Disordered Quantum Many-Body Systems

BIRS - Banff, October 27 - November 01 , 2013

Condensation in a Disordered Bose-Hubbard Model

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- **Lattice Bose-Gas and Bose-Einstein Condensation**
- **Bose-Hubbard Model and Mott-type Phase Transition**

Aizenman-Lieb-Seiringer-Solovej-Yngvason (2004)

- **BEC in the Infinite-Range-Hopping Model**

Bru-Dorlas (2003)

- **Random IRH Bose-Hubbard Model**

Dorlas-Pastur-VZ (2006)

- **Enhancement/Suppression of BEC by Randomness**

1. Lattice Bose-gas

- $\Lambda := \{x \in \mathbb{Z}^d : -L_\alpha/2 \leq x_\alpha < L_\alpha/2,\ \alpha = 1,\ldots,d\} \subset \mathbb{Z}^d$ with p.b.c., dual set $\Lambda^* := \{q_\alpha = 2\pi n/L_\alpha : n = 0, \pm 1, \pm 2, \ldots \pm (L_\alpha/2 - 1), L_\alpha/2,\ \alpha = 1,2,\ldots d\}$ to $\Lambda = L_1 \times L_2 \times \ldots \times L_d$, $|\Lambda| = V$.
- The *one-particle* Hilbert space $\mathfrak{h}(\Lambda) := \mathbb{C}^\Lambda$, basis $\{e_x\}_{x\in\Lambda}$, $e_x(y) = \delta_{x\,,\,y}$ and $u = \sum_{x\in\Lambda} u_x e_x \in \mathfrak{h}(\Lambda)$. The one-particle *kinetic-energy* (*hopping*) operator

$$(t_\Lambda u)(x) := \sum_{y\in\Lambda} t^\Lambda_{x\,,\,y}(u_x - u_y),\ t^\Lambda_{x\,y} = \frac{1}{V}\sum_{q\in\Lambda^*} \widehat{t}_q e^{iq(x-y)}\ ,\ \widehat{t}_q \geq 0.$$

- The *free boson* Hamilton in the Fock $\mathfrak{F}_B(\mathfrak{h}(\Lambda))$:

$$T_\Lambda := \sum_{x\,\in\Lambda} a^*_x(t_\Lambda a)_x = \frac{1}{2}\sum_{x,y\,\in\Lambda} t^\Lambda_{x\,y}(a^*_x - a^*_y)(a_x - a_y) = \sum_{q\in\Lambda^*}(\widehat{t}_0 - \widehat{t}_q)\widehat{a}^*_q\widehat{a}_q.$$

$\left[a_x, a^*_y\right] = \delta_{x\,,\,y}$, $\left[\widehat{a}_q, \widehat{a}^*_p\right] = \delta_{q\,,\,p}$. $N_\Lambda := \sum_{x\in\Lambda} n_x = \sum_{q\in\Lambda^*}\widehat{a}^*_q\widehat{a}_q$, $n_x := a^*_x a_x$.

2. Bose-Einstein Condensation: Bose-Hubbard Model

- Nearest neighbour (n.n.) hopping: the one-particle spectrum:

$$\epsilon(q) := (\hat{t}_0 - \hat{t}_q) = \sum_{\alpha=1}^{d} 4 \ \sin^2(q_\alpha/2) \geq 0 \ , \quad q \in \Lambda^* \ .$$

- Lattice free Bose-gas: the BEC occurs in the zero-mode:

$$\rho_{c,\ n.n.}^{free}(\beta) := \lim_{\mu\uparrow 0} \lim_{\Lambda} \frac{1}{V} \sum_{q\in\Lambda^*} \frac{1}{e^{\beta(\epsilon(q)-\mu)} - 1} = \int_0^\infty \frac{\mathcal{N}_d(d\epsilon}{e^{\beta\epsilon} - 1} < \infty \ ,$$

the density of states $\mathcal{N}_d(d\epsilon) = \{c_d \epsilon^{(d/2-1)} + o(\epsilon^{(d/2-1)})\}d\epsilon$, $d > 2$.

