The path spaces of a graph
Graph algebras workshop, BIRS

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A directed graph $E$ consists of a set $E^0$ of vertices and a set $E^1$ of directed edges, with direction determined by range and source maps $r, s : E^1 \to E^0$. A $k$-coloured graph is a directed graph with a map $c : E^1 \to \{c_1, \ldots, c_k\}$.

$E^0 = \{v, u\}$  
$E^1 = \{e, f, g, h\}$

$v = s(g) = s(h) = r(h) = r(f)$
$u = s(f) = s(e) = r(e) = r(g)$

$c(f) = c(g) = c_1$ (= blue)
$c(e) = c(h) = c_2$ (= red)
A sequence $\mu_1\mu_2\mu_3\ldots$ of edges is a path if $s(\mu_i) = r(\mu_{i+1})$ for all $i$.

\[
\begin{array}{c}
\mu_1 \\
r(\mu) \leftarrow \\
\mu_n \\
s(\mu) \rightarrow
\end{array}
\]

$E^n = \{ \mu : \mu \text{ is a path with } n \text{ (possibly } = \infty \text{) edges} \}$

$E^* = \{ \mu : \mu \text{ has finitely many edges} \}$. 
Higher-rank graphs

- A higher-rank graph, or $k$-graph, is a small category $\Lambda$ with a functor $d : \Lambda \to \mathbb{N}^k$ satisfying the unique factorisation property: if $\lambda \in \text{Mor}(\Lambda)$ has $d(\lambda) = m + n$, then there exists unique $\mu, \nu \in \text{Mor}(\Lambda)$ with $d(\mu) = m$, $d(\nu) = n$ and $\lambda = \mu \nu$.

- Call $d$ the degree functor.

Examples

1. Suppose $E$ is a directed graph. The path category $\mathcal{P}(E)$ of $E$ has $\text{Obj}(\mathcal{P}(E)) = E^0$, $\text{Mor}(\mathcal{P}(E)) = E^*$, range, source and composition inherited from $E$. With $d(\lambda) := |\lambda|$, $\mathcal{P}(E)$ is a 1-graph. Moreover, every 1-graph occurs as the path category of a directed graph.

2. Let $T_k$ be the category with a single object and morphisms $\mathbb{N}^k$. With $d = \text{id}_{\mathbb{N}^k}$, $T_k$ is a $k$-graph.
We may visualise a $k$-graph $\Lambda$ by its skeleton: the $k$-coloured directed graph $E_\Lambda$ with $E^0_\Lambda = \text{Obj}(\Lambda)$, $E^1_\Lambda = \bigcup_{i \leq k} d^{-1}(e_i)$, range and source as in $\Lambda$, and colouring $c^{-1}(e_i) = d^{-1}(e_i)$.

Examples

1. $\mathcal{P}(E)$ has skeleton isomorphic to $E$.
2. The skeleton of $T_k$ has single vertex, and a different coloured loop for each generator of $\mathbb{N}^k$:
Examples

3. For each \( m \in (\mathbb{N} \cup \{\infty\})^k \) there is a \( k \)-graph \( \Omega_{k,m} \) with objects \( \{p \in \mathbb{N}^k : p \leq m\} \), morphisms \( \{(p, q) \in \mathbb{N}^k \times \mathbb{N}^k : p \leq q \leq m\} \), \( r(p, q) = p \), \( s(p, q) = q \), \( (p, q)(q, t) = (p, t) \), and \( d(p, q) = q - p \). The skeleton of \( \Omega_{k,m} \) is denoted \( E_{k,m} \). The following 2-coloured graph is \( E_{2,(3,2)} \)

\[
\begin{array}{cccc}
(0, 0) & (1, 0) & (2, 0) & (3, 0) \\
| & | & | & \\
(0, 1) & (1, 1) & (2, 1) & (3, 1) \\
| & | & | & \\
(0, 2) & (1, 2) & (2, 2) & (3, 2) \\
\end{array}
\]
A coloured-graph morphism is a range, source and colour preserving map between two coloured graphs.

