Eigenvalue Statistics for Two-Particle Models

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Physical Situation

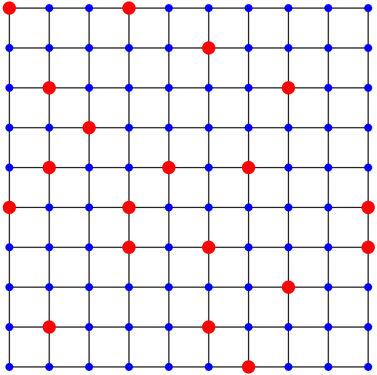
We wish to understand the statistics of eigenvalues for charge carriers in a random solid. In particular, we want the following two estimates:

Wegner Estimate

$\mathbb{P}(\mathcal{C}_\epsilon(H - E) \geq 1) \leq C_1|\Lambda|\epsilon$

Minami Estimate

$\mathbb{P}(\mathcal{C}_\epsilon(H - E) \geq 2) \leq C_2|\Lambda|^2\epsilon^2$



Here $\mathcal{C}_\epsilon$ counts the eigenvalues in $(-\epsilon, \epsilon)$.

Physical Situation

We use a discrete model, in which the evolution of the wave function ψ on the d-dimensional lattice $\mathbb{Z}^d$ is given by the Schrödinger equation:

Time-Dependent Schrödinger Equation

$i\hbar\dot{\psi}_t = H\psi_t$

where the Hamiltonian H acts on $\ell^2(\mathbb{Z}^d)$ by

$$(H\psi)(n) = (H_0\psi)(n) + g\,v(n)\psi(n) = \sum_{m\sim n} \psi(m) + g\,v(n)\psi(n).$$

Here the notation $m \sim n$ means that m is a lattice site adjacent to n, and the entries $v(n)$ of the potential (the so-called "single site" potentials) are independent random variables. The real parameter g is a coupling constant which describes the strength of the disorder.

Interacting Charge Carriers

- Eigenvalue statistics for single-electron Hamiltonians have been extensively studied. (Minami, 1996, Molchanov, 1988, etc.) Fewer results are known for Hamiltonians with correlated random variables or with interacting charge carriers. (But see Bellisard, Hislop and Stolz, 2007; Tautenhahn and Veselić, 2013)
- Previous work (Kirsch, Metzger and Müller, 2011; Gebert and Müller, 2013) has established a Wegner estimate for a model with an interacting electron-hole pair, with positive single-site potentials. (The second paper also proved localization for the same model.)
- We will give a second proof that allows sign-indefinite single-site potentials, while keeping the interacting charge carriers.
- We will also prove a weakened version of the Minami and higher-order estimates for the same model.

Setup

Our model comes from the Bogoliubov-de Gennes equation:

$$\begin{bmatrix} H_1 & B \\ B^* & -H_1 \end{bmatrix} \begin{bmatrix} \psi_+ \\ \psi_- \end{bmatrix} = E \begin{bmatrix} \psi_+ \\ \psi_- \end{bmatrix},$$

The 'particle' and 'hole' components ψ_+ and ψ_- of the quasi-particle state belong to the single-particle Hilbert space $\mathcal{H}$.
After a change of basis, our Hamiltonian will be of the form:

$$(H\psi)(x) = gA_i(x)\psi(x) + (K\psi)(x),$$

where $g > 0$ is a coupling constant, K is a deterministic self-adjoint operator on $L^2(\mathcal{V} \otimes \mathbb{C}^k)$, and i is chosen depending on the value of x. The A_i blocks have the form:

$$A_i = \begin{bmatrix} u_i & v_i \\ v_i & -u_i \end{bmatrix}$$

Main Theorem

Assumption A

For an integer n, let S be a given set of $2nk$ distinct integers. There exists an $\alpha > 0$ such that, for any integer $a \in S$, any $\epsilon \in [0,1]$ and arbitrary Hermitian $k \times k$ matrix J we have the bound

$$\mathbb{P}\left(\left|\det\left((A_i - a)^{-1} + (J + a)^{-1}\right)\right| \leq \epsilon\right) \leq K\epsilon^{\alpha}$$

Main Theorem

Theorem 1

Given Assumption A, we have

$$\mathbb{P}\left(\mathcal{C}_\epsilon(H-E) \geq m\right) \leq C\left(-ln(|\Lambda|(\epsilon/g)^\alpha)|\Lambda|(\epsilon/g)^\alpha\right)^m$$

for any $E \in \mathbb{R}$, for any $\epsilon \in [0, \min(2^{-k}, |\Lambda|^{-1/\alpha})]$ and for all $m \leq n$. Here the constant C depends on k, α and m but not on $|\Lambda|$ or ϵ.