- Bose-Hubbard model: **on-site** repulsive interaction

$$H_\Lambda := T_\Lambda + \lambda \sum_{x\in\Lambda} n_x(n_x - 1) \ , \ \lambda \geq 0 \ .$$

- **THEOREM** [Kennedy-Lieb-Shastry ('88)] There is zero-mode BEC in the Bose-Hubbard model with n.n. *hard-core* ($\lambda = +\infty$) interaction for the half-filled lattice, $\rho(\beta, \mu) \leq 1$.

3. BEC in the Infinite-Range-Hopping Model

• For the *Infinite-Range-Hopping* (IRH) Laplacian:

$$t^{\Lambda}_{x\,y} = \frac{1}{V}(1 - \delta_{x\,,\,y}) \ , \ \ x, y \in \Lambda.$$

the one-particle spectrum $\epsilon(q) := (\hat{t}_0 - \hat{t}_q) = (1 - \delta_{q\,,\,0})$ has a *gap*:

$$\lim_{q\to 0} \epsilon(q) = 1 \neq \epsilon(0) = 0 \ ,$$

but chemical potential $\mu \leq 0$. Since the density of states is zero in the gap, $\mathcal{N}_d(d\epsilon) = \delta_1(\epsilon)\, d\epsilon$, the *critical* particle density:

$$\rho^{free}_{c,\ i.r.}(\beta) = \int_0^{\infty} \frac{\mathcal{N}_d(d\epsilon)}{e^{\beta\epsilon} - 1} = \frac{1}{e^{\beta} - 1} \ , \quad \beta_c(\rho) = \ln(1 + 1/\rho).$$

• **THEOREM** [Bru-Dorlas ('03)] The IRH Bose-Hubbard model manifests the BEC, which is suppressed near the integer values $\rho = 1, 2, \ldots, k, k+1, \ldots$ of the total density ρ for positive repulsion parameters $\lambda \in [\lambda_k, \lambda_{k+1}]$, the "Mott insulator" phases.

4. Random IRH Bose-Habbard Model [DPZ(2006)]

• On the probability space,$(\Omega, \Sigma, \mathbb{P})$, consider the random Hamiltonian for disordered system:

$$H^{\omega}_{\Lambda} = \frac{1}{2V} \sum_{x,y\in\Lambda} (a^*_x - a^*_y)(a_x - a_y) + \sum_{x\in\Lambda} \lambda^{\omega}_x n_x(n_x - 1) + \sum_{x\in\Lambda} \varepsilon^{\omega}_x n_x,$$

where $\{\lambda^{\omega}_x \geq 0\}_{x\in\mathbb{Z}^d}$ $\left\{\varepsilon^{\omega}_x \in \mathbb{R}^1\right\}_{x\in\mathbb{Z}^d}$, for $\omega \in \Omega$, are real-valued stationary and ergodic random fields on $\mathbb{Z}^d$.

• **THEOREM 1** For almost all $\omega \in \Omega$,(a.s.), there exists a non-random thermodynamic limit of the pressure
$p^{\omega}_{\Lambda}(\beta,\mu) := \frac{1}{\beta V}\mathrm{Tr}_{\mathfrak{F}_B} \exp\left\{-\beta(H^{\omega}_{\Lambda} - \mu N_{\Lambda})\right\}$ exists and is equal to
$a.s.-\lim_{\Lambda} p^{\omega}_{\Lambda}(\beta,\mu) = p(\beta,\mu) := \sup_{r\geq 0}\{-r^2 +$
$\beta^{-1}\mathbb{E}[\ln \mathrm{Tr}_{(\mathfrak{F}_B)_x} \exp \beta[(\mu - \varepsilon^{\omega}_x - 1)n_x - \lambda^{\omega}_x n_x(n_x - 1) + r(a^*_x + a_x)]]\}$,
where $\mathbb{E}(\cdot)$ is expectation with respect to the measure $\mathbb{P}$. The BEC fraction $= -r^2$.

