Non-integrability criterion for homogeneous Hamiltonian systems via blowing-up theory of singularities

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Hamiltonian system and its integrability

- Hamiltonian system:

\[
\frac{dq_j}{dt} = \frac{\partial H}{\partial p_j}(p, q), \quad \frac{dp_j}{dt} = - \frac{\partial H}{\partial q_j}(p, q) \quad (j = 1, \ldots, k)
\]

where \( p = (p_1, \ldots, p_k), q = (q_1, \ldots, q_k), H : \mathbb{R}^{2k} \rightarrow \mathbb{R} \).

- Hamiltonian system (1) is integrable \iff there are \( k \) first integrals \( F_1 (= H), F_2, \ldots, F_k \) such that \( dF_1, \ldots, dF_k \) are linearly independent a.e. and that \( \{F_i, F_j\} = 0 \) for any \( i, j = 1, \ldots, k \).

- The dynamics of the integrable Hamiltonian systems are well understood because of the Liouville-Arnold theorem.

- The dynamics of the non-integrable Hamiltonian systems may be “chaotic”.

- Problem: distinguish between integrable and non-integrable Hamiltonian systems.
Brief history

• Bruns (1887) proved an algebraic first integral in the 3BP.
• Poincaré (around 1890) proved an analytic first integral in the R3BP.
• Kovalevskaya (1889) discovered an integrable parameter in the rigid body model by focusing on the property of the singularity.
• Ziglin (1982 –) provided a criterion for non-integrability by using Monodromy matrix.
• Yoshida (1986 –) provided a criterion for non-integrability of homogeneous Hamiltonian systems.
• Morales-Ruiz & Ramis (1999 –) extended the Ziglin analysis by using the differential Galois theory.
• Maciejewski (2011) proved meromorphic non-integrability of the P3BP for any masses by applying the Morales-Ramis theory.
Goal

- Goal: give a criterion of the non-integrability of the homogeneous Hamiltonian systems with two degrees of freedom from a new approach.
Homogeneous Hamiltonian system

Consider a homogeneous Hamiltonian system with two degrees of freedom:

\[ H(p, q) = \frac{1}{2} \|p\|^2 + U(q) \quad ((p, q) \in \mathbb{R}^2 \times \mathbb{R}^2) \]

where \( U \) is a homogeneous potential with degree \( \beta (\in \mathbb{R}) \):

\[ U(\lambda q) = \lambda^\beta U(q) \quad (\forall q \in \mathbb{R}^2 \setminus \{0\}, \forall \lambda > 0). \]

Let \( V(\theta) = U(\cos \theta, \sin \theta) \).
Example (The isosceles three-body problem)

Consider the isosceles three-body problem.

Assume that $m_1 = m_2, m_3 = \alpha m_1$.

This model is governed by the homogeneous Hamiltonian system with the potential energy

$$U(q) = -\frac{1}{q_1} - \frac{4\alpha^{3/2}}{\sqrt{\alpha q_1^2 + (\alpha + 2)q_2^2}}.$$ 

$\beta = -1$.

$$V(\theta) = -\sec \theta - \frac{4\alpha^{3/2}}{\sqrt{\alpha + 2 \sin^2 \theta}}.$$
Main result

**Theorem** Assume the following:

1. $\beta \in \mathbb{R} \setminus \{-2, 0\}$;
2. $\exists \theta_{-1} < \exists \theta_0 < \exists \theta_1$ s.t. $\frac{\partial V}{\partial \theta} (\theta_l) = 0$;
3. $V(\theta) < 0$ on $[\theta_{-1}, \theta_1]$;
4. $\frac{\partial V}{\partial \theta} (\theta) \neq 0$ on $(\theta_{-1}, \theta_0) \cup (\theta_0, \theta_1)$;
5. $\frac{\partial^2 V}{\partial \theta^2} (\theta_{\pm1}) < 0$;
6. $-\frac{1}{8} (\beta + 2)^2 V(\theta_0) < \frac{\partial^2 V}{\partial \theta^2} (\theta_0)$.

Then the homogeneous Hamiltonian system has no meromorphic first integral independent from $H$.
Remark

In the case of $\beta = -2$, the Hamiltonian system is always integrable. Because a function

$$G(p, q) = (q \cdot p)^2 - 2\|q\|^2 H(p, q)$$

is a first integral independent from $H$. 
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McGehee coordinates

We mainly consider the case of $\beta < 0$.

