s theorem. We then give $X_E \subseteq \prod_{\mathbb{Z}} E^1$ the subspace topology. The space $X_E$ is closed (and hence compact).

The pair $(X_E, \sigma_E)$ is a dynamical system.
Definition

The shift spaces \((X_E, \sigma_E)\) and \((X_F, \sigma_F)\) are **conjugate** if there exists a homeomorphism \(\phi : X_E \to X_F\) with

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**Definition**

If \(X_E\) is a shift space, the **suspension flow** is the quotient space

\[
SX_E := (X_E \times \mathbb{R})/\{(x, t) \sim (\sigma_E(x), t - 1)\}.
\]

There is a flow on \(SX_E\) induced by the flow \(\phi_t\) on \(X_E \times \mathbb{R}\) given by

\[
\phi_t(x, s) = (x, s + t).
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The shift spaces \((X_E, \sigma_E)\) and \((X_F, \sigma_F)\) are said to be **flow equivalent** if there is a homeomorphism \(h : SX_E \rightarrow SX_F\) carrying orbits of the flow on \(SX_E\) to orbits of the flow on \(SX_F\) and preserving the orientation.
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Conjugacy and Flow Equivalence are related to moves on the graphs.
**Move (O): Outsplitting**

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s^{-1}(v) = \{e, f\} \cup \{g\} \cup \{h\}
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\[ s^{-1}(v) = \{e, f\} \cup \{g\} \cup \{h\} \]

Move (I): Insplitting

\[ r^{-1}(v) = \{a\} \cup \{b\} \]
Move (R): Reduction

\[ s^{-1}(w) \text{ is a single edge } f \]
\[ s(r^{-1}(w)) \text{ is a single vertex } v \]
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Move (R): Reduction

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Move (R) is also sometimes called the “Parry-Sullivan Move”.

For each move there is also an inverse move.

Inverse of Outsplitting is called Outamalgamation.
Inverse of Insplitsitng is called Inamalgamation.
Inverse of Reduction is called Delay.
Suppose $E$ and $F$ are finite, strongly connected graphs and neither is a single cycle.

Williams proved:

$X_E$ is conjugate to $X_F \iff E$ can be transformed into $F$ via

Moves (O), (I), and their inverses

Work of Parry and Sullivan together with work of Franks shows $X_E$ is flow equivalent to $X_F$:

Parry-Sullivan\iff $E$ can be transformed into $F$ via

Moves (O), (I), (R), and their inverses\iff $E$ can be transformed into $F$ via

Franks\iff $\text{coker}(I - A_E) \sim = \text{coker}(I - A_F)$ and $\text{sgn}(\det(I - A_E)) = \text{sgn}(\det(I - A_F))$\iff $E$ can be transformed into $F$
Suppose $E$ and $F$ are finite, strongly connected graphs and neither is a single cycle.

Williams proved:

$$X_E \text{ is conjugate to } X_F \iff E \text{ can be transformed into } F \text{ via Moves (O), (I), and their inverses}$$

Work of Parry and Sullivan together with work of Franks shows

$$X_E \text{ is flow equivalent to } X_F \iff E \text{ can be transformed into } F \text{ via Moves (O), (I), (R), and their inverses}$$

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$$\iff \text{coker}(I - A_E) \cong \text{coker}(I - A_F) \text{ and } \text{sgn}(\det(I - A_E)) = \text{sgn}(\det(I - A_F))$$
Since $I - A_E$ and $I - A_F$ are finite matrices, their transposes have the same cokernels and determinants. Thus Franks’ result can be restated as

\[ E \text{ can be transformed into } F \text{ via Moves (O), (I), (R), and their inverses} \]

\[ \iff \quad \text{coker}(I - A_{E}^t) \cong \text{coker}(I - A_{F}^t) \text{ and } \text{sgn}(|\det(I - A_{E}^t)|) = \text{sgn}(|\det(I - A_{F}^t)|) \]

Cuntz and Krieger showed the moves preserve strong Morita equivalence of the associated $C^*$-algebra. (This can be seen easily from the graphs, using the universal property.)
Since $I - A_E$ and $I - A_F$ are finite matrices, their transposes have the same cokernels and determinants. Thus Franks’ result can be restated as

\[\begin{align*}
E \text{ can be transformed into } F \text{ via Moves (O), (I), (R), and their inverses } & \iff \text{coker}(I - A_E^t) \cong \text{coker}(I - A_F^t) \text{ and } \text{sgn}(\det(I - A_E^t)) = \text{sgn}(\det(I - A_F^t)) \\
& \iff K^\text{top}_0(C^* (E)) \cong K^\text{top}_0(C^* (F)) \text{ and } \text{sgn}(\det(I - A_E^t)) = \text{sgn}(\det(I - A_F^t))
\end{align*}\]
Since $I - A_E$ and $I - A_F$ are finite matrices, their transposes have the same cokernels and determinants. Thus Franks’ result can be restated as

\[ E \text{ can be transformed into } F \text{ via Moves (O), (I), (R), and their inverses } \iff \text{coker}(I - A^t_E) \cong \text{coker}(I - A^t_F) \text{ and } \text{sgn}(\det(I - A^t_E)) = \text{sgn}(\det(I - A^t_F)) \iff K_{\text{top}}^0(C^*(E)) \cong K_{\text{top}}^0(C^*(F)) \text{ and } \text{sgn}(\det(I - A^t_E)) = \text{sgn}(\det(I - A^t_F)) \]

