Graph Algebras: Bridges between graph C^*-algebras and Leavitt path algebras

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The topic of this workshop was graph algebras, specifically algebras and C^*-algebras constructed from directed graphs and their generalizations. This workshop brought together researchers from two mathematical disciplines, Algebra and Functional Analysis, with the objective of having these two groups share information regarding similar studies of graph algebras that are currently being undertaken in each discipline.

1 Overview of the Field

In the late 1990s researchers in Functional Analysis introduced a class of C^*-algebras constructed from directed graphs. If E is a graph, the graph C^*-algebra $C^*(E)$ is defined to be the universal C^*-algebra generated by elements satisfying relations determined by E. The collection of graph C^*-algebras includes C^*-algebras from many previously studied classes (e.g., AF-algebras, Cuntz-Krieger algebras, Kirchberg algebras, crossed products by $\mathbb{Z}$) and allows them to be studied by unified techniques. Very interestingly, it has also been found that the graph E not only determines the relations that the generators of $C^*(E)$ satisfy, but also many C^*-algebraic properties of $C^*(E)$ are reflected in the graph E. In fact, it can often be shown that a $C^*(E)$ has a particular C^*-algebraic property if and only if E has a corresponding property. In fact, much of the theory of graph C^*-algebras has been concerned with determining the correct graph-theoretic properties on E that are equivalent to important C^*-algebraic properties of $C^*(E)$.

Inspired by the success of the Functional Analysts, in 2005 Algebraists defined a method for taking a directed graph E and a field K and constructing an associated K-algebra $L_K(E)$, known as the Leavitt path algebra. As with the graph C^*-algebras, the Leavitt path algebras include many well-known classes of algebras and have been studied intensely in the algebra community since their introduction. In addition, as with graph C^*-algebras, the properties of the graph often correspond to properties of the Leavitt path algebra. The interplay between these two classes of graph algebras has been extensive and mutually beneficial — graph C^*-algebra results have helped to guide the development of Leavitt path algebras by suggesting what results are true and in what direction investigations should be focused, and Leavitt path algebras have given a better understanding of graph C^*-algebras by helping to identify those aspects of $C^*(E)$ that are algebraic, rather than C^*-algebraic, in nature. Moreover, results from each class have had nontrivial applications to the other, and the work of researchers from each side has guided discovery for the other.

One of the most intriguing aspects in the study of Leavitt path algebras is that it is almost always the case that the property of a graph that corresponds to a C^*-algebraic property of the graph C^*-algebra is the same graph property that corresponds to the analogous algebraic condition for the Leavitt path algebra; and moreover, this condition does not depend on the underlying field. For example, if E is graph, then the conditions on E that correspond to $C^*(E)$ being simple (i.e., have no *closed* two-sided ideals) are exactly the same conditions for $L_K(E)$ to be simple (i.e., have no two-sided ideals) for all fields K. There are two important things to notice here: (1) The conditions on the graph determine the same corresponding conditions (modulo the category) for Leavitt path algebras and graph C^*-algebras, and (2) These conditions depend only on the graph and not on the field. As a consequence, one can often take a graph C^*-algebra theorem, replace the term "graph C^*-algebra" with the term "Leavitt path algebra", and obtain a true statement. Despite these similarities, however, it has been found that the proofs required for the Leavitt path algebra theorems have needed to be very different from the proofs for the graph C^*-algebras and new techniques have been required. Currently, one of the major goals of the study of graph algebras is to understand the nature and extent of these similarities: Why do these similarities hold? Why is the underlying field irrelevant for many Leavitt path algebra results? What are properties where this similarity does not occur? Is there a "Rosetta Stone" that would allow one to take results for one class and immediately deduce corresponding results for the other class?

While there are many approaches to these questions, our workshop took a particular focus on classification results for graph C^*-algebras and Leavitt path algebras, which have used invariants from K-theory to classify isomorphism classes and Morita equivalence classes of these graph algebras. Classification of C^*-algebras is a program with Canadian origins, but there are many researchers worldwide (both algebraists and analysts) currently working in the subject. Recent results for classification of graph C^*-algebras and Leavitt path algebras have unveiled deep connections with dynamics, the gauge action and grading of the algebras, and various permanence properties. Consequently, we designed our conference to have these topics represented and to have individuals working on them present talks.