5.1 Limit of the Hard-Core Bosons: $\lambda_x^\omega = +\infty$

• This formally discards from the boson Fock space $\mathfrak{F}_B(\Lambda)$ all vectors with more than one particle at one site: there is orthogonal projection P_Λ such that $\mathfrak{F}_B^{h.c.}(\Lambda) := P_\Lambda\, \mathfrak{F}_B(\Lambda)$.

THEOREM 2

• $p_{h.c.}(\beta,\mu) = \sup_{r\geq 0}\{-r^2 +$
$\beta^{-1}\mathbb{E}\{\ln \mathrm{Tr}_{(\mathfrak{F}_B^{h.c.})_x} \exp(\beta P\,[(\mu-\varepsilon_x^\omega-1)n_x + r(a_x^*+a_x)]\,P)\}\}$

• Operators $c_x^* := P a_x^* P$, $c_x := P a_x P$ restricted to dom $c_x^* =$ dom $c_x = \mathfrak{F}_B^{h.c.}$, have commutation relations:

$$[c_x, c_y^*] = 0\,,\ (x \neq y)\ ,\quad (c_x)^2 = (c_x^*)^2 = 0\ ,\quad c_x c_x^* + c_x^* c_x = I\ .$$

• Taking the XY representation of these relations one gets:

$$p_{h.c.}(\beta,\mu) = \sup_{r\geq 0}\{-r^2 +$$
$$\mathbb{E}\left\{\tfrac{1}{2}(\mu-\varepsilon_x^\omega-1) + \beta^{-1}\ln\left[2\cosh\left(\tfrac{1}{2}\beta\sqrt{(\mu-\varepsilon_x^\omega-1)^2+4r^2}\right)\right]\right\}\}$$

5.2 Limit of the Perfect Bosons: $\lambda_x^\omega = 0$

THEOREM 3 Let $\varepsilon_x^\omega \geq 0$.

- $p_0(\beta, \mu < 0) = \sup_{r \geq 0}\{-r^2 + \beta^{-1}\mathbb{E}\{\ln \mathrm{Tr}_{(\mathfrak{F}_B)_x} \exp(\beta\,[(\mu - \varepsilon_x^\omega - 1)n_x + r(a_x^* + a_x)])\}\}$

- Let $\inf \varepsilon_x^\omega = 0$. Then
$p_0(\beta, \mu < 0) = \beta^{-1}\mathbb{E}\{\ln \mathrm{Tr}_{(\mathfrak{F}_B)_x} \exp(\beta\,[(\mu - \varepsilon_x^\omega - 1)n_x])\} =$
$\beta^{-1}\mathbb{E}\left\{\ln\left[1 - \exp\{\beta(\mu - \varepsilon_x^\omega - 1)\}\right]^{-1}\right\}$.

- For $\mu \to -0$:
$p_0(\beta, \mu = 0) := \beta^{-1}\mathbb{E}\left\{\ln\left[1 - \exp\{\beta(-\varepsilon_x^\omega - 1)\}\right]^{-1}\right\}$,
$\rho(\beta, \mu = 0) := \mathbb{E}\left[\frac{1}{e^{\beta(1+\varepsilon^\omega)} - 1}\right]$