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Since $I - A_E$ and $I - A_F$ are finite matrices, their transposes have the same cokernels and determinants. Thus Franks’ result can be restated as

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$$\iff K_{0}^{\text{top}}(C^*(E)) \cong K_{0}^{\text{top}}(C^*(F)) \text{ and } \text{sgn}(\text{det}(I - A^t_E)) = \text{sgn}(\text{det}(I - A^t_F))$$

- Cuntz and Krieger showed the moves preserve strong Morita equivalence of the associated $C^*$-algebra. (This can be seen easily from the graphs, using the universal property.)
- The sign of the determinant condition requires another move!
The Cuntz Splice

Move (CS): Cuntz Splice

$v$ is the base of two cycles
The Cuntz Splice

Move (CS): Cuntz Splice

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Let $E$ be a graph, and perform the Cuntz splice to obtain $F$.

\[
A_F = \begin{pmatrix}
1 & 1 & 0 & 0 & \cdots \\
1 & 1 & 1 & 0 & \cdots \\
0 & 1 & A_E & \\
0 & 0 & \vdots & \\
\vdots & \vdots & & \\
\end{pmatrix}
\]

Then $K_{0}^{\text{top}}(C^\ast(E)) \cong K_{0}^{\text{top}}(C^\ast(F))$, but $\det(I - A_F^t) = - \det(I - A_E^t)$. 
Work of Elliott together with work of Rørdam shows that the Cuntz splice preserves Morita equivalence of the associated $C^*$-algebra. However, unlike the other moves this cannot be shown explicitly, and relies on some "$C^*$-algebra magic".
Theorem (Cuntz and Krieger)

Suppose $E$ and $F$ are finite, strongly connected graphs and neither is a single cycle. Then $C^*(E)$ is strongly Morita equivalent to $C^*(F)$ if and only if $K_{0}^{\text{top}}(C^*(E)) \cong K_{0}^{\text{top}}(C^*(F))$.

Moreover, in this case one can transform $E$ into $F$ using moves (O), (I), (R), (CS), and their inverses.
Theorem (Cuntz and Krieger)

Suppose $E$ and $F$ are finite, strongly connected graphs and neither is a single cycle. Then $C^*(E)$ is strongly Morita equivalent to $C^*(F)$ if and only if $K_0^{\text{top}}(C^*(E)) \cong K_0^{\text{top}}(C^*(F))$.

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Proof:
$$K_0^{\text{top}}(C^*(E)) \cong K_0^{\text{top}}(C^*(F)) \implies \text{coker}(I - A^t_E) \cong \text{coker}(I - A^t_F)$$
Theorem (Cuntz and Krieger)

Suppose $E$ and $F$ are finite, strongly connected graphs and neither is a single cycle. Then $C^*(E)$ is strongly Morita equivalent to $C^*(F)$ if and only if $K^\text{top}_0(C^*(E)) \cong K^\text{top}_0(C^*(F))$.

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Proof:

$K^\text{top}_0(C^*(E)) \cong K^\text{top}_0(C^*(F)) \implies \text{coker}(I - A^t_E) \cong \text{coker}(I - A^t_F)$

(If $\text{sgn}\det(I - A^t_E) = \text{sgn}(\det(I - A^t_F))$, great.
If not, apply Cuntz splice.)
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(If $\text{sgn} \det(I - A^t_E) = \text{sgn} \det(I - A^t_F)$, great. If not, apply Cuntz splice.)

$\implies \text{coker}(I - A^t_E) \cong \text{coker}(I - A^t_F)$ and $\text{sgn} \det(I - A^t_E) = \text{sgn} \det(I - A^t_F)$
**Theorem (Cuntz and Krieger)**

Suppose $E$ and $F$ are finite, strongly connected graphs and neither is a single cycle. Then $C^* (E)$ is strongly Morita equivalent to $C^* (F)$ if and only if $K_{0}^{\text{top}}(C^* (E)) \cong K_{0}^{\text{top}}(C^* (F))$.

Moreover, in this case one can transform $E$ into $F$ using moves $(O), (I), (R), (CS)$, and their inverses.

**Proof:**

\[
K_{0}^{\text{top}}(C^* (E)) \cong K_{0}^{\text{top}}(C^* (F)) \implies \text{coker}(I - A_{E}^{t}) \cong \text{coker}(I - A_{F}^{t})
\]

(If $\text{sgn} \det(I - A_{E}^{t}) = \text{sgn} \det(I - A_{F}^{t})$, great. If not, apply Cuntz splice.)

\[
\implies \text{coker}(I - A_{E}^{t}) \cong \text{coker}(I - A_{F}^{t}) \text{ and } \text{sgn} \det(I - A_{E}^{t}) = \text{sgn} \det(I - A_{F}^{t})
\]

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Theorem (Cuntz and Krieger)

Suppose \( E \) and \( F \) are finite, strongly connected graphs and neither is a single cycle. Then \( C^*(E) \) is strongly Morita equivalent to \( C^*(F) \) if and only if \( K^\text{top}_0(C^*(E)) \cong K^\text{top}_0(C^*(F)) \).