2 Presentation Highlights and Scientific Progress Made

The talks at the workshop were designed to introduce students and researchers from other areas to the topics discussed while at the same time presenting recent results and giving an overview of the current status of these topics. The talks can roughly be divided into five categories: (1) Classification of Graph Algebras, (2) General Classification Results, (3) Symbolic Dynamics, (4) Structure Results for Graph Algebras, and (5) Generalizations of Graph Algebras. We summarize the results presented in talks for each of these categories.

(1) Classification of Graph Algebras

In his talk, M. Tomforde discussed ongoing efforts to classify unital simple Leavitt path algebras up to Morita equivalence and announced how he and E. Ruiz have obtained a complete classification for simple Leavitt path algebras coming from graphs with a finite number of vertices and an infinite number of edges. Interestingly, the invariant consists of the algebraic K_0-group together with the number of singular vertices in the graph, and this invariant may be replaced by the pair of the algebraic K_0-group and algebraic K_1-group when the underlying field is a field with no free quotients, but not in general. This result is surprising for two reasons: First, this is one of few results where the underlying field of the Leavitt path algebra makes a difference, and second, unlike in the C^*-algebra case the general invariant needs graph-theoretic data in addition to K-theory groups.

E. Ruiz in his talk discussed permanence properties that he has recently discovered for Cuntz-Krieger algebras and graph C^*-algebras. He also described how solving future classification problems would rely on identifying when a graph C^*-algebra is Morita equivalent to a unital graph C^*-algebra.

G. Restorff, discussed his work on classification of non-simple Cuntz-Krieger algebras through K-theory, which is one of the few complete classifications we have for a class of non-simple C^*-algebras. Restorff described the invariant coming from K-theory (which is sometimes called the K-web or filtrated K-theory), and he described certain redundancies that occur in the six-term exact sequences for Cuntz-Krieger algebras.

Adam Sørensen discussed extensions of his classification of unital simple graph C^*-algebras by graph moves. He announced a recent result that if E and F are finite graphs and $C^*(E)$ is Morita equivalent to

$C^*(F)$, then E can be transformed into F using the moves insplitting, outsplitting, source removal, delay, the Cuntz splice, and their inverses.

(2) General Classification

G. Elliott discussed classification efforts using the Elliott intertwining argument in an abstract classifying category. R. Bentmann descibed how the fact that gauge actions of Cuntz-Krieger algebras are KK-equivalent if and only if the corresponding matrices are shift equivalent over the integers, and how this follows from a general investigation of the relevant spectral sequences based on ideas of Bousfield. F. Perera announced his work on the structure of the set of dimension functions of certain $C(X)$-algebras using the Cuntz semigroup, which shows the Blackadar-Handelman conjectures hold for these examples. The talks of these speakers gave more insight into how general classification theory might be applied to graph algebras, and how KK-theory and the gauge action may be relevant to these efforts.

(3) Symbolic Dynamics

Mike Boyle discussed strong shift equivalence of matrices, a topic which has recently become of interest in graph algebras since it is known if E and F are two finite graphs with no sinks or sources, then strong shift equivalence of the adjacency matrices of E and F implies that $L_K(E)$ and $L_K(F)$ are isomorphic and that $C^*(E)$ and $C^*(F)$ are isomorphic. Strong shift equivalence is a well-known topic in symbolic dynamics, but is less well known among functional analysts and algebraists. In his talk, Boyle explained background and motivation for strong shift equivalence in terms of classification of symbolic dynamical systems and inverse problems for nonnegative matrices. He also discussed how existing strong shift equivalence theories are general enough to treat matrices over general rings, although most graph algebra results have currently only made use of matrices over integers.

In addition, in the past few years it has been found that flow equivalence of shift spaces is closely related to classification of unital graph C^*-algebras, and moreover, flow equivalence seems related to the current difficulties in the classification of unital Leavitt path algebras. Flow equivalence of shift spaces is a somewhat subtle and technical topic in symbolic dynamics, and many of the functional analysts and algebraists at this workshop have been eager to learn more about the topic. Boyle is one of the world's experts in symbolic dynamics, and throughout the workshop he engaged in many informal discussions and problem-solving sessions with various functional analysts and algebraists. Boyle helped many of the workshop participants to learn more about flow equivalence, and various invariants coming from symbolic dynamics, such as the zeta function and entropy, which may be able to be used in graph algebra work.