McGehee coordinates: $(r, \theta, v, w)$ and $\tau$

$$q = r(\cos \theta, \sin \theta),$$

$$p = r^{\beta/2}(v(\cos \theta, \sin \theta) + w(-\sin \theta, \cos \theta))$$

$$dt = r^{1-\beta/2}d\tau.$$

Then the canonical equations become

$$\frac{dr}{d\tau} = rv$$

(2)

$$\frac{d\theta}{d\tau} = w$$

(3)

$$\frac{dv}{d\tau} = -\frac{\beta}{2}v^2 + w^2 - \beta V(\theta)$$

(4)

$$\frac{dw}{d\tau} = -\left(\frac{\beta}{2} + 1\right)vw - \frac{\partial V}{\partial \theta}(\theta)$$

(5)

$q = 0$ is singularity but $r = 0$ is not singular in these differential equations (2)-(5).
Energy and Collision manifold

In these coordinates the total energy is

\[ h = r^\beta \left( \frac{v^2 + w^2}{2} + V(\theta) \right). \]  \hspace{1cm} (6)

We fix \( h \neq 0 \) and regard \( r \) as a function of \((\theta, v, w)\).

We consider the 3-dimensional dynamics.

The set

\[ \mathcal{M} = \left\{ (\theta, v, w) \mid \frac{v^2 + w^2}{2} + V(\theta) = 0 \right\} \]

is invariant. In the case of the \( n \)-body problem, \( \mathcal{M} \) is called collision manifold.

Since we fix the energy, as \( r \to 0 (q \to 0) \), the orbit converges to \( \mathcal{M} \) in the McGehee coordinates.
Equilibrium points

Recall that $\theta_\ell$ are a critical point of $V$, i.e. $\frac{\partial V}{\partial \theta}(\theta_\ell) = 0$.

Then $D_\ell^\pm = (\theta_\ell, \pm \sqrt{-2V(\theta_\ell)}, 0) \in \mathcal{M}$ are equilibrium points.
The case of the isosceles three-body problem

The invariant manifold (collision manifold) $\mathcal{M}$ for the isosceles three-body problem is like this figure:
Case of $\beta > 0$

In the case of $\beta > 0$, we replace $r$ with $R = r^{-1}$.

The equation $\frac{dr}{d\tau} = rv$ becomes $\frac{dR}{d\tau} = -Rv$.

We can define an invariant manifold corresponding to $R \to 0$ and we can discuss a similar argument as the case of $\beta < 0$. 
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Proof (homogeneous property)

We give the outline of the proof for $-2 < \beta < 0$. The other cases are similar (some signs change in the computation).

Assume that $\Phi(p, q)$ is a moromorphic first integral where $(p, q)$ are the original coordinates.

From the homogeneous property (if $(p(t), q(t))$ is a solution, so is $(c^{\beta/2}p(c^{\beta/2-2}t), c^q(c^{\beta/2-2}t))$ for any constant $c > 0$), we can assume that $\Phi$ satisfies $\Phi(c^{\beta/2}p, c^q) = c^\rho \Phi(p, q)$ without loss of generality.

In the McGehee coordinates, this property corresponds to the fact that $\Phi$ can be represented as $\Phi = r^\rho g(\theta, v, w)$. 
Proof(Coordinates)

We use the coordinates \((\theta, z, w)\) where \(z = \frac{v^2+w^2}{2} + V(\theta)\). These are analytic near the equilibrium points. The energy is

\[
h = r^\beta z. \tag{7}
\]

We consider the Laurent series of \(g\) at \(z = 0\) with respect to \(z\):

\[
g = \sum_{k=\mu}^{\infty} \gamma_k(\theta, w) z^k \quad (\gamma_\mu \neq 0).
\]

From (7), we get \(\Phi = \left(\frac{\hbar}{z}\right) \frac{\rho}{2\beta} \sum_{k=\mu}^{\infty} \gamma_k(\theta, w) z^k\).

The lowest order of \(z\) is \(\mu - \frac{\rho}{2\beta}\).
Proof (the case of $\mu - \frac{\rho}{2\beta} < 0$)

We first consider the case of $\mu - \frac{\rho}{2\beta} < 0$.