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Proof:

\[
K^\text{top}_0(C^*(E)) \cong K^\text{top}_0(C^*(F)) \implies \text{coker}(I - A^t_E) \cong \text{coker}(I - A^t_F)
\]

(If \( \text{sgn det}(I - A^t_E) = \text{sgn}(\text{det}(I - A^t_F)) \)), great.

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\[
\implies C^*(E) \text{ strongly Morita equiv. to } C^*(F)
\]
Abrams, Louly, Pardo, and Smith considered an analogous classification for Leavitt path algebras.
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- The $K$-theory of $L_K(E)$ can be computed. In fact, if $E$ is a finite graph with no sinks, then for any field $K$ we have

$$K^\text{alg}_0(L_K(E)) \cong \text{coker} \left( I - A_E^t : \bigoplus_{E^0} \mathbb{Z} \to \bigoplus_{E^0} \mathbb{Z} \right)$$

and $K^\text{top}_0(C^*(E)) \cong K^\text{alg}_0(L_K(E))$. 

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- One can show the moves (O), (I), and (R) preserve Morita equivalence of the associated Leavitt path algebra.
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- However, no one has been able to determine whether the Cuntz splice preserves Morita equivalence of the Leavitt path algebra or not!

Thus the best that can be accomplished is the following . . .
Theorem (Abrams, Louly, Pardo, and Smith)

Suppose $E$ and $F$ are finite, strongly connected graphs and neither is a single cycle. Also let $K$ be any field. If

$$K_0^\text{alg}(L_K(E)) \cong K_0^\text{alg}(L_K(F)) \quad \text{and} \quad \text{sgn}(\det(I - A_E^t)) = \text{sgn}(\det(I - A_F^t)),$$

then $L_K(E)$ is Morita equivalent to $L_K(F)$.

Moreover, in this case one can transform $E$ into $F$ using moves (O), (I), (R), and their inverses.
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Proof:

The hypotheses show

$$\text{coker}(I - A^t_E) \cong \text{coker}(I - A^t_F) \quad \text{and} \quad \text{sgn \det}(I - A^t_E) = \text{sgn}(\text{det}(I - A^t_F)).$$
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Franks’ theorem implies $E$ can be turned into $F$ via moves (O), (I), (R), and their inverses.
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Suppose $E$ and $F$ are finite, strongly connected graphs and neither is a single cycle. Also let $K$ be any field. If

$$K_0^{\text{alg}}(L_K(E)) \cong K_0^{\text{alg}}(L_K(F)) \quad \text{and} \quad \text{sgn}(\det(I - A^t_E)) = \text{sgn}(\det(I - A^t_F)),$$

then $L_K(E)$ is Morita equivalent to $L_K(F)$.

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Franks’ theorem implies $E$ can be turned into $F$ via moves $(O)$, $(I)$, $(R)$, and their inverses. These moves preserve Morita equivalence, so $L_K(E)$ is Morita equivalent to $L_K(F)$.
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- $K_0\text{alg}(L_K(E))$ and $\text{sgn}(\det(I - A_E^t))$ completely determine the Morita equivalence class of $L_K(E)$, and hence determine $K_n\text{alg}(L_K(E))$ for $n \in \mathbb{Z}$.
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- No one knows whether the “sign of the determinant condition” is necessary, or whether it can be removed from the theorem!
- No one knows if the Cuntz splice preserves Morita equivalence of the Leavitt path algebra.

We cannot even answer this in the simplest case:

Is $L_K(E_2)$ Morita equivalent to $L_K(E^-_2)$? No one knows.
What about other unital, purely infinite graph $C^*$-algebras?

Remember $C^*(E)$ is unital if and only if $E$ has a finite number of vertices. We could consider infinite graphs with a finite number of vertices that are strongly connected. Their $C^*$-algebras are purely infinite and simple. This was recently done, with great success, by Adam Sørensen.

We can't consider shift spaces here (the number of edges is infinite), but that's okay. Franks' result for finite, strongly connected graphs: $E$ can be transformed into $F$ via Moves (O), (I), (R) and their inverses $\iff \text{coker}(I - A^t_E) \cong \text{coker}(I - A^t_F)$ and $\text{sgn}(\det(I - A^t_E)) = \text{sgn}(\det(I - A^t_F))$ is a purely algebraic statement that does not rely on the notion of flow equivalence to state or prove.
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We can’t consider shift spaces here (the number of edges is infinite), but that’s okay. Franks’ result for finite, strongly connected graphs:

$$E \text{ can be transformed into } F \text{ via Moves (O), (I), (R) and their inverses } \iff \text{coker}(I - A^t_E) \cong \text{coker}(I - A^t_F) \text{ and } \text{sgn}(\det(I - A^t_E)) = \text{sgn}(\det(I - A^t_F))$$

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If $E$ has a finite number of vertices, but an infinite number of edges, the computation of the $K$-theory is a bit different:
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$E^0_{\text{reg}} = \text{vertices of } E \text{ that emit a finite and nonzero number of edges}$

$E^0_{\text{sing}} = \text{vertices that emit infinitely many edges or no edges}$