6. Phase Diagram for Interaction $\lambda > 0$: Non-Random and Random Models

• Recall [Bru-Dorlas ('03)]: Let $\lambda \geq 0$ and $\varepsilon_x^\omega = 0$. Let
$\tilde{p}(\beta,\mu,\lambda;r) := \frac{1}{\beta} \ln \mathrm{Tr}_{\mathcal{H}} \exp(-\beta\,[h_n(\mu,\lambda) - r(a^* + a)])$
$h_n(\mu,\lambda) := (1-\mu)n + \lambda n(n-1)$
Then critical temperature $\beta_c^{-1}(\rho,\lambda)$ and the critical chemical potential $\mu_c(\rho,\lambda)$ are defined by equations:

$$\tilde{p}''(\beta,\mu,\lambda;0) = 2\ , \quad \rho = \frac{1}{Z_0(\beta,\mu,\lambda)} \sum_{n=1}^{\infty} n\, e^{-\beta h_n(\mu,\lambda)}\ .$$

• If $\varepsilon_x^\omega \neq 0$ and $\lambda > 0$, then by [Dorlas-Pastur-V.Z.('06)] one gets equations:

$$\mathbb{E}\left[\tilde{p}''(\beta,\mu-\varepsilon^\omega,\lambda;0)\right] = 2,\ \rho = \mathbb{E}\left[\frac{1}{Z_0(\beta,\mu-\varepsilon^\omega,\lambda)} \sum_{n=1}^{\infty} n\, e^{-\beta h_n(\mu-\varepsilon^\omega,\lambda)}\right]$$

6.1 Random Perfect Bosons: $\lambda = 0$

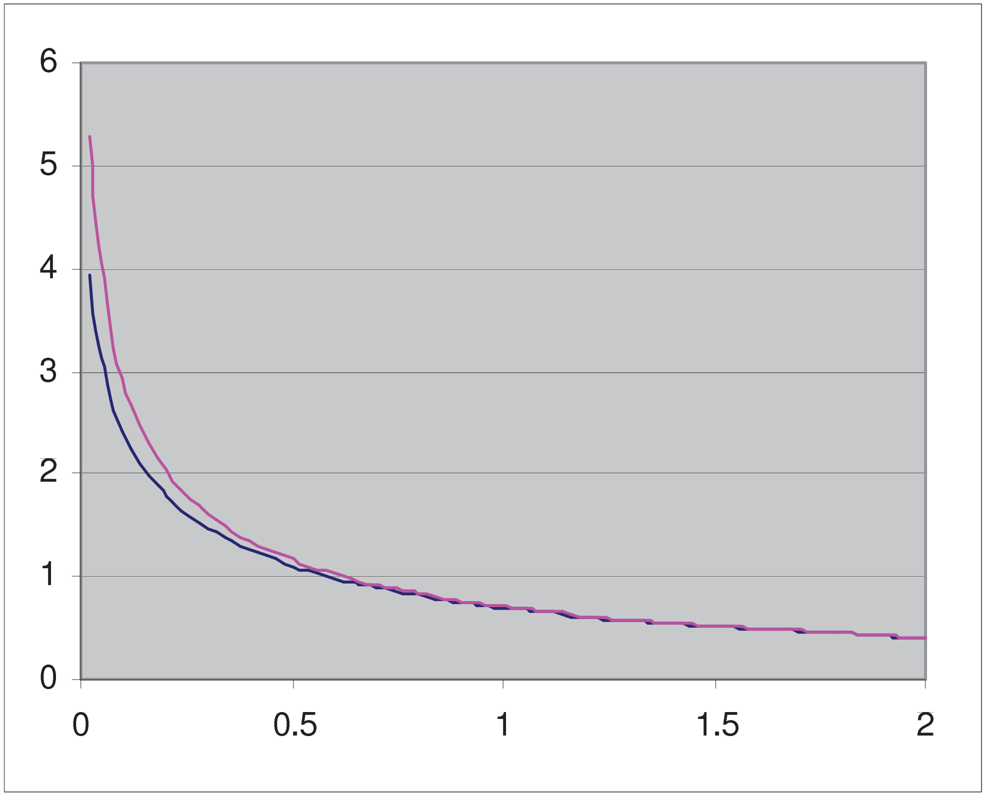
• Assume that random variable $\varepsilon^{\omega} \in [0, \varepsilon]$, then the maximal allowed (critical) value $\mu_c = 0$, and the *critical inverse* temperature $\beta_c := \beta_c(\rho, \lambda = 0)$ is given by equation:

$$\rho = \mathbb{E}\left[\frac{1}{e^{\beta_c(1+\varepsilon^{\omega})} - 1}\right].$$