**Lemma:** $\gamma_\mu$ is zero on $W^u(D^-_l)$.

$W^u(D^-_0)$ is an open set of $\mathcal{M}$. Hence $\gamma_\mu \equiv 0$.

This contradicts the assumption.
Proof (the case of $\mu - \frac{\rho}{2\beta} > 0$)

We consider the case of $\mu - \frac{\rho}{2\beta} > 0$.

**Lemma**: $\gamma_\mu$ is zero on $W^s(D^{-}_l)$

From assumption 6: \((\frac{-1}{8}(\beta + 2)^2V(\theta_0) < \frac{\partial^2 V}{\partial \theta^2}(\theta_0))\), the dynamics near $D^{-}_0$ on $\mathcal{M}$ is unstable focus. $W^s(D^{-}_1)$ is a spiral curve near $D^{-}_0$. Hence $\gamma_\mu \equiv 0$. 

![Diagram](image-url)
Proof (the case of $\mu - \frac{\rho}{2\beta} = 0$)

In the case of $\mu - \frac{\rho}{2\beta} = 0$,

Lemma: $\gamma_\mu$ is a constant on $W^{s/u}(D^-_l)$.

Therefore $\gamma_\mu \equiv c$. If $\Phi$ is not constant, by considering $\Phi - c$, this case can be reduced to the case of $\mu - \frac{\rho}{2\beta} > 0$. 
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Non-integrability of the isosceles three-body problem

The function $V$ of this problem is

$$V(\theta) = -\sec \theta - \frac{4\alpha^{3/2}}{\sqrt{\alpha + 2\sin^2 \theta}}.$$ 

By applying our theorem, we obtain the following:

Theorem 2

Assume that $\alpha < \frac{55}{4}$. Then the isosceles three-body problem is non-integrable. i.e. there is no meromorphic first integral independent from the energy.
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Yoshida coefficient

We call a point $c \in \mathbb{R}^2$ the Darboux point if $\nabla U(c) = c$. In the case of $n$-body problem, $c$ is called a central configuration. The eigenvalues of the Hessian matrix $D^2 U(c)$ at the Darboux point $c$ are called the Yoshida coefficients.

Since $U(c)$ is homogeneous with degree $\beta$, one of Yoshida coefficients is $\beta - 1$.

The other Yoshida coefficient is

$$\lambda = \beta^{-1} V(\theta_c)^{-1} \frac{\partial^2 V}{\partial \theta^2}(\theta_c) + 1$$

in the polar coordinates where $\frac{\partial V}{\partial \theta}(\theta_c) = 0$. 
Yoshida coefficient and integrability

The Morales-Ramis theorem (the differential Galois theory) proves non-integrability if one of the Yoshida coefficient is not in a certain set of rational numbers. For example, in the case of $\beta = -1$, according to the Moreles-Ramis theorem, the homogeneous Hamiltonian system is non-integrable if $\lambda$ is not in

$$\left\{-\frac{1}{2}p(p-3) \mid p \in \mathbb{Z}\right\} = \{1, 0, -2, -5, -9, \ldots \}.$$ 

In our theorem the assumption 6 is

$$-\frac{1}{8}(\beta + 2)^2 > (\lambda - 1)\beta \quad \text{for} \quad (\lambda > 9/8 \text{ if } \beta = -1).$$

In the case of the isosceles three-body problem,

- Our theorem: non-integrability for $\alpha < \frac{55}{4}$
- M-R theory: non-integrability for any $\alpha$. 
Our theorem v.s. Morales-Ramis theory

- Our theorem can be applied to $\beta \in \mathbb{R}\setminus\{-2, 0\}$ while M-R theory can be applied to $\beta \in \mathbb{Z}\setminus\{-2, 0\}$.
- In the case of integer $\beta$, M-R theory is stronger.
- Our theorem can be applied to two degrees of freedom while M-R theory can be applied to any degrees of freedom.
- Our function class of first integrals is bigger: we prove the non-existence of first integral which is meromorphic as a real function, while M-R theory prove the non-existence of first integrals which is meromorphic as a complex function.
- Our proof is simpler and based on dynamics (the behavior of stable and unstable manifolds). M-R’s method is far from the theory of the dynamics.
Thank you for your attention.