Dynamics of some symmetric $n$-body problems

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Introduction

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Remarks
We will try to study $n$-body problems which are symmetric with respect to the action of suitable extensions of finite rotation groups\(^{(1)}\). The space of symmetric configurations is the complement of an arrangement of linear subspaces in a Euclidean space, and blow-up, McGehee coordinates and variational methods can –in some cases– be used to understand local dynamics (around the space of collisions) and some properties of periodic orbits.

**Masses:** $m_1, m_2, \ldots, m_n > 0$

**Positions:** $q_1, q_2, \ldots, q_n \in \mathbb{R}^d$

**Homogeneity:** $-\alpha < 0$

**Potential:** \[
\sum_{i<j} \frac{m_i m_j}{\|q_i - q_j\|^\alpha}
\]

**Symmetries**

Two basic types of symmetries:

→ Involving time
  
  ➤ $t \mapsto t + \delta$: $x(t + \delta) = g x(t)$; 
  
  ➤ $t \mapsto -t$: $x(-t) = g x(t)$;

→ Not involving time $\forall t, x(t) \in X^G$.

Examples:

→ Antipodal symmetry $x(t + \delta) = -x(t)$.
→ Devaney isosceles\(^{(2)}\).
→ Sitnikov.
→ Chenciner Montgomery figure-eight and choreographies.
→ Delgado, Vidal, Venturelli, Ferrario, Terracini, Simò, Martinez, Chen, Salomone, Xia, Gronchi, Negrini, Fusco, ...

Two symmetric 3-choreographies
Consider now finite subgroups of $O(2)$ (planar case) and $SO(3)$ (spatial case). Recall the classification of such groups (*point groups*):

→ **Plane**:
  - Cyclic groups $C_n \subset SO(2)$ (of order $n$);
  - Dihedral groups $D_n \subset O(2)$ (of order $2n$).

→ **Space**:
  - Cyclic $C_n$ (of order $n$);
  - Dihedral $D_n$ (of order $2n$);
  - Tetrahedral $T \cong A_4$ (of order 12);
  - Octahedral $O \cong S_4$ (of order 24);
  - Icosahedral $Y \cong A_5$ (of order 60).

For subgroups of $O(3)$, one obtains full groups adding to the above the *inversion* $a : x \mapsto -x$, (which is in the center of $SO(3)$) and yields full groups $I \times C_n, I \times D_n$, with $I = \{1, a\} \ldots$ or the groups of *mixed type* (those without the inversion $a$).
Now consider a rotation group $K \subset SO(3)$ of order $n$, and $n$ bodies with equal masses “naturally” symmetric with respect to $K$. Here “naturally” means that the permutation action on $\{1, \ldots, n\}$ is the (natural) Cayley left action of $K$ on $K \approx \{1, \ldots, n\} \approx K$ by assigning indices to the elements of $K$. For each $g$, there exists a corresponding permutation $\sigma \in S_n$ defined by $gg_i = g_{\sigma_i}$. In other words, if $K = \{g_1, \ldots, g_n\}$, we consider configurations of $n$ points (with equal masses) $q_1, \ldots, q_n \in \mathbb{R}^3$. If $X$ is the $3n$-dimensional configuration space, then the induced symmetry $g: X \rightarrow X$ is defined by

$$
g \cdot (q_1, \ldots, q_n) = (gq_{\sigma^{-1}(1)}, gq_{\sigma^{-1}(2)}, \ldots, gq_{\sigma^{-1}(n)}) \cdot
$$

The space of symmetric configurations hence is

$$X^K = \{x \in X : Kx = x\}$$

$$= \{x = (q_1, \ldots, q_n) : q_i = g_i g_j^{-1} q_j \} \approx \{q_1\} = \mathbb{R}^3$$
(Configuration spaces)
(CONFIGURATION SPACES)
Dynamics of some symmetric $n$-body problems
(Configuration spaces)
Consider the binary collision subspace $\Delta_{ij} = \{q_i = q_j\} \subset X$. The projection $\pi_{ij}$ onto $\Delta_{ij}$ given by

$$
\pi_{ij}(x) = \pi_{ij}(q_1, \ldots, q_i, \ldots, q_j, \ldots, q_n)
= (q_1, \ldots, \frac{m_i q_i + m_j q_j}{m_i + m_j}, \ldots, \frac{m_i q_i + m_j q_j}{m_i + m_j}, \ldots, q_n)
$$

is well-defined, and orthogonal with respect to the mass-metric on $X$. Now, observe that if $\|x\|_M$ denotes the mass-metric on $X$

$$
\|x - \pi_{ij}(x)\|_M^2 = m_i \|q_i - \frac{m_i q_i + m_j q_j}{m_i + m_j}\|^2 + m_j \|q_j - \frac{m_i q_i + m_j q_j}{m_i + m_j}\|^2
= \ldots = \frac{m_i m_j}{m_i + m_j} \|q_i - q_j\|^2
$$
The potential
\[ \sum_{i<j} \frac{m_im_j}{\|q_i - q_j\|^\alpha} \]
can be therefore written as
\[ \sum_{i<j} \frac{(m_i + m_j)^{-\alpha/2}(m_im_j)^{1+\alpha/2}}{\|x - \pi_{ij}(x)\|_M}. \]

It is a weighted sum of powers of distances from \( x \) to binary collision subspaces \( \Delta_{ij} \).