• **Theorem I:** Irrespective of the ε^{ω}-distribution, this equation implies that the resulting $\beta_c(\rho, 0)$ is **lower** than $\ln\left(1 + \frac{1}{\rho}\right)$, for non-random case $\varepsilon_x^{\omega} = 0$.

• Disorder enhances Bose-Einstein condensation.

• **N.B.** This is no longer true when $\lambda > 0$, and even the opposite may hold, if λ is small enough !



$(\varepsilon = 0, \ \lambda = 0,1) \Rightarrow Pr = 1/\varepsilon \ (\varepsilon = 2, \lambda = 0,1)$

6.2 Discrete Random Potentials: Hard-Core Bosons

- Equations $\beta_c := \beta_c(\rho) = \beta_c(\rho, \lambda = +\infty)$ for a given density ρ:

$$\mathbb{E}\left[\frac{\tanh \beta(\mu - \varepsilon^{\omega} - 1)/2}{\mu - \varepsilon^{\omega} - 1}\right] = 1 \tag{1}$$

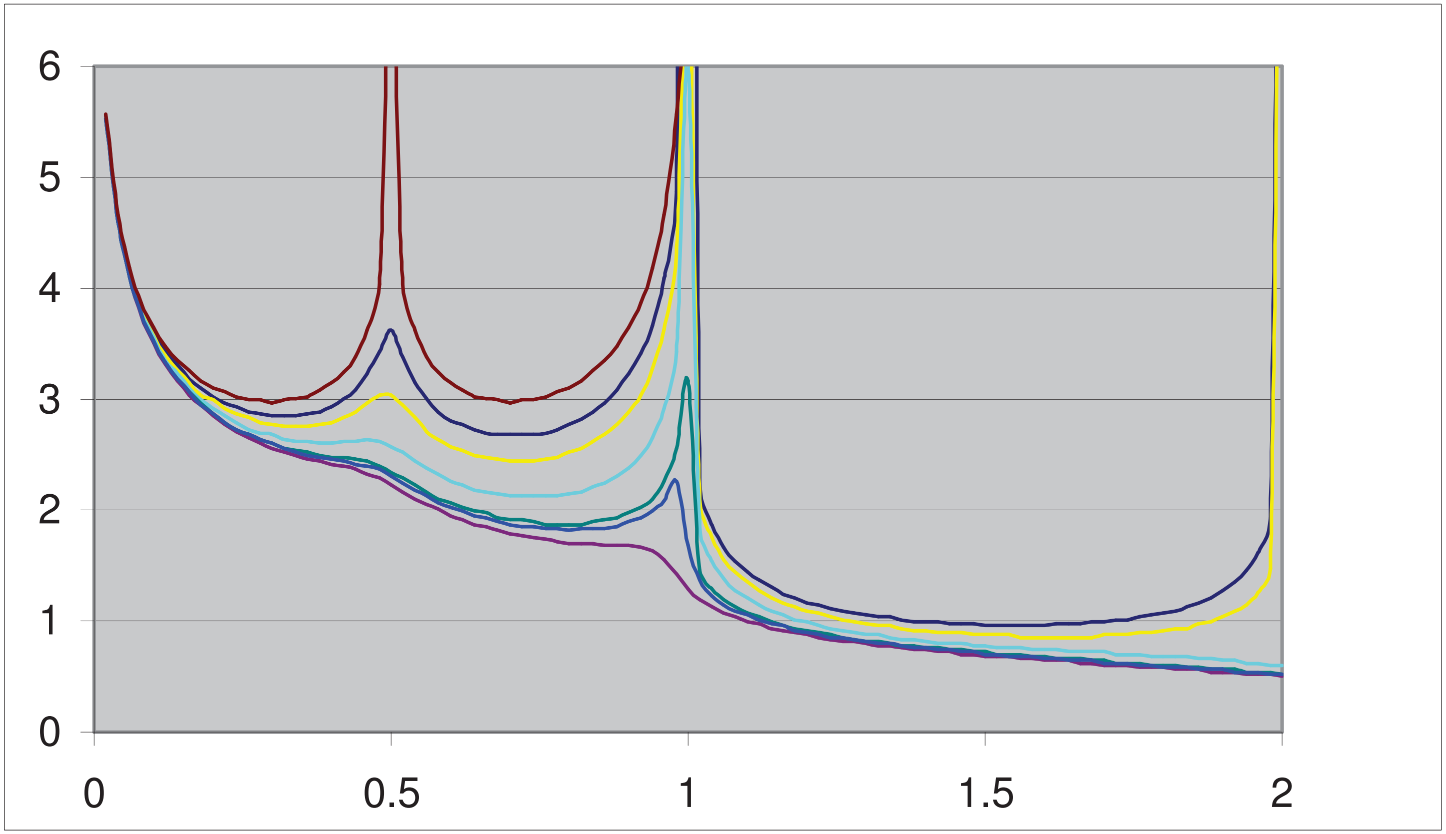
$$\rho = \frac{1}{2} + \frac{1}{2}\mathbb{E}\left[\tanh \frac{1}{2}\beta(\mu - \varepsilon^{\omega} - 1)\right] . \tag{2}$$