With respect to $E^0 = E^0_{\text{reg}} \cup E^0_{\text{sing}}$ we have

$$A_E = \begin{pmatrix} B_E & C_E \\ * & * \end{pmatrix}$$

where $B_E$ and $C_E$ have finite entries.
If $E$ has a finite number of vertices, but an infinite number of edges, the computation of the $K$-theory is a bit different:

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With respect to $E^0 = E_0^{\text{reg}} \cup E_0^{\text{sing}}$, we have

$$A_E = \begin{pmatrix} B_E & C_E \\ * & * \end{pmatrix}$$

where $B_E$ and $C_E$ have finite entries. Then

$$K_0^{\text{top}}(C^*(E)) \cong \text{coker } \left( \begin{pmatrix} I - B_E^t \\ - C_E^t \end{pmatrix} : \bigoplus_{E_0^{\text{reg}}} \mathbb{Z} \to \bigoplus_{E^0} \mathbb{Z} \right)$$

and

$$K_1^{\text{top}}(C^*(E)) \cong \text{ker } \left( \begin{pmatrix} I - B_E^t \\ - C_E^t \end{pmatrix} : \bigoplus_{E_0^{\text{reg}}} \mathbb{Z} \to \bigoplus_{E^0} \mathbb{Z} \right)$$
If $E$ is a finite number of vertices, we can still put the matrix in Smith Normal Form.

\[
\begin{pmatrix}
I - B^t_E \\
-C^t_E
\end{pmatrix}
\begin{array}{c}
\rightarrow \\
\end{array}
\begin{pmatrix}
d_1 \\
\vdots \\
\vdots \\
\vdots \\
0 \\
\end{pmatrix}
\begin{pmatrix}
d_1 \\
\vdots \\
\vdots \\
\vdots \\
0 \\
\end{pmatrix}
\begin{pmatrix}
\vdots \\
\vdots \\
\vdots \\
\vdots \\
\vdots \\
\end{pmatrix}
\begin{pmatrix}
0 \\
\ldots \\
0 \\
\ldots \\
0 \\
\end{pmatrix}
\end{pmatrix}
\]

We see that $K_{\text{top}}^0(C^*(E))$ no longer determines $K_{\text{top}}^1(C^*(E))$, so we will need the $K_{\text{top}}^1$-group in our invariant. Also, the number of singular vertices is a Morita equivalence invariant:

$|E_{\text{sing}}| = \text{rank } K_{\text{top}}^0(C^*(E)) - \text{rank } K_{\text{top}}^1(C^*(E)).$
If $E$ is has a finite number of vertices, we can still put the matrix in Smith Normal Form.

\[
\begin{pmatrix}
I - B_E^t \\
-C_E^t
\end{pmatrix}
\begin{pmatrix}
\vdots \\
\vdots \\
\vdots \\
\vdots \\
0 \\
0 \\
0 \\
0
\end{pmatrix}
= \begin{pmatrix}
0 \\
0 \\
0 \\
0 \\
0 \\
0 \\
0 \\
0
\end{pmatrix}
\]

Then

\[
K_0^{\text{top}}(C^*(E)) \cong \mathbb{Z}_{d_1} \oplus \ldots \oplus \mathbb{Z}_{d_k} \oplus \mathbb{Z}^m \quad \text{and} \quad K_1^{\text{top}}(C^*(E)) \cong \mathbb{Z}^n.
\]
If \( E \) is has a finite number of vertices, we can still put the matrix in Smith Normal Form.

\[
\begin{pmatrix}
I - B^t_E \\
-C^t_E
\end{pmatrix}
\leftrightarrow
\begin{pmatrix}
d_1 \\
\vdots \\
d_k \\
0 \\
\vdots \\
0 \\
0 \\
\vdots \\
0
\end{pmatrix}
\]

Then

\[
K_{0}^{\text{top}}(C^*(E)) \cong \mathbb{Z}_{d_1} \oplus \cdots \oplus \mathbb{Z}_{d_k} \oplus \mathbb{Z}^m \quad \text{and} \quad K_{1}^{\text{top}}(C^*(E)) \cong \mathbb{Z}^n.
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\[
\begin{pmatrix}
I - B^t_E \\
-C^t_E
\end{pmatrix} \quad \leftrightarrow \quad \begin{pmatrix}
d_1 \\
\vdots \\
d_k \\
0 \\
\vdots \\
0
\end{pmatrix}
\]

Then

\[K_{0}^{\text{top}}(C^*(E)) \cong \mathbb{Z}_{d_1} \oplus \ldots \oplus \mathbb{Z}_{d_k} \oplus \mathbb{Z}^m \quad \text{and} \quad K_{1}^{\text{top}}(C^*(E)) \cong \mathbb{Z}^n.\]

We see that $K_{0}^{\text{top}}(C^*(E))$ no longer determines $K_{1}^{\text{top}}(C^*(E))$. So we will need the $K_{1}^{\text{top}}$-group in our invariant.

Also, the number of singular vertices is a Morita equivalence invariant:

\[|E_{\text{sing}}^0| = \text{rank} \, K_{0}^{\text{top}}(C^*(E)) - \text{rank} \, K_{1}^{\text{top}}(C^*(E)).\]
What about the moves?