Its restriction to symmetric configurations \( X^K \subset X \) (all equal masses at the moment, but it can be easily generalized, e.g. isosceles or Sitnikov or multiple choreographies or ...)? If \( x \in X^K \), in general it is not true that \( \pi_{ij}(x) \in X^K \), but it happens that again it is a weighted sum of powers of distances from subspaces.
The subgroup $H$ ranges over all the isotropy subgroups of $K$. The orthogonal projection $p_H : E \to E_H$ project the configuration space $E$ onto the subspace $E_H$ fixed by $H$, and $C_H$ is a corresponding positive coefficient.
The subgroup $H \subset K$ ranges over all the isotropy subgroups of $K$. The orthogonal projection $\pi_H : E \rightarrow E^H$ project the configuration space $E$ onto the subspace $E^H$ fixed by $H$, and $C_H$ is a corresponding positive coefficient.

$$U = \sum_{H \subset K} \frac{C_H}{\|q - \pi_H(q)\|^\alpha}$$
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Let $q, p$ be the canonical coordinates, $(q, p) \in \text{phase space}$. Since $U$ is $-\alpha$-homogeneous, in McGehee coordinates (with mass-metric $\| \cdot \| = \| \cdot \|_M$) $\rho = \|q\|$, $s = \rho^{-1}q$, $z = \rho^{\alpha/2}p$ after rescaling time and defining $v = \langle z, s \rangle$, $w = z - \langle a, s \rangle s$

(where $w$ is tangent to the sphere) Newton equations become:

\[
\begin{align*}
\rho' &= \rho v \\
v' &= \|w\|^2 + \frac{\alpha}{2}v^2 - \alpha U(s) \\
s' &= w \\
w' &= -\|w\|^2 s + \left(\frac{\alpha}{2} - 1\right)v w + \nabla_s U(s),
\end{align*}
\]

where $\nabla_s U$ is the component of the gradient of $U$ tangent to the inertia ellipsoid $S = \{\|q\| = 1\}$. 
McGehee coordinates (cont.)

The coordinates $\rho, v, s, w$ yield a map (homeomorphism outside $\{\rho = 0\}$) defined on the phase space

$$(q, p) \mapsto (\rho, v, s, w) \in [0, +\infty) \times \mathbb{R} \times TS,$$

where $TS$ is the tangent bundle of $S$. The energy $H$ can be written as

$$2\rho^a H = v^2 + \|w\|^2 - 2U(s).$$

All trajectories going to a total collisions touch a submanifold of the boundary $\{\rho = 0\}$, termed the McGehee total collision manifold $M_0$, defined by the equation

$$v^2 + \|w\|^2 = 2U(s).$$

This equation defines also the projection of all parabolic trajectories as a subset of $\mathbb{R} \times TS$, where one eliminates $\rho$. (Hence, given a solution in $M_0$, one can integrate $\rho$ and obtain the full parabolic motion)
Partial collisions are a cone of a subset $\Delta \subset S$. $M_0$ is a sphere bundle on $S \setminus \Delta$, with fibers $\approx S$. The flow on $M_0$ is gradient-like (due to $v$), and stops at singular points in $\Delta \subset S$, or at equilibrium points, i.e., points satisfying the equations

$$\nu^2 = U(s), \quad \nabla_s U(s) = 0, \quad w = 0,$$

which correspond to central configurations: stationary points for the restricted potential $U (s \in S : \nabla_s U(s))$. Other equilibrium points in the phase space do not exist. Equilibrium points must be found, singular points must be regularized...
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Central configurations for $D_l$-symmetric configurations of $2l$ bodies
(1) If $G = D_l$ is the dihedral group with $2l$ elements, then central configurations for $D_l$-symmetric configurations are only those of the previous slide ($2l$-agon, $l$-prism and $l$-antiprism).

(2) Moreover, all the corresponding equilibrium points in the $M_0$ flow are hyperbolic\(^{(3)}\).

