Numerical study of a normally hyperbolic cylinder in the RTBP

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Mechanism of Instability

- Consider the three-body problem consisting of the Sun, Jupiter, and an Asteroid which moves on (approximate) ellipses.

- A possible source of instabilities are orbital resonances between the frequencies of Jupiter and the Asteroid.

- Jupiter and the Asteroid are regularly in the same relative position. Over a long time interval, Jupiter’s influence piles up and modifies the eccentricity of the Asteroid.

- According to Kepler’s third law, resonances take place when the semi-major axis $a$ satisfies

\[
a^{3/2} \approx \frac{\omega_J}{\omega_A} \in \mathbb{Q}.
\]
Kirkwood Gaps

- The Asteroid Belt is located between the orbits of Mars and Jupiter. The distribution of asteroids presents several gaps precisely at the resonances.
Kirkwood Gaps

- It is believed that these gaps are due to instability mechanisms.
- This motivates us to study the 3:1 resonance

\[ a^{3/2} \approx \frac{\omega_J}{\omega_A} = \frac{1}{3}. \]
Theorem 1 (FGKR, 2011) Consider the elliptic RTBP with mass ratio $\mu = 10^{-3}$ and eccentricity of Jupiter $e_0 > 0$.

For $e_0$ small enough, there exist $T > 0$ and a trajectory whose eccentricity $e(t)$ satisfies

$$e(0) < 0.55 \quad \text{and} \quad e(T) > 0.85,$$

while its semi-major axis stays almost constant

$$a(t) \approx 3^{-2/3}.$$
Summary of Proof

1. Prove the existence of a normally hyperbolic invariant cylinder $\Lambda$, which exists near the resonance.

2. Establish transversality of its stable and unstable invariant manifolds.

3. Compare inner and outer dynamics on $\Lambda$, and check that they do not have invariant circles.

4. Construct diffusing orbits by shadowing a composition of outer and inner maps.
• When $\mu > 0$, all known analytical techniques fail to estimate the splitting of separatrices (even for $e_0 = 0$).

• We set $\mu = 10^{-3}$, and we show numerically that the splitting is not too small.

• Since the splitting varies smoothly with respect to $e_0$, it suffices to estimate the splitting for $e_0 = 0$ (i.e. for the circular problem)!!
Ansatz 1 Consider the circular RTBP with mass ratio $\mu = 10^{-3}$ and Hamiltonian $H$.

In each energy level $H \in [H_-, H_+]$ there exists a hyperbolic periodic orbit $\lambda_H(t)$ which satisfies

$$|L_H(t) - 3^{-1/3}| < 50\mu \quad \text{for all} \quad t \in \mathbb{R}.$$ 

Each $\lambda_H$ has two branches of stable and unstable invariant manifolds $W^{s,j}(\lambda_H)$ and $W^{u,j}(\lambda_H)$ for $j = 1, 2$. For each $H \in [H_-, H_+]$ either

$$W^{s,1}(\lambda_H) \cap W^{u,1}(\lambda_H) \text{ transversally}$$

or

$$W^{s,2}(\lambda_H) \cap W^{u,2}(\lambda_H) \text{ transversally.}$$
Comments

• We verify the Ansatz numerically.

• Numerical analysis has several sources of error:
  – roundoff errors in computer arithmetic,
  – numerical approximation of ideal objects.

We evaluate such errors and check that they are appropriately small.

• Goal: to keep our numerics simple and convincing.

• Roldán and Zgliczynski are working towards a fully rigorous Computer-Assisted proof.
Choice of Coordinates

• Circular RTBP in rotating Cartesian coordinates

\[ H(x, y, p_x, p_y) = \frac{1}{2}(p_x^2 + p_y^2) + yp_x - xp_y - \frac{\mu_1}{r_1} - \frac{\mu_2}{r_2}, \]

\[ r_1^2 = (x - \mu_2)^2 + y^2, \]
\[ r_2^2 = (x + \mu_1)^2 + y^2. \]

• Sun is located to the left of the origin: \( \mu_1 = \mu \) is the small mass and \( \mu_2 = 1 - \mu \) is the large mass.
Symmetries of the System

- The system is reversible with respect to the involution

\[ R(x, y, p_x, p_y) = (x, -y, -p_x, y). \]

- Thus, a solution is symmetric if and only if it intersects the symmetry plane

\[ \{y = 0, \ p_x = 0\} \equiv \{y = 0, \ \dot{x} = 0\}. \]
Conservation of Energy

- The circular problem has a conserved quantity, the Jacobi constant $C$.
- When the Hamiltonian is constant $H = H_0$, we have

\[ H_0 = -\frac{C' - \mu_1 \mu_2}{2}. \]

- We will refer to $H_0$ as the energy of the system.
- It is natural to fix $H = H_0$ and perform our analysis for $H_0$. Then, we let $H$ vary and repeat our computations for $H \in [H_-, H_+]$. 
Computation of Periodic Orbits

- Fix $H = H_0$, and look for an (almost) resonant periodic orbit $\lambda_H(t)$ in this level of energy.