We have to make a few specifications about what is allowed for infinite graphs:

Move (O) can be performed at an infinite emitter, but when we partition outgoing edges, only one piece of the partition is allowed to have an infinite number of edges.

Move (I) can only be performed at a regular vertex.

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With these specifications, the moves still preserve Morita equivalence of the associated $C^*$-algebra.
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With these specifications, the moves still preserve Morita equivalence of the associated $C^*$-algebra.
Theorem (Sørensen)

Suppose $E$ and $F$ are strongly connected graphs that each have a finite number of vertices and an infinite number of edges. Then

$E$ can be transformed into $F$ via Moves $(O), (I), (R)$ and their inverses if and only if

$$K_{\text{top}}^0(C^*(E)) \cong K_{\text{top}}^0(C^*(F)) \text{ and } K_{\text{top}}^1(C^*(E)) \cong K_{\text{top}}^1(C^*(F))$$

Note:
The $K_{\text{top}}^1$-group is needed (as we would expect).
The "sign of the determinant condition" disappears!
(The matrices involved are not square, so we can't even take determinants.)
We do not need the Cuntz splice move!!!

In fact, this result implies . . .
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$$E \text{ can be transformed into } F \text{ via Moves } (O), (I), (R) \text{ and their inverses}$$

$$\iff K_{0}^{\text{top}}(C^{*}(E)) \cong K_{0}^{\text{top}}(C^{*}(F)) \text{ and } K_{1}^{\text{top}}(C^{*}(E)) \cong K_{1}^{\text{top}}(C^{*}(F))$$

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$$E \text{ can be transformed into } F \text{ via Moves (O), (I), (R) and their inverses} \iff K_{0}^{\text{top}}(C^{*}(E)) \cong K_{0}^{\text{top}}(C^{*}(F)) \text{ and } K_{1}^{\text{top}}(C^{*}(E)) \cong K_{1}^{\text{top}}(C^{*}(F))$$

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Note:

- The $K_1^{\text{top}}$-group is needed (as we would expect).
- The “sign of the determinant condition” disappears! (The matrices involved are not square, so we can’t even take determinants.)
- We do not need the Cuntz splice move!!!

In fact, this result implies . . .
Corollary

Suppose $E$ is a strongly connected graph that has a finite number of vertices and an infinite number of edges, and if $F$ is the graph obtained by performing the Cuntz splice to $E$, then $F$ may be obtained by performing Moves (O), (I), (R), and their inverses to $E$. 

Thus, while we cannot turn

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using Moves (O), (I), (R), and their inverses, we can turn

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Corollary

Suppose $E$ is a strongly connected graph that has a finite number of vertices and an infinite number of edges, and if $F$ is the graph obtained by performing the Cuntz splice to $E$, then $F$ may be obtained by performing Moves (O), (I), (R), and their inverses to $E$.

Thus, while we cannot turn $E_2$ into $E_2^-$

\[ E_2 \xrightarrow{\text{Moves (O), (I), (R), and their inverses}} E_2^- \]

using Moves (O), (I), (R), and their inverses,
Corollary

Suppose $E$ is a strongly connected graph that has a finite number of vertices and an infinite number of edges, and if $F$ is the graph obtained by performing the Cuntz splice to $E$, then $F$ may be obtained by performing Moves (O), (I), (R), and their inverses to $E$.

Thus, while we cannot turn $E_2$ into $E_2^-$

Thus, while we cannot turn $E_2$ into $E_2^-$

using Moves (O), (I), (R), and their inverses, we can turn $E_\infty$ into $E_\infty^-$

using Moves (O), (I), (R), and their inverses.
Theorem (Sørensen)

Suppose $E$ and $F$ are strongly connected graphs that each have a finite number of vertices and an infinite number of edges. Then the following are equivalent:

1. $C^*(E)$ is Morita equivalent to $C^*(F)$
2. $K_{0}^{\text{top}}(C^*(E)) \cong K_{0}^{\text{top}}(C^*(F))$ and $K_{1}^{\text{top}}(C^*(E)) \cong K_{1}^{\text{top}}(C^*(F))$.
3. $K_{0}^{\text{top}}(C^*(E)) \cong K_{0}^{\text{top}}(C^*(F))$ and $|E_{\text{sing}}^0| = |F_{\text{sing}}^0|$.

Moreover, in this case one can transform $E$ into $F$ using moves (O), (I), (R), and their inverses.
Efren Ruiz (University of Hawai‘i at Hilo) and I have considered how we can use Sørensen’s result

\[
E \text{ can be transformed into } F \text{ via Moves } (O), (I), (R) \text{ and their inverses}
\]

\[
\iff K_{0}^{\text{top}}(C^{*}(E)) \cong K_{0}^{\text{top}}(C^{*}(F)) \text{ and } K_{1}^{\text{top}}(C^{*}(E)) \cong K_{1}^{\text{top}}(C^{*}(F))
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to classify Leavitt path algebras.
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to classify Leavitt path algebras.