A square is a coloured-graph morphism from the coloured graph on the right into $E$. We think of this as a labelling of the picture on the right with elements of our graph.
**k-coloured graphs**

A *coloured-graph morphism* is a range, source and colour preserving map between two coloured graphs.

A *square* is a coloured-graph morphism from the coloured graph on the right into $E$. We think of this as a labelling of the picture on the right with elements of our graph.
Given a $k$-coloured graph $E$, we say a collection of squares $C$ is **complete** if for each $c_i; c_j$-coloured path $x \in E^2$, there exists a unique square in $C$ of which $x$ is a subpath.
Given a $k$-coloured graph $E$, we say a collection of squares $\mathcal{C}$ is *complete* if for each $c_i; c_j$-coloured path $x \in E^2$, there exists a unique square in $\mathcal{C}$ of which $x$ is a subpath.

For example: these two squares are a complete collection for $E$. Such a collection is not typically unique.
Associativity of \( C \)

Let \( E \) be a \( k \)-coloured graph and \( C \) be a complete collection of squares. Given a 3-coloured path \( fgh \in E^3 \), the squares in \( C \) give \( f_i, g_i, h_i, f^i, g^i, h^i \in E^1 \) as shown in the following diagram.

We say that \( C \) associative if \( f^2 = f_2 \), \( g^2 = g_2 \) and \( h^2 = h_2 \).
Suppose that $E$ is a $k$-coloured graph and $C$ complete collection of squares which is associative.

- For each $\Lambda$, $\{\lambda \in \Lambda : d(\lambda) = e_i + e_j, i \neq j\}$ determines a complete collection of squares $C_{\Lambda}$ for $E_{\Lambda}$ which is associative.
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For each $\Lambda$, $\{\lambda \in \Lambda : d(\lambda) = e_i + e_j, i \neq j\}$ determines a complete collection of squares $C_\Lambda$ for $E_\Lambda$ which is associative.

**Theorem (Hazlewood-Raeburn-Sims-W)**

There is a $k$-graph $\Lambda_{E,C}$ and an isomorphism $\psi : E_{\Lambda_{E,C}} \cong E$ such that $\psi \circ \phi \in C_{\Lambda_{E,C}}$ for each $\phi \in C$ (i.e. $\psi$ preserves squares).
Suppose that $E$ is a $k$-coloured graph and $C$ complete collection of squares which is associative.

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**Theorem (Hazlewood-Raeburn-Sims-W)**

*Suppose $\Lambda$ is a $k$-graph. If $E \cong E_{\Lambda}$ preserves squares, then $\Lambda \cong \Lambda_{E,C}$.*
Suppose that $E$ is a $k$-coloured graph and $C$ complete collection of squares which is associative.

For each $\Lambda$, \( \{ \lambda \in \Lambda : d(\lambda) = e_i + e_j, i \neq j \} \) determines a complete collection of squares $C_\Lambda$ for $E_\Lambda$ which is associative.

**Theorem (Hazlewood-Raeburn-Sims-W)**

There is a $k$-graph $\Lambda_{E,C}$ and an isomorphism $\psi : E_{E,C} \cong E$ such that $\psi \circ \phi \in C_{E,C}$ for each $\phi \in C$ (i.e. $\psi$ preserves squares).

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**Theorem (Hazlewood-Raeburn-Sims-W)**

Let $\sim$ be the equivalence relation on $\mathcal{P}(E)$ generated by $C$. Then $\mathcal{P}(E)/\sim$ is a $k$-graph which is isomorphic to $\Lambda_{E,C}$. 
Notation and Nomenclature

- A *k-graph morphism* is a degree preserving functor between two *k*-graphs.
- Each $\lambda \in \text{Mor}(\Lambda)$ may be uniquely identified with a *k*-graph morphism $x_{\lambda}: \Omega_{k,d(\lambda)} \to \Lambda$: for $m \leq n \leq d(\lambda)$ the factorisation property gives us a unique $x_{\lambda}(m, n) \in d^{-1}(m - n)$ satisfying $\lambda = \lambda' x_{\lambda}(m, n) \lambda''$. Then $x_{\lambda}(0, d(\lambda)) = \lambda$. 
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Hence we define $\Lambda^m$ for $m \in (\mathbb{N} \cup \{\infty\})^k$ to be the set of $k$-graph morphisms $\Omega_{k,m} \to \Lambda$ and identify $\Lambda^m$ and $d^{-1}(m)$. 