(For the hard-core interaction the total particle density $\rho \leq 1$).

- Bernoulli random potential: $\varepsilon_x^{\omega} = \varepsilon$ with probability p and $\varepsilon_x^{\omega} = 0$ with probability $1 - p$.
- New Phenomenon: Let $\rho = p = 1/2$. Then (1) and (2) $\Rightarrow$

$$\tanh \frac{\beta_c \varepsilon}{4} = \frac{1}{2}\varepsilon \ , \ \lambda = +\infty \ .$$

This equation has *no solution* for $\varepsilon \geq 2 \Rightarrow$ no Bose-Einstein condensation for Bernoulli potential, if particle density $\rho = p$ and $\varepsilon \geq$ (some critical value) $\varepsilon_{cr} = 2$. **Strong randomness is able to destroy BEC** for the fractional density $\rho = p = 1/2$.



$Pr = 1/2,\ \varepsilon = 4,\ \lambda = 3, 3.3, 4, 4.5, 6, 10, \infty$

• **Theorem II:** One obtains the same phenomenon for $\rho = 1 - p$, although $\beta_c(\rho \neq 1 - p,\ \lambda = +\infty) < \infty$.

• **Strong randomness is able to suppress (not to destroy) BEC** for fractional densities $\rho \neq 1 - p$.

6.3 Bernoulli random potential for the case $\lambda < +\infty$.

The critical temperature for free bosons increases due to disorder. For the interacting system this is a more subtle matter, since it depends on the value of repulsion: For a not very large repulsions close to $\lambda_{c,\rho=1}(\varepsilon = 0) = 3$, we get $\beta_c(\rho = 1; \lambda = 3, \varepsilon > 0) < \beta_c(\rho = 1; \lambda = 3, \varepsilon = 0) = +\infty$.