Problem: Algebraic \( K \)-theory not the same as the topological \( K \)-theory.
With respect to $E^0 = E^0_{\text{reg}} \cup E^0_{\text{sing}}$ we have

$$A_E = \begin{pmatrix} B_E & C_E \\ * & * \end{pmatrix}$$

where $B_E$ and $C_E$ have finite entries.
With respect to $E^0 = E^0_{\text{reg}} \cup E^0_{\text{sing}}$ we have

$$A_E = \begin{pmatrix} B_E & C_E \\ * & * \end{pmatrix}$$

where $B_E$ and $C_E$ have finite entries. If $K$ is any field, then

$$K^\text{alg}_0(L_K(E)) \cong \text{coker} \left( \begin{pmatrix} I - B^t_E \\ -C^t_E \end{pmatrix} : \bigoplus_{E^0_{\text{reg}}} \mathbb{Z} \to \bigoplus_{E^0} \mathbb{Z} \right)$$

and

$$K^\text{alg}_1(L_K(E)) \cong \text{ker} \left( \begin{pmatrix} I - B^t_E \\ -C^t_E \end{pmatrix} : \bigoplus_{E^0_{\text{reg}}} \mathbb{Z} \to \bigoplus_{E^0} \mathbb{Z} \right) \oplus \text{coker} \left( \begin{pmatrix} I - B^t_E \\ -C^t_E \end{pmatrix} : \bigoplus_{E^0_{\text{reg}}} K^\times \to \bigoplus_{E^0} K^\times \right)$$
With respect to $E^0 = E^0_{\text{reg}} \cup E^0_{\text{sing}}$ we have

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where $B_E$ and $C_E$ have finite entries. If $K$ is any field, then

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and

$$K_1^{\text{alg}}(L_K(E)) \cong \text{ker} \left( \begin{pmatrix} I - B^t_E \\ -C^t_E \end{pmatrix} : \bigoplus_{E^0_{\text{reg}}} \mathbb{Z} \to \bigoplus_{E^0} \mathbb{Z} \right) \oplus \text{coker} \left( \begin{pmatrix} I - B^t_E \\ -C^t_E \end{pmatrix} : \bigoplus_{E^0_{\text{reg}}} K^\times \to \bigoplus_{E^0} K^\times \right)$$

We see the field matters in $K_1^{\text{alg}}(L_K(E))$. 
Efren Ruiz and I considered a certain property of fields.

**Definition**

An abelian group **has no free quotients** if no nonzero quotient of the group is a free abelian group.

**Theorem**

*The following are equivalent:*

1. **\( G \) has no free quotients.**
2. **\( G \) is not a direct sum of a free abelian group and another group.**
3. **\( \text{Hom}_\mathbb{Z}(G, F) = \{0\} \) for every free abelian group \( F \).**

**Definition**

A field **\( K \) has no free quotients** if the abelian group \((K^\times, \cdot)\) has no free quotients.
Example
The following are examples of fields with no free quotients:

- $\mathbb{C}$
- $\mathbb{R}$
- All finite fields.
- All algebraically closed fields.
- All fields that are perfect with characteristic $p > 0$.
- All fields $K$ such that $(K^\times, \cdot)$ is a torsion group.
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The following are examples of fields with no free quotients:

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Example

\( \mathbb{Q} \) is an example of a field with free quotients:

\[
\mathbb{Q}^\times \cong \mathbb{Z}_2 \oplus \mathbb{Z} \oplus \mathbb{Z} \oplus \ldots.
\]
Let $E$ and $F$ be graphs, and let $K$ be a field with no free quotients.

1. If $K_0^\text{alg}(L_K(E)) \cong K_0^\text{alg}(L_K(F))$, then $K_0^\text{top}(C^*(E)) \cong K_0^\text{top}(C^*(F))$.

2. If $K_1^\text{alg}(L_K(E)) \cong K_1^\text{alg}(L_K(F))$, then $K_1^\text{top}(C^*(E)) \cong K_1^\text{top}(C^*(F))$.

3. If $K_0^\text{alg}(L_K(E)) \cong K_0^\text{alg}(L_K(F))$ and $K_1^\text{alg}(L_K(E)) \cong K_1^\text{alg}(L_K(F))$, then $|E_0^\text{sing}| = |F_0^\text{sing}|$.

These implications do not hold if the hypothesis that $K$ has no free quotients is dropped.

We can put this together with Sørensen’s result to obtain a classification for unital Leavitt path algebras of infinite graphs.
Theorem (Ruiz and T)

Let $E$ and $F$ be strongly connected graphs with a finite number of vertices and an infinite number of edges. If $K$ is a field with no free quotients, then the following are equivalent:

1. $L_K(E)$ is Morita equivalent to $L_K(F)$.
2. $K^\text{alg}_0(L_K(E)) \cong K^\text{alg}_0(L_K(F))$ and $K^\text{alg}_1(L_K(E)) \cong K^\text{alg}_1(L_K(F))$.
3. $K^\text{alg}_0(L_K(E)) \cong K^\text{alg}_0(L_K(F))$ and $|E^\text{sing}_0| = |F^\text{sing}_0|$.

Moreover, in this case $E$ can be transformed into $F$ via the moves $(O)$, $(I)$, $(R)$, and their inverses.
This implies that for simple unital Leavitt path algebras of infinite graphs over a field with no free quotients, all algebraic $K$-theory information is contained in the $K^\text{alg}_0$-group and $K^\text{alg}_1$-group.