- Note 1: for the case $m = 1$ we can prove a stricter estimate, without the log factor.
- Note 2: the assumption will turn out to hold for our model provided that the joint distribution μ of the u, v variables is uniformly α-Hölder continuous.

Strategy of the Proof

Ultimately, we want $\mathcal{C}_\epsilon(H-E) \geq m$ to imply an inequality of the form

$$|det(H-E)| \leq C\epsilon^m \tag{1}$$

Given this implication, we will be able to calculate the probability of the above event to get the desire bounds from Theorem 1.

However, we have two problems:

- The number of factors in this determinant is proportional to $|\Lambda|$. (Hence C will depend on $|\Lambda|$.)
- The individual factors can be arbitrarily large.

To solve these problems we will need to replace the original Hamiltonian $(H-E)$ with a succession of modified operators with similar eigenvalue statistics.

Auxiliary Theorem

Theorem 2

Consider the following two statements:

(A)

$$\mathcal{C}_{K_1\epsilon}(H-E) \geq m \tag{2}$$

(B) There exist index subsets

$$\alpha_m = \{i_1, \ldots, i_m\}, \quad \beta_m = \{j_1, \ldots, j_m\}$$

of $\{1, \ldots, |\Lambda|k\}$ such that

$$P_{\alpha_m}(H-E)^{-1}P_{\beta_m}(H-E)^{-1}P_{\alpha_m} \geq \frac{(K_2)^2}{\epsilon^2} I_m \tag{3}$$

Auxiliary Theorem

Theorem 2 (continued)

Then (A) with $K_1 = 1$ implies (B) with

$$K_2 = \frac{C_m}{|\Lambda| k}, \quad C_m = \frac{2^{-m}}{m! 2^{m+1}} \tag{4}$$

Conversely, (B) with $K_2 = 1$ implies (A) with $K_1 = 1$.

This result will allow us to reduce the dimensionality of the original problem.

Rewriting the Problem

Define γ_m to be a third index set that includes the indices for all blocks with at least one index in α_m or β_m.

Then repeated application of Theorem 2 gives the following:

$$
\begin{aligned}
&\mathcal{C}_\epsilon(H-E) \geq m \\
&\implies P_{\alpha_m}(H-E)^{-1}P_{\beta_m}(H-E)^{-1}P_{\alpha_m} \geq \frac{(K_2)^2}{\epsilon^2} I_m \\
&\implies P_{\alpha_m}\left(P_{\gamma_m}(H-E)^{-1}P_{\gamma_m}\right)P_{\beta_m}\left(P_{\gamma_m}(H-E)^{-1}P_{\gamma_m}\right)P_{\alpha_m} \geq \frac{(K_2)^2}{\epsilon^2} I_m \\
&\implies \mathcal{C}_{\epsilon/K_2}\left(P_{\gamma_m}(H-E)^{-1}P_{\gamma_m}\right)^{-1} \geq m
\end{aligned}
$$

Rewriting the Problem

- Now
$$D_\omega = \left(P_{\gamma_m}(H-E)^{-1}P_{\gamma_m}\right)^{-1} \tag{5}$$
can be evaluated using the block matrix inversion formula. D_ω has the same functional form as H, i.e. it can be decomposed as the sum of the block diagonal matrix D_1—whose blocks are independent copies of A_i—and the D_1-independent matrix D_2.
- Now we can construct a new matrix:
$$\hat{D}_\omega = (D_1 - aI_n)^{-1} + (D_2 + aI_n)^{-1}$$
We still have similar eigenvalue behavior:
$$\mathcal{C}_{9n^2\epsilon/K_2}(\hat{D}_\omega) \geq m$$
but the new matrix is bounded:
$$\|\hat{D}_\omega\| \leq 2$$

Estimating the Probability

- Thus we have $|\det \hat{D}_\omega| \leq 2^{kn-m}(9n^2\epsilon/K_2)^m$
- The proof of Theorem 1 now comes down to verifying that for any $\epsilon \in [0, 2^{-k}]$

$$\mathbb{P}(\mathcal{E}) \leq K\alpha^m ln^m((\epsilon/K_2)^{-1})(\epsilon/K_2)^{\alpha m}, \tag{6}$$

  with K that depends only on n.

- The proof uses induction. For $n = 1$ the result follows from **(A)**. Suppose that (6) holds for n blocks. Let $\hat{D}_{11}$ denote the first diagonal block in $\hat{D}_\omega$. Then

$$\det \hat{D}_\omega = \det \hat{D}_{11} \det(\hat{D}_\omega/\hat{D}_{11}).$$

  Both determinants on the right-hand side are random, but the first one depends only on A_1.