T. Carlsen in his talk discussed work in progress related to orbit equivalence and when there is an isomorphism between two graph C^*-algebras that maps the diagonal subalgebra of one graph C^*-algebra onto the diagonal subalgebra of the other. Carlsen discussed how these topics are related to the groupoid of the graph and to the notion of strong shift equivalence, which Boyle discussed in his talk.

(4) Structure results for graph algebras

G. Abrams gave a talk discussing prime and primitive Leavitt path algebras, and described recent results showing that these two notions coincide for Leavitt path algebras of countable graphs, but differ for Leavitt path algebras of uncountable graphs. He discussed how uncountable graphs can be used to give exotic examples of algebras, including an example of a von Neumann regular ring that is prime but not primitive — an object known to exist through a complicated construction in the past, but now more accessible through the easier techniques of graph algebras.

R. Hazrat gave a talk on the graded structure of a Leavitt path algebra, and described how the algebraic K_0-group of a Leavitt path algebra $L_K(E)$ carries a $K[x, x^{-1}]$-module structure. He also described how this module structure on $K_0(L_K(E))$ determines the graded isomoporhism class of $L_K(E)$ in certain situations. Hazrat is one of the first people to notice the role of the grading on Leavitt path algebras plays in their classification. This has led to investigations of the corresponding role of the gauge action in the classification of graph C^*-algebras, as discussed in part by Carlsen in his talk. In addition, there were many informal discussions of workshop participants with Hazrat to discuss his approaches to classification.

(5) Generalizations of Graph Algebras

There were four important generalizations of graph algebras discussed in the talks at the workshop: graph algebras of separated graphs, Cuntz-Pimsner rings, C^*-algebras of k-graphs, and analogues of Cuntz algebras on L^p spaces.

Separated graphs were introduced by Ara and Goodearl in [4, 5], and they described how to construct analogues of the Leavitt path algebras and graph C^*-algebras from them. The algebras of separated graphs include Leavitt algebras $L(m,n)$ of type (m,n), which are not all included in the class of Leavitt path algebras. At this workshop Ara gave a talk explaining an application of separated graph algebras to paradoxical compositions, answering a recent question asked independently by of Rørdam and Sierakowski, and by Kerr and Novak, in the negative.

A Cuntz-Pimsner algebra is a C^*-algebra constructed from Hilbert bimodule. The class of Cuntz-Pimsner algebras vastly generalizes the class of graph C^*-algebras. In [9] Carlsen and Ortega introduced algebraic analogues of these C^*-algebras, which they call Cuntz-Pimsner rings. At the workshop, Otega described results describing when Cuntz-Pimsner rings are simple, and used these results to give applications to specific subclasses, namely a uniqueness result for the Toeplitz algebra of a directed graph and characterizations of when crossed products of a ring by a single automorphism and fractional skew monoid rings of a single corner isomorphism are simple.

The k-graphs are higher-dimensional versions of graphs described in terms of categories with a degree functor that gives a notion of "length". One can construct a C^*-algebra from a k-graph in a way that generalizes the graph C^*-algebra construction. There has been extensive work done on k-graph C^*-algebras over the past 10 years. At the workshop, there were three talks on k-graph C^*-algebras. S. Webster discussed different characterizations of the path spaces of k-graphs, and A. Kumjian and A. Sims each gave a talk on joint work concerning twisted products of k-graph C^*-algebras and their K-theories.

At the workshop, C. Phillips introduced a new generalization of graph C^*-algebras, using Banach algebras of operators on L^p spaces. Using this framework, Phillips introduced L^p-analogues $\mathcal{O}_d^p$ of the Cuntz algebra $\mathcal{O}_d$ for each $p \in [1, \infty]$, as well as L^p-analogues of the UHF algebras. He also surveyed fundamental results he has obtained for these C^*-algebras, and raised a number of open questions for future study.

3 Resources Created and List of Open Problems Generated

After all the talks at the workshop ended, we had a one-hour problem session during which we asked participants to generate a list of important outstanding problems in the field. Many of these are difficult problems, whose solutions would have important applications to the subject and direct future investigations in graph algebras.