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- Hence we define \( \Lambda^m \) for \( m \in (\mathbb{N} \cup \{\infty\})^k \) to be the set of *k*-graph morphisms \( \Omega_{k,m} \to \Lambda \) and identify \( \Lambda^m \) and \( d^{-1}(m) \).
- Unique factorisation property implies that \( \Lambda^0 = \{\text{id}_v : v \in \text{Obj}(\Lambda)\} \), hence we identify \( \Lambda^0 \) with \( \text{Obj}(\Lambda) \).
- We identify \( \text{Mor}(\Lambda) \) and \( \Lambda \). Refer to elements of \( \Lambda \) as *paths*, and elements of \( \Lambda^0 \) *vertices*. 

S.B.G. Webster

The path spaces of a graph
A *k-graph morphism* is a degree preserving functor between two *k*-graphs.

Each $\lambda \in \text{Mor}(\Lambda)$ may be uniquely identified with a *k*-graph morphism $\chi_\lambda : \Omega_{k,d(\lambda)} \to \Lambda$: for $m \leq n \leq d(\lambda)$ the factorisation property gives us a unique $\chi_\lambda(m, n) \in d^{-1}(m - n)$ satisfying $\lambda = \lambda' \chi_\lambda(m, n) \lambda''$. Then $\chi_\lambda(0, d(\lambda)) = \lambda$.

Hence we define $\Lambda^m$ for $m \in (\mathbb{N} \cup \{\infty\})^k$ to be the set of *k*-graph morphisms $\Omega_{k,m} \to \Lambda$ and identify $\Lambda^m$ and $d^{-1}(m)$.

Unique factorisation property implies that $\Lambda^0 = \{\text{id}_v : v \in \text{Obj}(\Lambda)\}$, hence we identify $\Lambda^0$ with $\text{Obj}(\Lambda)$.

We identify $\text{Mor}(\Lambda)$ and $\Lambda$. Refer to elements of $\Lambda$ as *paths*, and elements of $\Lambda^0$ *vertices*.

Given a subset $F \subset \Lambda$ and a vertex $v \in \Lambda^0$, define $vF := r^{-1}(v) \cap F$ and $Fv := s^{-1}(v) \cap F$. 

S.B.G. Webster  The path spaces of a graph
Let $\lambda$ be the path of degree $(3, 2)$ with range $v$ in the $k$-graph $\Lambda$ represented on the left.

- Unique factorisation forces $\lambda = fgfee = feghf = hhfgf = \ldots$
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Unique factorisation forces $\lambda = fgfee = feghf = hhfgf = \ldots$

$\lambda$ is represented by the $k$-graph morphism $\Omega_{2,(3,2)} \to \Lambda$ encoded by the labelling of $\Omega_{2,(3,2)}$ on the right.

The path $\lambda((2, 1), (3, 2)) = fe = hf$, the square on the top right.
The path space

- Given a $k$-graph $\Lambda$, we call $W_\Lambda := \bigcup_{m \in (\mathbb{N} \cup \{\infty\})^k} \Lambda^m$ the path space of $\Lambda$.
- We endow $W_\Lambda$ with the cylinder set topology (or initial topology) given by the indicator function $\chi : W_\Lambda \to \{0, 1\}^\Lambda$, where $\chi_x(\lambda) = 1$ if $x(0, d(\lambda)) = \lambda$ and 0 otherwise [PW].
The path space

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- A base for this topology on $W_\Lambda$ consists of the sets

$$Z(\mu \setminus G) := Z(\mu) \setminus \bigcup_{\nu \in G} Z(\mu \nu),$$

where $Z(\mu) := \{\lambda \in W_\Lambda : \lambda(0, d(\mu)) = \mu\}, \mu \in \Lambda$, and $G \subset \Lambda$. We may insist that $G \subset \bigcup_{i \leq k} \Lambda^{e_i}$. [W]
- With this topology $W_\Lambda$ is a locally compact, Hausdorff space [W, PW].
Minimal common extensions

Given $\mu, \nu \in \Lambda$, we say that $\lambda$ is a *minimal common extension of $\mu$ and $\nu$* if $\lambda \in \mathcal{Z}(\mu) \cap \mathcal{Z}(\nu)$ and $d(\lambda) = d(\mu) \lor d(\nu)$. We denote the set of all such $\lambda$ by $\text{MCE}(\mu, \nu)$.