(3) For the $4$-body Klein group, and any $\alpha \in (0, 2)$, there are $12$ square central configurations ($4$ for each coordinate plane), and $8$ tetrahedra, which are minima for $U$.
Dimensions of the stable and unstable manifolds in $M_0$: $2$ and $2$ for the tetrahedral CC’s, $3$ and $1$ ($v > 0$) or $1$ and $3$ ($v < 0$) for the squares.

(4) For the $l$-dihedral $2l$-body problem and $\alpha \in (0, 2)$, the three families of central configurations have dimensions of the stable and unstable manifolds in $M_0$ equal to: prism and planar the same as square CC for the $4$-body, all antiprisms the same as tetrahedral CC.

\(^{(3)}\)Ferrario/Portaluri: On the dihedral $n$-body problem (see n. (1)).
Minimal CC’s for T (of order 12), the O (of order 24) and Y (of order 60) and their 2-covers.
Recall that for a rotation group, $S \approx S^2$ and $M_0$ is a four-dimensional $S^2$-bundle over $S \setminus \Delta$.
For each rotation in the symmetry group $G$, there is a collision axis, and two antipodal collision points in $S$. Coxeter planes contain pairs of rotation axes, and are invariant in the flow. That is, each of the symmetry planes gives rise to an invariant surface in $M_0$ containing $l$-agon collisions, with a rectangular flow analogous to the square flow.
For any $\alpha$, a bouncing regularization is possible, but only locally within the plane, by setting for the horizontal plane

$$u = \frac{\sin^\alpha(2\theta)}{\sqrt{W(\theta)}} w$$

with $W(\theta) = \sin^\alpha(2\theta)U(\theta)$ and changing time accordingly. Here $\theta \approx s$ and $w \approx w$. Similar formulas hold for the prism and tetrahedral case.

For $\alpha = 1$ a Levi-Civita double covering map can be defined, which gives the “bouncing” regularization on invariant planes. But, as far as we know, not explicitly for any symmetry group (cfr. Lemaitre-Moeckel-Montgomery).
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COVERING OF THE TETRAHEDRAL SECTION
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In the negative energy region, one can expect to find (many?) periodic collisionless orbits. A few can be proven to exist by applying previous results\(^{(4)}\)(\(^{(5)}\)), minimizing the Lagrangean action on the Sobolev space of \(G\)-equivariant loops, for suitable \(G\). Let \(\sigma\), \(\tau\) and \(\rho\) be the permutation, time and space representation of \(G\), and \(X\) the configuration space.

\[(6)\] Let \(K = \ker \tau\). If \(\rho(K) \subset SO(3)\) is a finite group of rotations acting transitively on the index set \(\{1, \ldots, n\}\), and if \(X^G = \{0\}\), then there exists a \(G\)-equivariant collisionless minimizer.

How to define group actions satisfying this condition?
(7) Corollary. Given \( K \subset SO(3) \) a subgroup of order \( n \), with permutation regular representation \( \hat{\sigma} : K \to \Sigma_n \), if \( g \in N_{O(3)}K \) is such that \( (\mathbb{R}^3)^g = 0 \), and \( s \in \Sigma_n \) is the permutation on \( K \) defined by conjugation with \( g \), then the subgroup \( G \) of \( SO(3) \times \Sigma_n \) generated by the graph of \( \hat{\sigma} \) and the element \( (g, s) \) satisfies the hypotheses of (6), with \( \rho, \sigma \) natural projections and \( \tau \) defined as \( \tau(K) = 0, \tau((g, \sigma)) = 1 \).

(8) Corollary. Let \( K \subset SO(3) \) be a subgroup of order \( n \) as above. Then the antipodal map \( g = -I \in O(3) \) normalizes \( K \) and induces the trivial conjugation permutation \( s \).
**Examples**

Klein group, \( g = -I \) (but the minimizer is also \( \mathbb{Z}_3 \)-symmetric): [▶]  

Klein group, \( g = \text{Hip-Hop rotation} \): [▶]
Examples: Tetrahedral Group of Order 12

\[ K = \text{tetrahedral group}, \; g = \text{Hip-Hop 4-rotation: } \]

\[ K = \text{tetrahedral group}, \; g = \text{Hip-Hop 3-rotation: } \]
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It is possible to consider multiple copies of the same symmetric minimizing orbit, and a minimizer will exist (eight 3-choreographies + 21 singletons: \([\triangleright]\), a 3-choreography + a 5-choreography + a 7-choreography + a 9-choreography and 3 singletons - \(|G| = 630\) \([\triangleright]\)).
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Suitable symmetry groups occur in the problem of constellations of satellites (Walker delta pattern, [▷]...).
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Take two subspaces, fixed by involutions, with a single intersection. Minimize in the space of all paths going from one component of a subspace to a component of the other \(\implies\) there exists a collisionless minimizer, yielding a symmetric minimizer (periodic or quasi-periodic ...).


Thank you