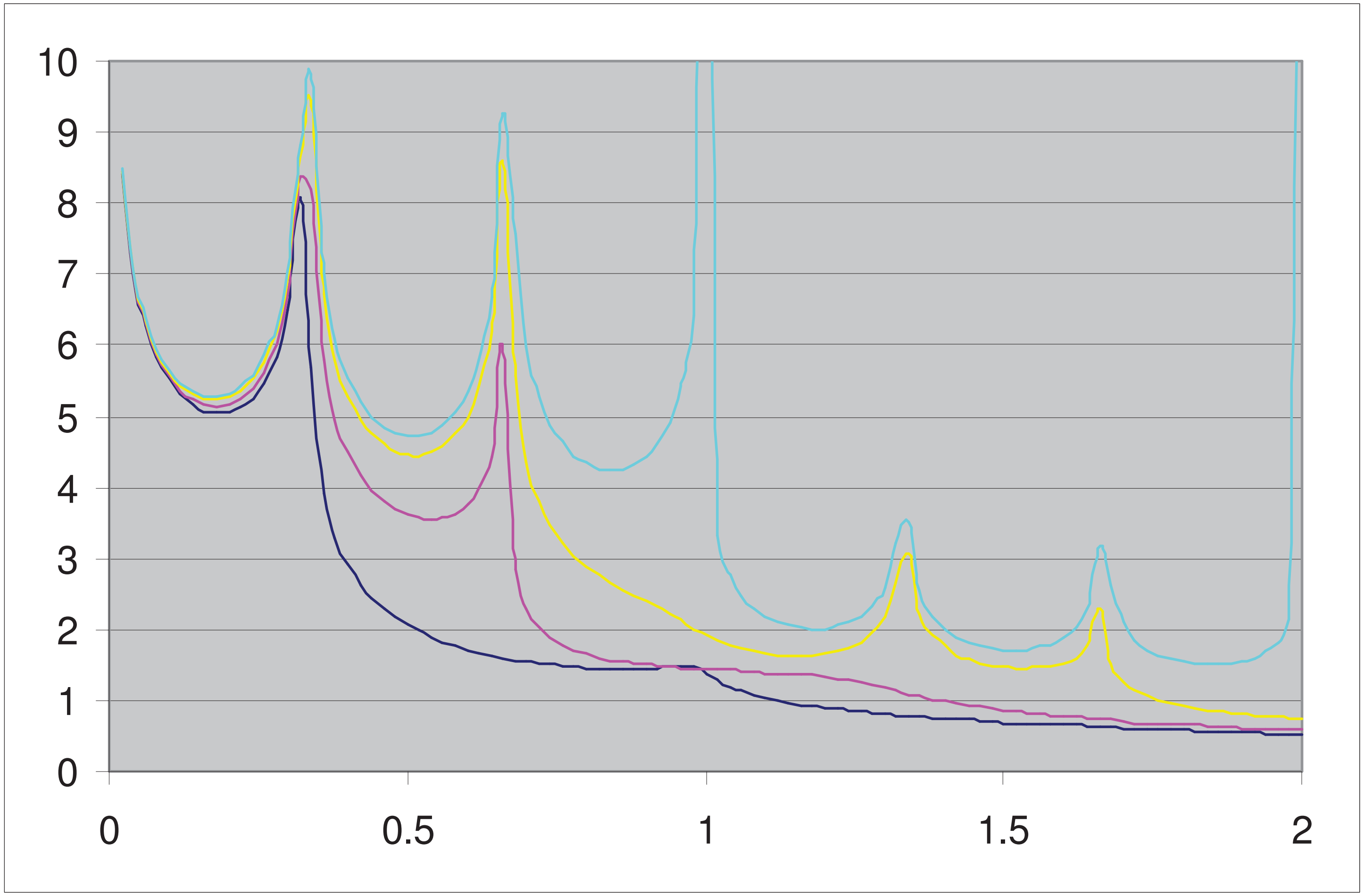
This *lowering* of $\beta_c(\rho = 1)$, which *favourites* the BEC can be explained intuitively as follows:

At density $\rho = 1$, there is one particle per site, if $\varepsilon > 0$, then the lattice splits (by the Bernoulli random potential) into two parts with energies 0 and ε. A particle jumping from a site with ε to a site with $\varepsilon = 0$ loses the energy ε, which counteracts the gain of λ. This creates more freedom of movement promoting BEC. See **Fig.**.