**Example (1)**

Given a directed graph $E$, and two paths $\mu, \nu \in E^*$, then

$$\text{MCE}(\mu, \nu) = \begin{cases} \{\mu\} & \text{if } \mu \in \mathcal{Z}(\nu) \\ \{\nu\} & \text{if } \nu \in \mathcal{Z}(\mu) \\ \emptyset & \text{otherwise.} \end{cases}$$
Given \( \mu, \nu \in \Lambda \), we say that \( \lambda \) is a \textit{minimal common extension of} \( \mu \) \textit{and} \( \nu \) if \( \lambda \in \mathcal{Z}(\mu) \cap \mathcal{Z}(\nu) \) and \( d(\lambda) = d(\mu) \lor d(\nu) \). We denote the set of all such \( \lambda \) by \( \text{MCE}(\mu, \nu) \).

\textbf{Example (2)}

\[ \text{MCE}(\mu, \nu) = \{ \mu \alpha_1, \mu \alpha_2 \} = \{ \nu \beta_1, \nu \beta_2 \} \]
Minimal common extensions

Given $\mu, \nu \in \Lambda$, we say that $\lambda$ is a minimal common extension of $\mu$ and $\nu$ if $\lambda \in Z(\mu) \cap Z(\nu)$ and $d(\lambda) = d(\mu) \vee d(\nu)$. We denote the set of all such $\lambda$ by $\text{MCE}(\mu, \nu)$.

Example (3)

\[
\begin{align*}
\text{MCE}(g_0, h_0) &= \emptyset \\
\text{MCE}(x_0x_1, h_0) &= \{x_0x_1h_2\} \\
\text{MCE}(x_0, x_0g_1) &= \{x_0g_1\}.
\end{align*}
\]
Finite exhaustive sets

Given $\nu \in \Lambda^0$, a subset $E \subset \nu \Lambda$ is exhaustive at $\nu$ if for each $\mu \in \nu \Lambda$, there exists $\nu \in E$ such that $\text{MCE}(\mu, \nu) \neq \emptyset$. We denote the set of all finite exhaustive sets at $\nu$ by $\nu \mathcal{FE}(\Lambda)$.

Example (1)

$$\xymatrix{ (f_i)_{i \in \mathbb{N}} \ar[r] & \vdots \ar[l]_{\nu} \ar[r] & w }$$

We have $\nu \in E$ for every $E \in \nu \mathcal{FE}(\Lambda)$, and $w \mathcal{FE}(\Lambda) = \{ \{w\} \}$. 
Finite exhaustive sets

Given \( v \in \Lambda^0 \), a subset \( E \subset v\Lambda \) is exhaustive at \( v \) if for each \( \mu \in v\Lambda \), there exists \( \nu \in E \) such that \( \text{MCE}(\mu, \nu) \neq \emptyset \). We denote the set of all finite exhaustive sets at \( v \) by \( vFE(\Lambda) \).

Example (2)

\[
\{v\}, \{\nu\}, \{\mu\}, \{\nu, \mu\}, \{\mu\alpha_1, \mu\alpha_2\} \in vFE(\Lambda)
\]

\[
\{\mu\alpha_1\}, \{\nu\beta_2\} \notin vFE(\Lambda)
\]
Finite exhaustive sets

Given \( v \in \Lambda^0 \), a subset \( E \subseteq v\Lambda \) is exhaustive at \( v \) if for each \( \mu \in v\Lambda \), there exists \( \nu \in E \) such that \( \text{MCE}(\mu, \nu) \neq \emptyset \). We denote the set of all finite exhaustive sets at \( v \) by \( vFE(\Lambda) \).