- As a first approximation, consider the 2BP and look for the resonant periodic orbit $\tilde{\lambda}_H(t)$ in the level of energy $H_{2BP} = H_0$.

- To simplify numerics, we choose a symmetric periodic orbit.

- Refine $\tilde{\lambda}_H(t)$ into $\lambda_H(t)$ in the R3BP using a Newton method.
Poincaré Map

• Consider the RTBP in Cartesian coordinates.

• Define the Poincaré section

\[ \Sigma_+ = \{ y = 0, \; \dot{y} > 0 \} \]

with Poincaré map

\[ P : \Sigma_+ \rightarrow \Sigma_+. \]

• On the section, the variable \( p_y \) can be eliminated. We can recover it from the energy condition

\[ H(x, y, p_x; p_y) = H_0, \]

since \( \partial_{p_y} H = \dot{y} \neq 0. \)

• Hence, at each energy level, \( P = P(x, p_x) \) is a 2-dimensional symplectic map.
Fixed Point Equation

- In the rotating frame, a 3:1 resonant periodic orbit makes 2 turns around the origin.

- One can look for a periodic point \( a = (x, p_x) \) of the Poincaré map
  \[
a = P^2(a),
  \]
  or equivalently, a fixed point of the \textit{iterated Poincaré map} \( \mathcal{P} \)
  \[
a = \mathcal{P}(a).
  \]

- However, we want a \textit{symmetric} periodic orbit. Thus, after half a period, it must intersect the symmetry plane \( \{y = 0, p_x = 0\} \):
  \[
  \Pi_{p_x} \circ P(a) = 0.
  \]

- Solve this 1-d equation using a Newton method.
Family of Periodic Orbits

- Finally, let $H$ vary in the range $[H_-, H_+] = [-1.733, -1.405]$ to obtain the family of (almost) resonant periodic orbits

$$\Lambda_0 = \bigcup_{H \in [H_-, H_+]} \lambda_H.$$ 

- $\Lambda_0$ is a family of symmetric periodic orbits around the Sun.
- Accuracy in the computation of periodic orbits: $10^{-14}$. 

Family of Periodic Orbits

\[ H = -1.733, \quad C = 3.467 \]

[Graph showing a circle with labeled axes and coordinates]
Family of Periodic Orbits

H = -1.729, C = 3.460
Family of Periodic Orbits

\[ H = -1.719, \ C = 3.439 \]
Family of Periodic Orbits

$H=-1.640$, $C=3.281$
Family of Periodic Orbits

H = -1.535, C = 3.071
Family of Periodic Orbits

$H = -1.456, \ C = 2.913$

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Family of Periodic Orbits

\[ H = -1.405, \ C = 2.811 \]
In the Loop

• When $H \approx -1.6$, the periodic orbit develops loops. The reason is the following:

• Near the apohelion, the sidereal velocity of Asteroid becomes smaller than the velocity of rotating frame $\Rightarrow$ relative velocity is negative, and orbit is direct.

• At other parts of the orbit, the sidereal velocity of Asteroid is larger than the velocity of rotating frame $\Rightarrow$ relative velocity is positive, and orbit is retrograde.

• Loops are inherent to this resonant family of periodic orbits in the rotating system, even for the 2BP.
**In the Loop**

- When the loops appear, there is one more iterate of the Poincaré map. However, the family is continuous with respect to the period $T_H$.

- This is an artifact produced by rotating coordinates. One can get rid of this technical problem by redefining the Poincaré map in a suitable way.
Numerical Bounds

- The period stays close to the resonant period of the unperturbed system
  \[ |T_H - 2\pi| < 15\mu. \]
- \( L_H(t) \) stays close to the resonant value \( 3^{-1/3} \):
  \[ \max_{t \in [0,T_H]} |L_H(t) - 3^{-1/3}| < 50\mu. \]
Stability of Periodic Orbits

- Compute eigenvalues $\lambda, \lambda^{-1}$ of $D\mathcal{P}(a)$. 

![Graph showing log(λ) vs. H]


Stability of Periodic Orbits

- The family of periodic orbits is
  - less hyperbolic when $H \to H_-$, or equivalently $e \to 0$.
  - more hyperbolic when $H \to H_+$, or equivalently $e \to 1$.

- Since the system is close to integrable ($\mu$ is small), one expects eigenvalues $\lambda, \lambda^{-1}$ close to unity.

- Nevertheless, non-integrability is noticeable in the picture. This is due to the effect of the perturbing body (Jupiter) on the Asteroid.
Computation of Invariant Manifolds

- Fix \( H = H_0 \), and look for the (1-d) invariant manifolds \( W^u(a), W^s(a) \) of the hyperbolic fixed point \( a \) in this level of energy.