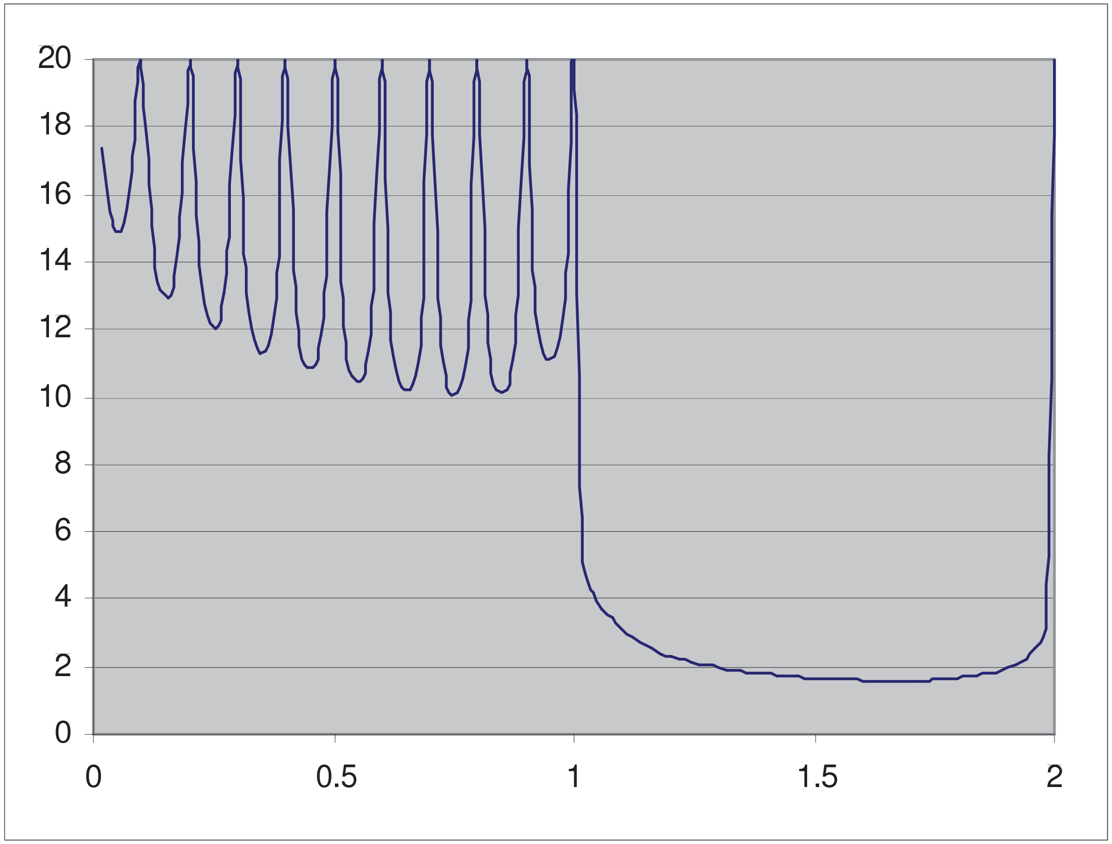
6.4 Trinomial distribution and beyond: $\lambda < +\infty$

$$\varepsilon^{\omega} = \left\{ \begin{array}{ll} 0 & \mathrm{Pr} = 1/3 \\ \frac{1}{2}\varepsilon & \mathrm{Pr} = 1/3 \\ \varepsilon & \mathrm{Pr} = 1/3 \quad . \end{array} \right\}$$

See **Fig**.



$Pr = 1/3, \ \varepsilon = 10, \ \lambda = 3, 4, 6, 8$



$Pr = 1/10, \ \varepsilon = 10, \ \lambda = 8$

THANK YOU FOR YOUR ATTENTION !