Example (3)

\[
\begin{align*}
\{ h_0, x_0, g_0 \}, \{ g_0, x_0 \}, \{ g_0, x_0g_1, x_0x_1 \}, \{ h_0, g_0, x_0g_1, x_0x_1g_2 \} & \in vFE(\Lambda) \\
\{ x_0 \}, \{ x_0, h_0 \}, \{ g_0, x_0g_1, x_0x_1g_2 \} & \notin vFE(\Lambda)
\end{align*}
\]
Boundary paths

A path $x \in W_\Lambda$ is a boundary path if for each $n \in \mathbb{N}^k$ with $n \leq d(x)$ and $E \in x(n)\mathcal{FE}(\Lambda)$, there exists $m \in \mathbb{N}^k$ such that $x(n, m) \in E$. Denote the set of all boundary paths by $\partial \Lambda$.

Examples (1)

$\Lambda^\infty = \{x : \Omega_k,(\infty)^k \to \Lambda : x$ is a $k$-graph morphism$\} \subset \partial \Lambda$.

$\partial \Lambda = \Lambda^\infty$ if $0 < |v\Lambda^m| < \infty$ for all $v \in \Lambda^0$ and $m \in \mathbb{N}^k$.

If $k=1$, then $\partial \Lambda = \Lambda^\infty \cup \{x \in \Lambda : |s(x)\Lambda^1| = 0$ or $\infty\}$. E.g. if $\Lambda$ is the 1-graph

$$
\begin{array}{c}
\downarrow \\
\vdots \\

(f_i)_{i \in \mathbb{N}}
\end{array}$$

then $\partial \Lambda = \{v, w\} \cup \{f_i : i \in \mathbb{N}\}$.
Boundary paths

A path $x \in \mathcal{W}_\Lambda$ is a boundary path if for each $n \in \mathbb{N}^k$ with $n \leq d(x)$ and $E \in x(n) \mathcal{FE}(\Lambda)$, there exists $m \in \mathbb{N}^k$ such that $x(n, m) \in E$. Denote the set of all boundary paths by $\partial \Lambda$.

Example (2)

$$\partial \Lambda = \Lambda u_1 \cup \Lambda u_2,$$
where $\Lambda v := s^{-1}(v)$. 
A path \( x \in W_\Lambda \) is a **boundary path** if for each \( n \in \mathbb{N}^k \) with \( n \leq d(x) \) and \( E \in x(n)\mathcal{FE}(\Lambda) \), there exists \( m \in \mathbb{N}^k \) such that \( x(n, m) \in E \). Denote the set of all boundary paths by \( \partial \Lambda \).

**Example (3)**

\[
\nu \partial \Lambda = \{ x_0 x_1 h_2, \ g_0, x_0 g_1, \ x_0 x_1 g_2 \}
\]
A path $x \in W_{\Lambda}$ is a **boundary path** if for each $n \in \mathbb{N}^k$ with $n \leq d(x)$ and $E \in x(n)\mathcal{FE}(\Lambda)$, there exists $m \in \mathbb{N}^k$ such that $x(n, m) \in E$. Denote the set of all boundary paths by $\partial\Lambda$.

**Example (4)**

$$v \partial\Lambda = \{x_0 \ldots x_{i-1}g_i : i \in \mathbb{N}\} \cup \{h_0f_0 \ldots, x_0x_1 \ldots\}$$
Boundary paths

Let \( \sigma \) be the shift action of \( \mathbb{N}^k \) partially defined by
\[
\sigma_n(\lambda)(p, q) = \lambda(n + p, n + q) \quad \text{for} \quad d(\lambda) \geq n.
\]

\( \sigma_n(x) \in \partial \Lambda \) for each \( n \in \mathbb{N}^k \) and \( x \in \partial \Lambda \) with \( d(x) \geq n \).

\( \lambda x \in \partial \Lambda \) for every \( \lambda \in \Lambda \) and \( x \in s(\lambda)\partial \Lambda \).

\( \nu \partial \Lambda \neq \emptyset \) for all \( \nu \in \Lambda^0 \).