**Corollary**

If $E$ and $F$ are strongly connected graphs with a finite number of vertices and an infinite number of edges, $K$ is a field with no free quotients, and

$$K^\text{alg}_0(L_K(E)) \cong K^\text{alg}_0(L_K(F)) \text{ and } K^\text{alg}_1(L_K(E)) \cong K^\text{alg}_1(L_K(F)),$$

then

$$K^\text{alg}_n(L_K(E)) \cong K^\text{alg}_n(L_K(F)) \text{ for all } n \in \mathbb{Z}.$$
What happens when the underlying field has free quotients?

Example (An interesting (counter)example)

Let $E$ and $F$ be the following graphs and let $K = \mathbb{Q}$.

Then $K_{\text{alg}}(L_{\mathbb{Q}}(E)) \sim K_{\text{alg}}(L_{\mathbb{Q}}(F)) \sim \mathbb{Z} \oplus \mathbb{Z}$

but ... $K_{\text{alg}}(L_{\mathbb{Q}}(E)) \not\sim K_{\text{alg}}(L_{\mathbb{Q}}(F))$.

so $L_{K}(E)$ and $L_{K}(F)$ are not Morita equivalent.
What happens when the underlying field has free quotients?

Example (An interesting (counter)example)

Let $E$ and $F$ be the following graphs and let $K = \mathbb{Q}$.

\[ E \quad \infty \quad \bullet \quad \overset{\circlearrowleft}{\longrightarrow} \quad \bullet \quad \overset{\circlearrowleft}{\longrightarrow} \quad \infty \]

\[ F \]

Then $K_{alg}^0(L \mathbb{Q}(E)) \cong Z \oplus Z = K_{alg}^0(L \mathbb{Q}(F)) \cong Z \oplus Z \oplus \ldots$

but $K_{alg}^2(L \mathbb{Q}(E)) \not\cong K_{alg}^2(L \mathbb{Q}(F))$.

so $L_K(E)$ and $L_K(F)$ are not Morita equivalent.
What happens when the underlying field has free quotients?

**Example (An interesting (counter)example)**

Let $E$ and $F$ be the following graphs and let $K = \mathbb{Q}$.

Then

$$K_0^{\text{alg}}(L_{\mathbb{Q}}(E)) \cong K_0^{\text{alg}}(L_{\mathbb{Q}}(F)) \cong \mathbb{Z} \oplus \mathbb{Z}$$

$$K_1^{\text{alg}}(L_{\mathbb{Q}}(E)) \cong K_1^{\text{alg}}(L_{\mathbb{Q}}(F)) \cong \mathbb{Z} \oplus \mathbb{Z} \oplus \ldots$$

but . . .

$$K_2^{\text{alg}}(L_{\mathbb{Q}}(E)) \ncong K_2^{\text{alg}}(L_{\mathbb{Q}}(F)).$$

Mark Tomforde (University of Houston)  
Classification of Leavitt path algebras  
April 22, 2013  41 / 46
Example

Observations: For general fields

- $K_n^{\text{alg}}(L_K(E)) \cong K_n^{\text{alg}}(L_K(F))$ for $n = 0, 1$ does not imply that $L_K(E)$ and $L_K(F)$ are Morita equivalent.
Observations: For general fields

- \( K_n^{\text{alg}}(L_K(E)) \cong K_n^{\text{alg}}(L_K(F)) \) for \( n = 0, 1 \) does not imply that \( L_K(E) \) and \( L_K(F) \) are Morita equivalent.

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Observations: For general fields

- \( K_n^{\text{alg}}(L_K(E)) \cong K_n^{\text{alg}}(L_K(F)) \) for \( n = 0, 1 \) does not imply that \( L_K(E) \) and \( L_K(F) \) are Morita equivalent.

- \( K_n^{\text{alg}}(L_K(E)) \cong K_n^{\text{alg}}(L_K(F)) \) for \( n = 0, 1 \) does not imply that \( K_n^{\text{alg}}(L_K(E)) \cong K_n^{\text{alg}}(L_K(F)) \) for \( n \in \mathbb{Z} \).

- The number of singular vertices in \( E \) cannot be determined from the two groups \( K_0^{\text{alg}}(L_K(E)) \) and \( K_1^{\text{alg}}(L_K(E)) \).
Theorem (Ruiz and T)

Let $E$ and $F$ be strongly connected graphs with a finite number of vertices and an infinite number of edges. If $K$ has no free quotients, TFAE:

1. $L_K(E)$ is Morita equivalent to $L_K(F)$.
2. $K_0^{\text{alg}}(L_K(E)) \cong K_0^{\text{alg}}(L_K(F))$ and $K_1^{\text{alg}}(L_K(E)) \cong K_1^{\text{alg}}(L_K(F))$.
3. $K_0^{\text{alg}}(L_K(E)) \cong K_0^{\text{alg}}(L_K(F))$ and $|E^0_{\text{sing}}| = |F^0_{\text{sing}}|$.

But our example shows in general $(2) \not\Rightarrow (1)$. Remarkably, we can prove the following . . .
Theorem (Ruiz and T)

Let $E$ and $F$ be strongly connected graphs with a finite number of vertices and an infinite number of edges. If $K$ has no free quotients, TFAE:

1. $L_K(E)$ is Morita equivalent to $L_K(F)$.
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But our example shows in general $\not\Rightarrow$ (1). Remarkably, we can prove the following . . .