- We note now that the matrix $\hat{D}_\omega/\hat{D}_{11}$ is of the form $\tilde{D}_1 + \tilde{D}_2$, where

$$\tilde{D}_1 = \mathrm{diag}((A_2 - a)^{-1}, \ldots, (A_{n+1} - a)^{-1}),$$

and $\tilde{D}_2$ is independent of the random variables $\{A_i\}_{i=2}^{n+1}$. By the induction hypothesis, we have

$$\mathbb{P}\left(|\det(\hat{D}_\omega/\hat{D}_{11})| \le x\right) \le \hat{K}\alpha^n ln^n(x^{-1})x^\alpha, \quad x \in [0, 2^{-k}], \tag{7}$$

where $\hat{K}$ depends only on n.

- Let $S := \{\omega : \ |\det \hat{D}_\omega| \le \epsilon\}$, and let

$$F_\omega = |\det \hat{D}_{11}|, \quad G_\omega = |\det(\hat{D}_\omega/\hat{D}_{11})|.$$

We set $Q := \{\omega : \ F_\omega \le \epsilon\}$, then by Assumption **(A)**

$$\mathbb{P}(Q) \le K\epsilon^\alpha. \tag{8}$$

Estimating the Probability

- On the other hand, we have

$$
\begin{aligned}
\chi(S \setminus Q) &= \int_0^{\epsilon} \delta(F_\omega G_\omega - t)\chi(F_\omega > \epsilon) dt \\
&= \int_0^{\epsilon} dt \int_{\epsilon}^{2^k} \delta(s G_\omega - t)\delta(F_\omega - s) ds \\
&= \int_{\epsilon}^{2^k} \chi(s G_\omega \leq \epsilon)\delta(F_\omega - s) ds, \qquad (9)
\end{aligned}
$$

where we have used the bound $F_\omega \leq 2^k$ which follows from our assumptions on $\hat{D}_{1,2}$.

Estimating the Probability

- Taking expectations on both sides and using (7), we obtain

$$
\begin{aligned}
\mathbb{E}\chi(S \setminus Q) &\leq \mathbb{E} \int_{\epsilon}^{2^k} ds\, \delta(F_\omega - s)\mathbb{E}\left(\chi(sG_\omega \leq \epsilon) \,\middle|\, A\right) \\
&\leq K\alpha^n \epsilon^\alpha \mathbb{E} \int_{\epsilon}^{2^k} \frac{ln^n(\frac{s}{\epsilon})\delta(F_\omega - s)}{s^\alpha} ds \\
&= K\alpha^n \epsilon^\alpha \mathbb{E} \frac{ln^n(\epsilon^{-1} F_\omega)\chi(F_\omega > \epsilon)}{(F_\omega)^\alpha} \\
&\leq K\alpha^n \epsilon^\alpha ln^n(\epsilon^{-1} 2^k)\mathbb{E} \frac{\chi(F_\omega > \epsilon)}{(F_\omega)^\alpha} \\
&\leq K\alpha^n \epsilon^\alpha \left(ln(2^k) + ln(\epsilon^{-1})\right)^n \mathbb{E} \frac{\chi(F_\omega > \epsilon)}{(F_\omega)^\alpha} \\
&\leq K\alpha^n \epsilon^\alpha \left(2ln(\epsilon^{-1})\right)^n \mathbb{E} \frac{\chi(F_\omega > \epsilon)}{(F_\omega)^\alpha} \qquad (10)
\end{aligned}
$$

for $\epsilon \in [0, 2^{-k}]$.

Estimating the Probability

- Using now **(A)**, the fact that $F_\omega \leq 2^k$ and the layer cake representation, we get

$$\mathbb{E}\,\frac{\chi(F_\omega > \epsilon)}{(F_\omega)^\alpha} = \int_{2^{-k\alpha}}^{\epsilon^{-\alpha}} \mathbb{P}\left((F_\omega)^{-\alpha} \geq t\right) dt \tag{11}$$

$$\leq K \int_{2^{-k\alpha}}^{\epsilon^{-\alpha}} \frac{1}{t} dt \leq 2K\alpha ln\left(\epsilon^{-1}\right) \tag{12}$$

Combination of (8), (10), and (12) yields the induction step.

Wegner k-orbital Model

- The Wegner k-orbital model uses state functions from $\ell^2(\mathbb{Z}^d) \otimes \mathbb{C}^k$ and has the Hamiltonian:

$$(H_\omega \psi)(n) = \sum_{m \sim n} \psi(m) + g\, V(n)\psi(n)$$

  where $\omega = \{V(n)\}_{n \in \mathbb{Z}^d}$ is a family of $k \times k$ i.i.d. self-adjoint Gaussian random matrices.
- It can be verified that the conditions of Theorem 1 will still hold for this model, so that the n-level Wegner estimate still holds for any fixed value of k.
- However, the constant C from those estimates depends very strongly on k, so passage to the $k \to \infty$ limit is another question.

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