Notice that
\[
W_\Lambda \setminus \partial \Lambda = \bigcup_{\lambda \in \Lambda} \left( \bigcup_{E \in s(\lambda)\mathcal{F}\mathcal{E}(\Lambda)} \mathcal{Z}(\lambda \setminus E) \right),
\]

so \( \partial \Lambda \) is closed in \( W_\Lambda \), and hence a locally compact Hausdorff space.
Give $W_\Lambda$ a partial order $\leq$ defined by $\mu \leq \lambda \iff \lambda \in Z(\mu)$.

A filter in $W_\Lambda$ is a subset $U \subset W_\Lambda$ such that

1. if $\lambda \in U$ and $\mu \leq \lambda$, then $\mu \in U$, and
2. if $\mu, \nu \in U$, then there exists $\lambda \in U$ with $\mu, \nu \leq \lambda$.

Denote the set of all filters by $\hat{\Lambda}$. Say $U$ is an ultrafilter if $U$ is a maximal filter. Denote the set of ultrafilters by $\hat{\Lambda}_\infty$. 
Filters [Exel]

- Give $W_{\Lambda}$ a partial order $\leq$ defined by $\mu \leq \lambda \iff \lambda \in \mathcal{Z}(\mu)$.
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- Each $x \in W_{\Lambda}$ determines a filter $U_x := \{x(0, n) : n \in \mathbb{N}^k, n \leq d(x)\}$
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Each $x \in W_\Lambda$ determines a filter

$U_x := \{x(0, n) : n \in \mathbb{N}^k, n \leq d(x)\}$

Conversely, each filter $U \in \hat{\Lambda}$ determines a $k$-graph morphism: let $m = \vee\{d(x) : x \in U\}$. Then for $p, q \in \mathbb{N}^k$ with $p \leq q \leq m$, define $x_U : \Omega_{k,m} \rightarrow \Lambda$ by $x_U(p, q) = \lambda(p, q)$, where $\lambda \in U$ and $d(\lambda) \geq q$. This is well defined since $U$ is a filter.
Give $W_{\Lambda}$ a partial order $\leq$ defined by $\mu \leq \lambda \iff \lambda \in \mathcal{Z}(\mu)$.

A filter in $W_{\Lambda}$ is a subset $U \subset W_{\Lambda}$ such that

1. if $\lambda \in U$ and $\mu \leq \lambda$, then $\mu \in U$, and
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Denote the set of all filters by $\hat{\Lambda}$. Say $U$ is an ultrafilter if $U$ is a maximal filter. Denote the set of ultrafilters by $\hat{\Lambda}_\infty$.

Each $x \in W_{\Lambda}$ determines a filter $U_x := \{x(0, n) : n \in \mathbb{N}^k, n \leq d(x)\}$.

Conversely, each filter $U \in \hat{\Lambda}$ determines a $k$-graph morphism: let $m = \bigvee\{d(x) : x \in U\}$. Then for $p, q \in \mathbb{N}^k$ with $p \leq q \leq m$, define $x_U : \Omega_{k,m} \to \Lambda$ by $x_U(p, q) = \lambda(p, q)$, where $\lambda \in U$ and $d(\lambda) \geq q$. This is well defined since $U$ is a filter.

$\hat{\Lambda}$ has similar looking topology, replacing $\mathcal{Z}(\mu)$ with $\hat{\mathcal{Z}}(\mu) := \{U \in \hat{\Lambda} : \mu \in U\}$.

$\hat{\Lambda} \cong W_{\Lambda}$. 

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S.B.G. Webster
Filters

Example (1)

\[ (f_i)_{i \in \mathbb{N}} \]

\[ \therefore \text{Don't need infinite receivers to see this.} \]

\[ \hat{\Lambda}_\infty = \{ \{ w \} \} \cup \{ U_{f_i} : i \in \mathbb{N} \} \]

\[ f_i \rightarrow v \text{ in } W_\Lambda. \]

\[ \hat{\Lambda}_\infty \text{ is not closed!} \]

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The path spaces of a graph
Filters

Example (2)

\begin{align*}
U_{g_0}, & \quad U_{h_0f_0f_1...} U_{x_0g_1}, \quad U_{x_0x_1g_2}, \quad \ldots \in \hat{\Lambda}_\infty \\
\rightarrow & \quad x_0 \ldots x_{n-1}g_n \rightarrow x_0x_1 \ldots \\
\notin & \quad U_{x_0x_1x_2...} \notin \hat{\Lambda}_\infty !!!
\end{align*}
In path-space terminology, the anologue of $\hat{\Lambda}_\infty$ is denoted $\Lambda^{\leq\infty}$ (Definition in RSY2004).