After the workshop, we created an online resource for these problems, which we have called the "Graph Algebra Problem Page". The "Graph Algebra Problem Page" is located at the URL

`http://www.math.uh.edu/∼tomforde/GraphAlgebraProblems/GraphAlgebraProblemPage.html`

This website is meant to serve as a repository for these problems, with the following three goals:

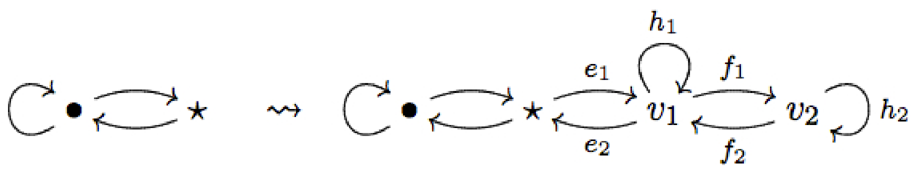
1. Making these open problems available to the public;
2. Providing clear statements of the problems with links to relevant resources or background material; and
3. Maintaining the list by reporting any progress made and providing references to solutions or partial solutions as they are developed.

Our hope is that others working on graph algebras will use this page to find out about current problems in the subject, and we also hope others working in the area will contribute open problems to the list, so that it can direct attention to trends in the subject and also become a public record of progress made in the discipline. After the workshop we sent an announcement to several of our colleagues working on graph algebras making them aware of the webpage, and we received a number of responses providing feedback. The webpage also contains links to the BIRS workshop website, and links to the videos and slides of the talks that BIRS maintains. In this way, the website also helps to advertise the BIRS resources and make them available to the graph algebra community.

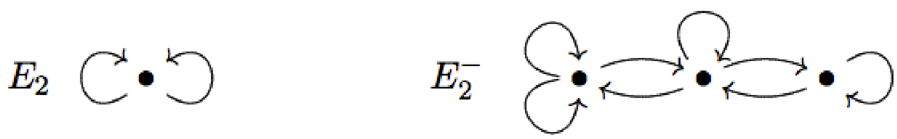
Here is a list of the open problems generated from the workshop, which are also on the webpage.

List of Open Problems in Graph Algebras

1. **The Cuntz Splice for Leavitt path algebras.** If E is a graph, the operation of the Cuntz splice attaches a portion to the graph that changes the sign of $\det(I - A^t)$, where A is the vertex matrix of E. Here is an example: In the following graph, we attach the Cuntz Splice at the starred vertex, and label the adjoined vertices and edges.



The Cuntz splice is important in the classification of C^*-algebras of finite graphs. As a special case of the construction, we can consider the graph with a single vertex and two edges, which we denote E_2, and the graph formed by performing a Cuntz splice to it, denoted E_2^-.



In [19] Rørdam showed that $C^*(E_2)$ is isomorphic to $C^*(E_2^-)$. It automatically follows from this that if E is any finite graph whose associated algebra is simple and purely infinite, and if F is the graph formed by performing the Cuntz splice at any vertex of F, then $C^*(E)$ is Morita equivalent to $C^*(F)$. (More generally, Eilers, Ruiz, and Sørensen have shown in a forthcoming preprint that if E is any graph with a finite number of vertices and if F is the graph formed by performing the Cuntz splice at any vertex of E that is the base point of two distinct cycles, then $C^*(E)$ is Morita equivalent to $C^*(F)$.)

The classification for Leavitt path algebras of finite graphs has been hindered by the lack of analogous results concerning the Cuntz splice. In particular, the following are important open problems.

Question: Let E be a graph whose associated algebra is simple and purely infinite, and let K be any field. If F is the graph formed by performing the Cuntz splice to any vertex of E, then is $L_K(E)$ Morita equivalent to $L_K(F)$?

The answer to the above question is the last remaining piece in the classification of unital Leavitt path algebras. See [20, Section 8], for a discussion of these issues.

Question: If K is any field, then are $L_K(E_2)$ and $L_K(E_2^-)$ isomorphic as rings? (Note: Sometimes in the notation, the field K is suppressed and one writes L_2 and L_2- in place of $L_K(E_2)$ and $L_K(E_2^-)$, respectively. The question is then often stated as: Are L_2 and L_2- isomorphic?)

