Disordered Quantum Many-Body Systems

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1 Overview

The central goal of the workshop "Disordered Quantum Many-Body Systems" was to bring together for an exchange of views and perspectives, researchers working on open problems related to disordered many-body quantum systems. The workshop featured presentations of recent advances in many-body theory and in the theory of disordered systems, and of techniques and approaches from the different areas. Attention was paid to the existing challenges and the possibility of further progress through possible cross fertilization between related areas and directions of research.

The theoretical physics perspective was provided by several researchers who have recently contributed to a better conceptual understanding of the many-body localization transition, supported by numerical evidence. A more mathematical perspective was offered by mathematical physicists working in the field of many-body systems, such as quantum spin systems. New insights on the effects of disorder were presented by experts working on the theory of disordered quantum systems such as the Anderson model and related random Schrödinger-type operators. The workshop has included a mix of the leading experts and junior researchers.

Among the topics discussed during the meeting were:

- Resonant delocalization for the Anderson model on tree graphs
- Disorder effects on Bose-Einstein condensation and superfluidity in interacting systems
- Phase transitions in random matrices, e.g. in terms of their eigenvalue statistics
- Disorder effects on classical many-body oscillator systems
- Investigation of Hartree-Fock theory in stochastic crystals

2 Some Recent Developments and Open Problems

Among the research questions which were selected for discussion at the workshop were:

1. What is many-body localization?

2. How can resonances in many body quantum systems be analyzed, and what is their effect on spectra and dynamics?

The first question has received strong attention in the recent physics literature, including works by Basko, Aleiner and Altshuler [9], Oganesyan and Huse [52], Znidaric, Prosen and Prelovsek [62], Pal and Huse [54], Hastings [29], addressing many-body Anderson-type models as well as random quantum spin systems. However, elucidation of the issue is still quite called for.

The recognition that resonances (to which the second question refers) play a key role in quantum conduction is not new, having certainly been expressed in works of Mott, Anderson and Landauer. It reappeared recently in the mathematical work of Aizenman and Warzel [3, 4], which has clarified the conditions for absolutely continuous spectrum on tree graphs (correcting the previously held pictures of the phase diagram). Themes which are developed in that work also suggest directions in which it will be of interest to see further progress in the understanding of the possible effects of resonant delocalization in many body quantum systems.

Following are more comments on a number of specific research topics where important progress was made within the years leading up to the workshop. Here we will first mention topics most closely related to the main workshop goals, starting with some recent (and some not so recent) physical perspectives on many-body localization, followed by a discussion of some simple mathematical models. We will also discuss some open problems as well as recent results on resonance induced delocalization on tree graphs, a phenomenon which may also play a role in systems with voluminous state space, such as the configuration space of many body systems.

2.1 A physics perspective on the many-body localization transition

A starting point for the discussion of quantum effects of disorder is provided by the Anderson model [7], which features a discretized Schrödinger operator with random potential, modeling the dynamics of a quantum particle in a disordered environment. It has been applied to describe electrons in metal as well as photons in optical lattices. In the former case it is to be viewed as only a step towards the quantum-mechanical theory of transport problems of the electron gas in metals, since the effects of the many-body interactions were underrepresented by a static effective potential.

The theory of random Schrödinger operators has attracted a great deal of work by mathematical physicists, starting with [24]. While the rigorous proof of the existence of extended states for the three-dimensional Anderson model at low disorder remains an open problem, several methods have been developed to prove localization phenomena for Schrödinger-type operators (one-dimensional and multi-dimensional, on a lattice or in the continuum). For recent introductory reviews, and additional references, see [34, 58, 5]. Delocalization in the presence of homogeneous disorder was accomplished only for tree graphs (Klein [36, 37], Aizenman and Warzel [3, 4]). While that is still far from the desired understanding of three dimensional systems, arguments were presented that tree graphs share certain features with the configuration spaces of many particle systems, Altshuler et. al. [6]. This raises the possibility that the recently found surprising effects of resonant delocalization [4] may be of relevance also in that context.

It took rather long before research of the disorder effects on quantum systems started to move beyond the one-particle theory towards a more complete understanding of disorder effects on quantum many-body systems [1]. More recently the topic of disorder effects in many-body systems started to draw considerable attention in the physics literature, see for example [25, 9, 52, 62, 54, 15, 60]. The central goal is to understand the many-body localization transition, which includes a proper description of both sides of the transition, namely of the concepts of many-body localization as well as of thermalization (the role of heat bath being served by the bulk of the system), or of equidistribution of the total energy and particle numbers.