Define $\partial\hat{\Lambda}$ to be the filters $U \in \hat{\Lambda}$ such that for each $\mu \in U$, $E \subset s(\mu)FE(\Lambda)$, there exists $\nu \in E$ such that $\mu \nu \in x$.

Then $\partial\hat{\Lambda} = \{U_x : x \in \partial\Lambda\}$. 

Filters
Filters

- In path-space terminology, the anologue of $\hat{\Lambda}_\infty$ is denoted $\Lambda_\leq \infty$ (Definition in RSY2004).
- Define $\partial \hat{\Lambda}$ to be the filters $U \in \hat{\Lambda}$ such that for each $\mu \in U$, $E \subset s(\mu) FE(\Lambda)$, there exists $\nu \in E$ such that $\mu \nu \in x$
- Then $\partial \hat{\Lambda} = \{ U_x : x \in \partial \Lambda \}$.
- $\hat{\Lambda}_\infty = \partial \hat{\Lambda}$
In path-space terminology, the anologue of $\hat{\Lambda}_\infty$ is denoted $\Lambda^{\leq\infty}$ (Definition in RSY2004).

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Then $\partial\hat{\Lambda} = \{U_x : x \in \partial\Lambda\}$.

$\hat{\Lambda}_\infty = \partial\hat{\Lambda}$

$\hat{\Lambda}_\infty = \partial\hat{\Lambda}$ if $\Lambda$ is row-finite and locally convex:

$\Lambda$ is row-finite if $\nu\Lambda^m$ is finite for each $\nu \in \Lambda^0$ and $m \in \mathbb{N}^k$, and

$\Lambda$ is locally-convex if for each $i \neq j$, $\mu \in \Lambda^{e_i}$ and $\nu \in r(\mu)\Lambda^{e_j}$, the sets $s(\mu)\Lambda^{e_j}$ and $s(\nu)\Lambda^{e_i}$ are nonempty.
A $k$-graph $\Lambda$ is **finitely aligned** if $\text{MCE}(\mu, \nu)$ is finite (possibly empty) for all $\mu, \nu \in \Lambda$.

Given a finitely aligned $k$-graph $\Lambda$, a **Cuntz-Krieger $\Lambda$-family** in a $C^*$-algebra $B$ is a map $s : \Lambda \to B$ such that each $s_\lambda$ is a partial isometry, and that

- **CK1.** $\{s_\nu : \nu \in \Lambda^0\}$ are mutually orthogonal projections,
- **CK2.** $s_\mu s_\nu = s_{\mu \nu}$ if $\mu \nu \in \Lambda$,
- **CK3.** $s_\mu^* s_\nu = \sum_{\mu \alpha = \nu \beta \in \text{MCE}(\mu, \nu)} s_\alpha s_\beta^*$, and
- **CK4.** $\prod_{\lambda \in E}(s_\nu - s_\lambda s_\lambda^*) = 0$ for all $\nu \in \Lambda^0$ and $E \in \nu \mathcal{F} \mathcal{E}(\Lambda)$.

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\begin{enumerate}
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\end{enumerate}

$C^*(\Lambda)$ is the universal $C^*$-algebra for Cuntz-Krieger $\Lambda$-families.