Comment: Since the algebraic K_0-groups of $L_K(E_2)$ and $L_K(E_2^-)$ are both zero, these algebras will be Morita equivalent if and only if they are isomorphic. In addition, we mention that all algebraic K-groups of these two algebras are zero, so the K-groups cannot be used to distinguish between the two.

Comment: One can verify that the homogeneous zero components of $L_K(E_2)$ and $L_K(E_2^-)$ have different K_0-groups and hence are not isomorphic. Thus there does not exist a graded isomorphism from $L_K(E_2)$ onto $L_K(E_2^-)$.

Comment: Unlike in the C^*-algebra case, an affirmative answer to the second question does not immediately imply an affirmative answer to the first. For a discussion of what extra properties are needed, see [1, Section 2], particularly "The Hypothesis" listed there.

2. ***K*-groups of unital Leavitt path algebras.** In [20, Example 11.2] an example was given of graphs E and F that each have associated algebras that are unital, simple, purely infinite, and have the property

that $K_0(L_{\mathbb{Q}}(E)) \cong K_0(L_{\mathbb{Q}}(F))$ and $K_1(L_{\mathbb{Q}}(E)) \cong K_1(L_{\mathbb{Q}}(F))$, but $K_2(L_{\mathbb{Q}}(E))$ is not isomorphic to $K_2(L_{\mathbb{Q}}(F))$.

Question: For a given field K and any natural number N, do there exist graphs E and F, each with a finite number of vertices, an infinite number of edges, and associated algebras that are simple and purely infinite, such that $K_i(L_K(E)) \cong K_i(L_K(F))$ for all $1 \leq i \leq N-1$, but $K_N(L_K(E))$ is not isomorphic to $K_N(L_K(F))$?

It follows from [20, Theorem 7.1] that such a field K must necessarily be a field with free quotients, as defined in [20, Section 6].

3. **Twisted k-graph algebras.** In [12] and [11] it was described how a $\mathbb{T}$-valued 2-cocycle c on a k-graph Λ can be incorporated into the relations defining the associated C^*-algebra to obtain a twisted k-graph C^*-algebra $C^*(\Lambda, c)$. (Here $\mathbb{T}$ denotes the unit circle consisting of complex numbers of modulus one.) It was shown in [13] that if $C^*(\Lambda)$ is a Kirchberg algebra, then $C^*(\Lambda, c) \cong C^*(\Lambda)$ for any 2-cocycle c. In [7] the authors showed that for a k-graph Λ and a ring R one can define a "higher-rank Leavitt path algebra" $\mathrm{KPR}(\Lambda)$, which they call a Kumjian-Pask algebra over R. For a 2-cycle c it is possible to mimic the definition of a twisted k-graph C^*-algebra to define a "twisted Kumjian-Pask algebra" $\mathrm{KPR}(\Lambda, c)$. This raises the following question.

   Question: Let $\mathbb{C}$ denote the field of complex numbers. Does there exist a 2-graph Λ and a $\mathbb{T}$-valued 2-cocycle c on Λ such that $C^*(\Lambda)$ is a Kirchberg algebra, and $\mathrm{KPC}(\Lambda)$ is not isomorphic to $\mathrm{KPC}(\Lambda, c)$?

4. **Moves on Graphs.** If E and F are graphs with a finite number of vertices, it follows from results of [10], [19], and [22] that $C^*(E)$ is Morita equivalent to $C^*(F)$ if and only if the graph E may be transformed into the graph F using the following five moves and their inverses: (S) Source Removal, (O) Outsplitting, (I) Insplitting, (R) Reduction, and (CS) Cuntz Splice.

   Question: Is there a set of graph moves that generates Morita equivalence for all graph C^*-algebras? In other words, is there a set of graph moves such that if E and F are (possibly infinite) graphs, then $C^*(E)$ is Morita equivalent to $C^*(F)$ if and only if E may be transformed into F using these moves?

   Question: Is there a set of graph moves that generates isomorphism for graph C^*-algebras? In other words, is there a set of moves such that if E and F are (possibly infinite) graphs, then $C^*(E)$ is isomorphic to $C^*(F)$ if and only if E may be transformed into F using these moves? (An answer to this question would be interesting even under the hypotheses that E and F are finite graphs whose associated C^*-algebras are simple.)