Consequences of thermalization are:

- Quantum information is dispersed over the entire system and can not be stored in a finite subsystem.

[1] We draw here distinction between *multi-particle systems* and *many-body systems*. In the former the number of particles N is finite and fixed. For that case localization theory has been extended in [17, 18, 2], at large enough disorder and weak interactions. The available methods do not yield results for the "thermodynamic limit" in which $N \to \infty$.

- The long-time entanglement of any initial pure state is extensive, i.e. the von Neumann entropy

$$\text{Trace}\{\rho_S \log \rho_S\} \tag{1}$$

  is proportional to the size of S.

There is currently great interest in proving statement of this type, either for specific models or for suitable general classes of systems. So far, only weaker statements can be proved mathematically, and additional assumptions have to be imposed [47].

Many-body localization is best characterized by what it is not, meaning that any property which shows that a system does not thermalize can be considered as a form of many-body localization (allowing for a multitude of choices familiar already from the theory of one-body localization, where spectral localization, dynamical localization and Poisson statistics of eigenvalues are among the commonly accepted indicators). In particular, the absence of information transport in a many-body system or an area law for the entanglement entropy of suitable states (which shows that $\text{Trace}\{\rho_S \log \rho_S\}$ grows like the surface area of S) can be considered as many-body localization properties.

Physicists have looked at verifying the above conditions, either for localization or for thermalization, in two types of concrete many-body systems: systems of interacting electrons and of interacting spins. The paper [9] considers interacting electrons in an Anderson-type background potential. In the regime where the non-interacting system has no extended states, it is argued that the weakly interacting system has vanishing DC electrical conductivity at sufficiently low positive temperature (and at arbitrary temperature if the electron density is sufficiently low). A mathematical proof of the conclusions drawn in [9], which are based on elaborate and not always mathematically rigorous physical tools, seems currently out of reach.

More promising for a mathematical treatment of many-body localization seem to be disordered quantum spin systems. Results from theoretical physics, including numerical studies, which indicate the existence of regimes in which thermalization breaks down in such models were presented at the workshop by D. Abanin and A. Pal, see Section 3.1 below as well as an introductory account in [32]. The relation of the one particle theory with quantum spin systems was emphasized early on and may as well be described by quoting from [7]:

> "We assume that we have sites j distributed in some way, regularly or randomly, in three-dimensional space; the array of sites we call the 'lattice.' We then assume we have entities occupying these sites. They may be spins or electrons or perhaps other particles, but let us call them spins here for brevity. [...] and the simple process we study is the motion of a single "up" spin among "down" spins."

A concrete model in which this exact situation is realized is the d-dimensional XXZ spin system in random exterior magnetic field in Z-direction (or 3-direction), whose Hamiltonian is given by

$$H = H(\nu) = \sum_{(x,y)} \left\{ \frac{1}{\Delta}(S_x^1 S_y^1 + S_x^2 S_y^2) + S_x^3 S_y^3 \right\} + \sum_x \nu_x S_x^3. \tag{2}$$

Here the first sum is over all next neighbors (x, y) in $\mathbb{Z}^d$ (or rather, at least initially, a finite subset of $\mathbb{Z}^d$). S^1, S^2 and S^3 are the standard spin-$1/2$ matrices, and $\nu = (\nu_x)_{x\in\mathbb{Z}^d}$ is an array of real-valued i.i.d. random variables. For the value $\Delta = 1$ of the anisotropy parameter this becomes the isotropic Heisenberg model.