- Approximate the local invariant manifolds using a linear segment. The error committed in the linear approximation is controlled:

\[
err(\eta) = \| P(a + \eta v) - (a + \lambda \eta v) \| \in O(\eta^2).
\]

- Globalize the manifolds using the Poincaré map.

- Choose a displacement \( \eta \) such that \( err(\eta) < 10^{-8} \) uniformly in \( H \).
Invariant Manifolds for $H = -1.733$

The graph shows the invariant manifolds for the specified value of $H$. The graph includes:

- Unstable manifold
- Stable manifold
- Symmetry axis

The graph also highlights points $a_1$ and $a_2$.
New Poincaré Section

- Notice that the fixed points $a_1, a_2$ are in the symmetry plane by construction.

- Unfortunately, the homoclinic points are not in the symmetry plane.

- Consider the new Poincaré section

$$\Sigma_- = \{y = 0, \dot{y} < 0\}.$$  

- In the new section $\Sigma_-$, the fixed points $a_1, a_2$ are reversible:

$$R(a_1) = a_2.$$  

Hence, the homoclinic points are now in the symmetry plane.
Invariant Manifolds on the section $\Sigma$.
Homoclinic Points

- Thanks to reversibility, the intersection of the manifolds with the symmetry axis $p_x = 0$ is a homoclinic point.

- We consider two homoclinic points:
  - $z_1$ corresponds to the “inner” splitting,
  - $z_2$ corresponds to the “outer” splitting.

- Compute $z_1, z_2$ using a standard bisection method.

- We verify that $z_1, z_2$ lie on the symmetry axis with tolerance $10^{-10}$ uniformly in $H$. 
Inner Splitting for $H = -1.405$
Computation of Splitting Angle

- Look for the tangent vectors $w_u$ and $w_s$ to the manifolds at $z$. The splitting angle is the oriented angle between them.

- We use two different methods to compute the tangent vectors at $z$. This way we can validate the numerical accuracy of the splitting angle.
**First Method**

- Let $p_0 \in W^u_{\text{loc}}(a)$ be the preimage of the homoclinic point $z$ in the local manifold

\[ \mathcal{P}^n(p_0) = z. \]

- Let $v_0$ be the tangent vector to the manifold at $p_0$ (i.e. the eigenvector).

- Transport $v_0$ by the Jacobian $D\mathcal{P}$ at the successive iterates of $p_0$

\[ w_u = \prod_{i=0}^{n-1} D\mathcal{P}(p_i)v_0. \]
Second Method

- Let \( z = (x^*, 0) \) be the homoclinic point.
- Look at the manifold \( W^u(a) \) as a graph over the vertical line \( x = x^* \).
- Sample the manifold \( W^u(a) \) at different values of \( p_x \):
  \[
p_x = \frac{j}{10^5}, \quad j \in (-2, -1, 1, 2).
\]
- Apply numerical differentiation to these values, using central differences centered at \( p_x = 0 \):
  \[
d_1 = \frac{x(0.00001) - x(-0.00001)}{0.00002},
  \]
  \[
d_2 = \frac{x(0.00002) - x(-0.00002)}{0.00004}.
\]
- Use Richardson extrapolation to improve the precision of derivative:
  \[
d = \frac{4d_1 - d_2}{3}.
\]
Accuracy of Computations

- Let $H = H_0 = -1.405$, for example.

- According to the first method, the splitting angle is $\sigma^{(1)} = -9.780327341442923e - 05$.

- According to the second method,

<table>
<thead>
<tr>
<th>$p_x$</th>
<th>$x^u$</th>
</tr>
</thead>
<tbody>
<tr>
<td>-0.00002</td>
<td>$-8.703373796876306e - 02$</td>
</tr>
<tr>
<td>-0.00001</td>
<td>$-8.703373845681261e - 02$</td>
</tr>
<tr>
<td>0.00001</td>
<td>$-8.703373943484494e - 02$</td>
</tr>
<tr>
<td>0.00002</td>
<td>$-8.703373992482412e - 02$</td>
</tr>
</tbody>
</table>
\[
d_1 = -4.890161608983589e\times05
\]
\[
d_2 = -4.890152657810453e\times05
\]
\[
d = -4.890164592707968e\times05
\]
\[
\sigma^{(2)} = -9.780329177619804e\times05
\]

- Compare the splitting angle computed using the two methods:

\[
\sigma^{(1)} = -9.780327341442923e\times05,
\]
\[
\sigma^{(2)} = -9.780329177619804e\times05.
\]

They differ by less than $10^{-10}$ (total numerical error).
Validation of Splitting Angle

- The splitting angle is several orders of magnitude larger than the total numerical error for a large range of energies $H \approx [-1.6, -1.4]$. 

![Graph showing the relationship between $H$ and $\sigma$]