5. **Dependence of Isomorphism Class on the Field.**

   Question: Do there exist graphs E and F and fields K and K' such that $L_K(E) \cong L_K(F)$, but $L_{K'}(E)$ is not isomorphic to $L_{K'}(F)$? Likewise, do there exist graphs E and F and a field K such that $L_K(E) \cong L_K(F)$, but $L_{\mathbb{Z}}(E)$ is not isomorphic to $L_{\mathbb{Z}}(F)$, where $\mathbb{Z}$ denotes the ring of integers?

   See [23] for the definition of and basic results for Leavitt path algebras over rings.

6. **The Isomorphism and Morita Equivalence Conjectures.** The following conjectures were first posed by Abrams and Tomforde in [2].

   The Isomorphism Conjecture: Let $\mathbb{C}$ denote the field of complex numbers. If E and F are graphs and if $L_{\mathbb{C}}(E)$ and $L_{\mathbb{C}}(F)$ are isomorphic as rings, then are $C^*(E)$ and $C^*(F)$ isomorphic as C^*-algebras?

   The Morita Equivalence Conjecture: Let $\mathbb{C}$ denote the field of complex numbers. If E and F are graphs and if $L_{\mathbb{C}}(E)$ and $L_{\mathbb{C}}(F)$ are Morita equivalent as rings, then are $C^*(E)$ and $C^*(F)$ strongly Morita equivalent as C^*-algebras?

   See [2] for the first appearance of these conjectures. Also see [2] and [21] for partial progress, and lists of classes of graphs where the conjectures are known to hold.

7. **Continuous Orbit Equivalence.** If E is a graph, the subalgebra $D(E)$ is defined to be the closure of $\mathrm{span}\{S_\alpha S_\alpha^* : \alpha \text{ is a finite path}\}$. One can show that if the graph E satisfies Condition (L), then $D(E)$ is a MASA of $C^*(E)$. Given a graph E, we let Σ_E denote the one-sided shift space consisting of one-sided infinite paths in E together with the canonical shift map. In [14, Theorem 1], Matsumoto

proves that if E and F finite graphs with no sinks and whose associated C^*-algebras are simple, then Σ_E and Σ_F are continuously orbit equivalent if and only if there an isomorphism $\phi : C^*(E) \to C^*(F)$ with $\phi : (D(E)) = D(F)$. (The definition of "continuously orbit equivalent" can be found in [14].) This raises the following questions.

Question: If E and F are graphs and $C^*(E) \cong C^*(F)$, then is there an isomorphism $\phi : C^*(E) \to C^*(F)$ with $\phi(D(E)) = D(F)$?

Question: If E and F are graphs and $C^*(E)$ is strongly Morita Equivalent to $C^*(F)$, then is there a Morita equivalence between $C^*(E)$ and $C^*(F)$ that preserves $D(E)$ and $D(F)$?

In [15, Theorem 4.3] Matsumoto proved that if E and F are finite graphs with no sinks and whose associated C^*-algebras are simple, and if the sign of $\det(I - A^t)$ is equal to the sign of $\det(I - B^t)$ (where A and B are the vertex matrices of E and F, respectively), then $C^*(E)$ is isomorphic to $C^*(F)$ if and only if Σ_E and Σ_F are continuously orbit equivalent. In [15, Section 6] Matsumoto states that there are no known examples of one-sided shifts Σ_E and Σ_F that are continuously orbit equivalent and with sign of $\det(I - A^t)$ not equal to the sign of $\det(I - B^t)$. This has led Matsumoto to make the following conjecture.

Matsumoto's Conjecture: If E is a graph with one-sided shift Σ_E and vertex matrix A, then $\det(I - A^t)$ is an invariant of the continuous orbit equivalence class of Σ_E.

Matsumoto points out in [15, Section 6] that if this conjecture is true, the triple

$$(K_0(C^*(E)), [1]_0, \det(I - A^t))$$

would be a complete invariant for the continuous orbit equivalence class of the one-sided shift Σ_E. This would imply that two one-sided topological Markov shifts Σ_E and Σ_F are continuously orbit equivalent if and only if the graph C^*-algebras $C^*(E)$ and $C^*(F)$ are isomorphic and $\det(I - A^t) = \det(I - B^t)$.