$C^*(\Lambda)$ is nonzero since the representation $S : \Lambda \to B(\ell^2(\partial \Lambda))$ given by

$$S_\lambda \xi_x = \begin{cases} 
\xi_\lambda x & \text{if } s(\lambda) = r(x) \\
0 & \text{otherwise}
\end{cases}$$

yields a nonzero Cuntz-Krieger $\Lambda$-family.
We call $D_\Lambda := C^*\left(\{s_\lambda s^*_\lambda : \lambda \in \Lambda\}\right) \subset C^*(\Lambda)$ the diagonal $C^*$-subalgebra of $C^*(\Lambda)$. One can show that $D_\Lambda = \overline{\text{span}}\left\{s_\lambda s^*_\lambda : \lambda \in \Lambda\right\}$. 
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**Theorem (W)**

$D_\Lambda \cong C_0(\partial \Lambda)$
Diagonal Subalgebra

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The path spaces of a graph
Diagonal Subalgebra

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- For each $n \in \mathbb{N}^k$, $\{s_\lambda s^*_\lambda : \lambda \in \Lambda^n\}$ is a family of mutually orthogonal projections.
- Notice that $\mu \leq \lambda \iff s_\lambda s^*_\lambda \leq s_\mu s^*_\mu$. 

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- So \( \{\lambda : \phi(s_\lambda s^*_\lambda) = 1\} \in \hat{\Lambda} \), and so determines a unique path \( x \in W_\Lambda \).
Diagonal Subalgebra

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**Theorem (W)**

$D_\Lambda \cong C_0(\partial \Lambda)$

- Let $\phi$ be a character of $D_\Lambda$.
- For each $n \in \mathbb{N}^k$, $\{s_\lambda s_\lambda^* : \lambda \in \Lambda^n\}$ is a family of mutually orthogonal projections.
- Notice that $\mu \leq \lambda \iff s_\lambda s_\lambda^* \leq s_\mu s_\mu^*$.
- So $\{\lambda : \phi(s_\lambda s_\lambda^*) = 1\} \in \hat{\Lambda}$, and so determines a unique path $x \in W_\Lambda$.
- For each $n \leq d(x)$ and $E \in x(n)\mathcal{F}\mathcal{E}(\Lambda)$, (CK4) says that $\prod_{\lambda \in E}(s_{x(n)} - s_\lambda s_\lambda^*) = 0$, and it follows that $x \in \partial \Lambda$. 

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The path spaces of a graph
Removing Sources

Farthing defined a process which, given an row-finite $k$-graph $\Lambda$, constructs a row-finite $k$-graph $\Gamma$ with no sources such that $C^*(\Lambda) \sim_{SME} C^*(\Gamma)$. This process extends the non-infinite boundary paths of $\Lambda$ to infinite paths [F,W].
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Here, $w_n := x_0 \ldots x_{n-1} g_n$ and any path of degree $(1, \infty)$ are all elements of $\partial \Lambda$. The idea is to extend these paths to be infinite in all directions (degrees, colours, ...).
Boundary path starting with $h_0$

The path spaces of a graph
Boundary path $w_3$
Putting it all together

\[ v_0 \xrightarrow{g_0} v_1 \xrightarrow{g_1} v_2 \xrightarrow{g_2} v_3 \ldots \]

\[ h_0 \xleftarrow{x_0} h_1 \xleftarrow{x_1} h_2 \xleftarrow{x_2} h_3 \ldots \]
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Putting the bits together

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- The projection \( \pi : \Lambda^0 \Gamma^\infty \to \partial \Lambda \) is a homeomorphism.
- The isomorphism \( C^* (\Lambda) \cong \rho C^* (\Gamma) \rho \) induces a homeomorphism \( \rho : \hat{pD}_\Gamma \rho \to \hat{D}_\Lambda \).
Putting the bits together

- The projection $\pi : \Lambda^0 \Gamma^\infty \to \partial \Lambda$ is a homeomorphism.
- The isomorphism $C^*(\Lambda) \cong pC^*(\Gamma)p$ induces a homeomorphism $\rho : pD_\Gamma p \to \hat{D}_\Lambda$.
- Then the following diagram commutes:

$$
\begin{array}{ccc}
\Lambda^0 \Gamma^\infty & \xrightarrow{\pi} & \partial \Lambda \\
\downarrow{\eta} & & \downarrow{h_\Lambda} \\
pD_\Gamma p & \xrightarrow{\rho} & \hat{D}_\Lambda
\end{array}
$$

Where $\eta$ is essentially a restriction of $h_\Gamma : \Gamma^\infty \to \hat{D}_\Gamma$ to paths with range in $\Lambda^0$. 

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