Theorem (Ruiz and T)

Let $E$ and $F$ be strongly connected graphs with a finite number of vertices and an infinite number of edges. Let $K$ be any field. Then $L_K(E)$ is Morita equivalent to $L_K(F)$ if and only if $K_0^{\text{alg}}(L_K(E)) \cong K_0^{\text{alg}}(L_K(F))$ and $|E_0^{\text{sing}}| = |F_0^{\text{sing}}|$.

So, in general, $(1) \iff (3) \implies (2)$. 
So the proper invariant for $L_K(E)$ when $E$ has an infinite number of edges is

$$(K_0^{\text{alg}}(L_K(E)), |E_{\text{sing}}^0|)$$

and when $K$ has no free quotients this can be replaced by

$$(K_0^{\text{alg}}(L_K(E)), K_1^{\text{alg}}(L_K(E))).$$
So the proper invariant for $L_K(E)$ when $E$ has an infinite number of edges is

$$(K_0^{\text{alg}}(L_K(E)), |E_0^{\text{sing}}|)$$

and when $K$ has no free quotients this can be replaced by

$$(K_0^{\text{alg}}(L_K(E)), K_1^{\text{alg}}(L_K(E))).$$

Combining the theorem of Abrams, Louly, Pardo, and Smith with the theorem of Ruiz and Tomforde gives a nearly complete classification of unital simple Leavitt path algebras.
Theorem (Classification of simple Unital Leavitt Path Algebras)

Let $L_K(E)$ and $L_K(F)$ be simple unital Leavitt path algebras.

1. If $E$ and $F$ both have a finite number of edges, and if

$$K_0^\text{alg}(L_K(E)) \cong K_0^\text{alg}(L_K(F)) \text{ and } \text{sgn}(\det(I - A_E^t)) = \text{sgn}(\det(I - A_F^t)),$$

then $L_K(E)$ is Morita equivalent to $L_K(F)$.

2. If $E$ and $F$ both have an infinite number of edges, then $L_K(E)$ is Morita equivalent to $L_K(F)$ if and only if

$$K_0^\text{alg}(L_K(E)) \cong K_0^\text{alg}(L_K(F)) \text{ and } |E_0^{\text{sing}}| = |F_0^{\text{sing}}|.$$

3. If one of $E$ and $F$ has a finite number of edges, and one has an infinite number of edges, then $L_K(E)$ and $L_K(F)$ are not Morita equivalent.
Theorem (Classification of simple Unital Leavitt Path Algebras)

Let $L_K(E)$ and $L_K(F)$ be simple unital Leavitt path algebras.

(1) If $E$ and $F$ both have a finite number of edges, and if

$$K_0^{\text{alg}}(L_K(E)) \cong K_0^{\text{alg}}(L_K(F)) \text{ and } \text{sgn} (\det(I - A_E^t)) = \text{sgn} (\det(I - A_F^t)),$$

then $L_K(E)$ is Morita equivalent to $L_K(F)$.

(2) If $E$ and $F$ both have an infinite number of edges, then $L_K(E)$ is Morita equivalent to $L_K(F)$ if and only if

$$K_0^{\text{alg}}(L_K(E)) \cong K_0^{\text{alg}}(L_K(F)) \text{ and } |E_0^{\text{sing}}| = |F_0^{\text{sing}}|.$$

(3) If one of $E$ and $F$ has a finite number of edges, and one has an infinite number of edges, then $L_K(E)$ and $L_K(F)$ are not Morita equivalent.

The only missing part is to determine if the “sign of the determinant condition” is necessary in (1).
Theorem (Classification of simple Unital Leavitt Path Algebras)

Let \( L_K(E) \) and \( L_K(F) \) be simple unital Leavitt path algebras.

1. If \( E \) and \( F \) both have a finite number of edges, and if

\[
K_0^{\text{alg}}(L_K(E)) \cong K_0^{\text{alg}}(L_K(F)) \quad \text{and} \quad \text{sgn}(\det(I - A^t_E)) = \text{sgn}(\det(I - A^t_F)),
\]

then \( L_K(E) \) is Morita equivalent to \( L_K(F) \).

2. If \( E \) and \( F \) both have an infinite number of edges, then \( L_K(E) \) is Morita equivalent to \( L_K(F) \) if and only if

\[
K_0^{\text{alg}}(L_K(E)) \cong K_0^{\text{alg}}(L_K(F)) \quad \text{and} \quad |E^0_{\text{sing}}| = |F^0_{\text{sing}}|.
\]

3. If one of \( E \) and \( F \) has a finite number of edges, and one has an infinite number of edges, then \( L_K(E) \) and \( L_K(F) \) are not Morita equivalent.

The only missing part is to determine if the “sign of the determinant condition” is necessary in (1). When \( K \) has no free quotients, we can replace \( |E^0_{\text{sing}}| = |F^0_{\text{sing}}| \) in (2) with \( K_1^{\text{alg}}(L_K(E)) \cong K_1^{\text{alg}}(L_K(F)) \).
Thank you!