8. **Analogues of $\mathcal{O}_2 \otimes \mathcal{O}_2 \cong \mathcal{O}_2$.** A famous and important theorem of Elliott (which was exposited by Rørdam in [18]) states that $\mathcal{O}_2 \otimes \mathcal{O}_2 \cong \mathcal{O}_2$. It is natural to ask if an analogue of this result holds for the Leavitt algebra with two generators. If K is a field, and L_2 denotes the Leavitt algebra with two generators over K (i.e., the Leavitt path algebra of the graph with one vertex and two edges), then Ara and Cortiñas showed in [3] that $L_2 \otimes L_2$ and L_2 are not isomorphic, and indeed not even Morita equivalent. However, one can still ask the following question.

   Question: Is $L_2 \otimes L_2$ isomorphic to a subalgebra of L_2?

   Equivalently, one may ask if there exists an injective homomorphism $\phi : L_2 \otimes L_2 \to L_2$, and since $L_2 \otimes L_2$ is simple it suffices to produce a nonzero homomorphism $\phi : L_2 \otimes L_2 \to L_2$. Moreover, since eL_2e is isomorphic to L_2 for any nonzero idempotent e in L_2, the existence of a nonzero homomorphism from $L_2 \otimes L_2$ into L_2 implies the existence of a unital injective homomorphism from $L_2 \otimes L_2$ into L_2.

9. **L^p-versions of the Cuntz algebras and UHF algebras.** Chris Phillips has defined L^p-versions of the Cuntz algebras for $p \in [1, \infty]$, which are denoted by $\mathcal{O}_d^p$. Definitions and basic facts for these objects can be found in his paper [16] as well as the slides from his talk at our workshop. Phillips has created a long list of open problems, which we have made available in PDF form at the following URL:

   `http://www.math.uh.edu/~tomforde/GraphAlgebraProblems/Phillips-Lp-Problems.pdf`

   We highlight one problem from this list (see Problem 5.3 in the PDF), which is particularly intriguing.

   Question: For which values of $p \in [1, \infty]$ is it true that $\mathcal{O}_2^p \otimes \mathcal{O}_2^p$ is isomorphic to $\mathcal{O}_2^p$?

10. **One-sided and two-sided shift spaces.** If E is a finite graph with no sinks or sources, we let Σ_E denote the one-sided shift space consisting of one-sided infinite paths in E, and we let X_E denote the two-sided shift space consisting of two-sided bi-infinite paths in E. It is well known that for irreducible graphs, the Morita equivalence class of $C^*(E)$ is closely related to flow equivalence class of X_E (see [19]). In particular, if E and F are finite graphs with no sinks whose associated algebras are simple, then X_E is flow equivalent to X_F if and only if $C^*(E)$ is Morita equivalent to $C^*(F)$ and the sign

of $\det(I - A^t)$ is equal to the sign of $\det(I - B^t)$ (where A and B are the vertex matrices of E and F, respectively). One may ask how other dynamical properties of Σ_E and X_E are related to the isomorphism class and Morita equivalence class of $C^*(E)$.

Question: Let E and F be finite graphs with no sinks or sources. It is known that if the two-sided shifts X_E and X_F are conjugate, then there is a gauge-invariant Morita equivalence between $C^*(E)$ and $C^*(F)$. Can a converse, or partial converse, to this result be obtained? In other words, if there is a gauge-invariant Morita equivalence between $C^*(E)$ and $C^*(F)$, what additional hypotheses are needed to ensure that X_E and X_F are conjugate?

Question: Let E and F be finite graphs with no sinks. It is known that if the one-sided shifts Σ_E and Σ_F are conjugate, then there is a gauge-invariant isomorphism between $C^*(E)$ and $C^*(F)$. Can a converse, or partial converse, to this result be obtained? In other words, if there is a gauge-invariant isomorphism between $C^*(E)$ and $C^*(F)$, what additional hypotheses are needed to ensure that Σ_E and Σ_F are conjugate?

Question: Let E and F be finite graphs with no sinks. Is there a notion of flow equivalence for the one-sided shift Σ_E? If not, can one be developed? Is the flow equivalence class of Σ_E related to the isomorphism class or the Morita equivalence class of $C^*(E)$?

11. **Separated Graph Algebras.** In [4] and [5] Ara and Goodearl introduced separated Leavitt path algebras (generalizing the Leavitt algebras $L(m,n)$), and they also introduced their C^*-algebra counterparts. If (E,C) is a separated graph, we let $M(E,C)$ denote the abelian monoid with generators $\{av : v \in E^0\}$ satisfying the relations $a_v = \sum_{e\in X} a_{r(e)}$ for all $v \in E^0$ and all $X \in C_v$. It was shown in [5, Theorem 4.3] that there is a natural map $M(E,C) \to V(L(E,C))$ sending a_v to $[v] \in V(L(E,C))$, where $V(L(E,C))$ is the abelian monoid of Murray-von Neumann equivalence classes of projections in matrices over $L(E,C)$. If we let $V(C^*(E,C))$ denote the abelian monoid of Murray-von Neumann equivalence classes of projections in matrices over $C^*(E,C)$, there is a similar natural map $M(E,C) \to V(C^*(E,C))$ sending a_v to $[v] \in V(C^*(E,C))$.

    Question: Is the natural map $M(E,C) \to V(C^*(E,C))$ an isomorphism? (This is equivalent to asking if the natural map $V(L(E,C)) \to V(C^*(E,C))$ is an isomorphism.)

    The result is certainly true for non-separated graphs [6, Theorem 7.1]. In addition, if the answer to this question is positive, it would follow, as in [5, Corollary 4.5], that every conical abelian monoid is isomorphic to $V(C^*(E,C))$ for some finitely separated graph (E,C). If the answer is negative, one would still like to know whether this map is always injective.

12. **Phantom Cuntz-Krieger Algebras.** If A is a C^*-algebra, we say that A "looks like a Cuntz-Krieger algebra" if all of the following conditions are satisfied:

    - A is unital, purely infinite, nuclear, separable, and has real rank zero.
    - A has finitely many ideals.
    - Every subquotient of A has finitely generated K_0-group, finitely generated and free K_1-group, and rank of the K_0-group equal to rank of the K_1-group.
    - The simple subquotients of A are in the bootstrap class of Rosenberg and Schochet.

    A "phantom Cuntz-Krieger algebra" is then defined to be a C^*-algebra that looks like a Cuntz-Krieger algebra but is not isomorphic to a Cuntz-Krieger algebra.

    Question: Do phantom Cuntz-Krieger algebras exist?

    Arklint has obtained a number of results concerning phantom Cuntz-Krieger algebras in [8]. She has also shown that simple phantom Cuntz-Krieger algebras do not exist [8, Corollary 3.3], and that phantom Cuntz-Krieger algebras with exactly one ideal do not exist [8, Corollary 3.7].

4 Outcome of the Meeting

This workshop provided researchers in graph algebras the opportunity to share results, initiate new collaborations, and learn more about techniques used in the subject. In particular, having groups of researchers from both functional analysis and algebra fostered great cross-fertilization of ideas, promoted collaboration between researchers who do not frequently meet with each other, and led to a deeper understanding of phenomena found in both areas.

The algebraists and functional analysts in attendance each worked to educate members from the other group about techniques in their own area. In addition, participants who were knowledgeable about symbolic dynamics were very productive in teaching members of both groups about invariants and notions of equivalence from dynamics that could be useful in further graph algebra investigations.

We also point out that although many of the functional analysts had been to BIRS before (presumably because of Canada's prominence in functional analysis and C^*-algebras), there were at least four very established algebraists who said this was their first time at BIRS. Consequently, this workshop allowed BIRS to influence mathematicians that the station has not reached previously, and provided an opportunity for these prominent researchers to contribute to the BIRS environment.

Finally, by creating a list of major open problems in the field of graph algebras and using these problems to create the "Graph Algebra Problem Page" website, this workshop has helped to direct future research in the subject. The problems collected give researchers a summary of the current directions of investigations, and highlight important issues that must be addressed for the subject to move forward.

BIRS support for this workshop has provided an invaluable contribution to the understanding of graph algebras, and has helped to promote interaction between researchers in functional analysis and algebra. This has allowed for communication and collaboration between groups whose work has often been done in parallel.

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