After properly re-normalizing the ground state energy (so that it becomes zero for the restriction of H to any finite subsystem), H can be introduced as a self-adjoint Hamiltonian directly in the thermodynamic limit of a spin system over $\mathbb{Z}^d$. In this explicit construction of the GNS representation the infinite volume Hilbert space is given by the span of $\varphi_{(j_1,\ldots,j_N)}$, $N = 0, 1, 2, \ldots$, $j_1, \ldots, j_N \in \mathbb{Z}^d$, where $\varphi_{(j_1,\ldots,j_N)}$ represents the state with up-spins at $j_1, \ldots, j_N$ in a sea of down-spins on $\mathbb{Z}^d$, making the all-spins-down state $\phi_\emptyset$ the vacuum vector. H preserves the particle number, i.e. the subspaces $\mathcal{H}^N$ spanned by the N-spins-up states $\varphi_{(j_1,\ldots,j_N)}$ are invariant under H for each fixed N. In particular, for $N = 1$ the restriction of H to $\mathcal{H}^1$ can be seen to be equivalent to the d-dimensional Anderson model

$$-\frac{1}{2\Delta} h_0 + \nu, \tag{3}$$

where h_0 is the next-neighbor hopping Laplacian (or adjacency operator) on $\mathbb{Z}^d$ and φ_j, $j \in \mathbb{Z}^d$, are viewed as the canonical basis vectors in $\ell^2(\mathbb{Z}^d)$. For a mathematical introduction to this see [26].

Thus the Anderson model describes the motion of a single up spin among down spins under the XXZ Hamiltonian H. What it does not account for are the interactions between multiple (or many) spins. Understanding the latter for XXZ spin systems in random field (meaning the restrictions of H to $\mathcal{H}^N$ for arbitrary N) or, much more generally, for other disordered quantum many-body systems is still a wide open challenge, both physically and mathematically.

Among the basic and very useful general results on discrete spin systems with local interactions is the proof by Lieb and Robinson [46] that the sound, and more generally the effects of local perturbations, propagate only at bounded velocity (i.e. waves propagate with finite group velocity). Some of the models are also exactly solvable, such as the antiferromagnetic Heisenberg model [10] (whose solution ushered in the Bethe ansatz), the XY spin chain [45], the Bose gas and the Hubbard model [8, 61, 43].

In the most recent decade quantum spin systems have seen a renewed surge of interest, in theoretical physics as well as in quantum information theory, where quantum spin systems are used to model information transport in systems of qubits. In particular, Lieb-Robinson bounds have been found to be an important tool in proving other properties of quantum many-body systems, such as exponential decay of ground state correlations [30, 48] and an area law for the entanglement entropy of the ground state of one-dimensional systems [28] (in both cases it is assumed that the system is gapped, i.e. that the ground state energy is separated from the energy of excited states, uniformly in the system size). For more results which were found in this context as well as many more references see the survey [49].

2.2 Simple Models I: The XY Spin Chain with Disorder

There are two particularly simple toy models of quantum many-body systems, both long known in physics, which can be used as test cases for the concepts which describe many-body localization and to provide some first examples where rigorous results can be obtained. Both of these models have in common that they can be studied in terms of an effective one-particle Hamiltonian (with the Anderson model appearing prominently in both cases), meaning, in essence, that known results on one-particle localization can be used to verify many-body localization properties.

First, let us consider the XY model which is found from the XXZ model by dropping the S^3 contribution to the next-neighbor interactions. For $d = 1$, in fact on the finite chain $\{1, \dots, n\}$, one gets

$$H_n = \sum_{x=1}^{n-1} \left\{ (1+\gamma) S_x^1 S_{x+1}^1 + (1-\gamma) S_x^2 S_{x+1}^2 \right\} + \sum_{x=1}^{n} \nu_x S_x^3, \tag{4}$$

where we allow for anisotropy in the interaction term measured by the parameter γ and, as in (2), include a random field in the 3-direction. For constant field this model has been found to be exactly solvable via the Jordan-Wigner transform in [45]. For general field, the Jordan-Wigner transform maps H_n to a free Fermion field which can be solved via diagonalization of the effective one-particle Hamiltonian

$$h_n = \begin{pmatrix} \nu_1 J & -S(\gamma) & & \\ -S(\gamma)^t & \ddots & \ddots & \\ & \ddots & \ddots & -S(\gamma) \\ & & -S(\gamma)^t & \nu_n J \end{pmatrix}, \tag{5}$$

whose two-by-two block entries are given by

$$J = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad S(\gamma) = \begin{pmatrix} 1 & \gamma \\ -\gamma & -1 \end{pmatrix}, \tag{6}$$

see [27] for a detailed review. Due to this explicit connection of the many-body Hamiltonian H_n with the one-particle Hamiltonian h_n, the XY chain has become the prototypical toy model at the beginning of many investigations of quantum spin systems. For example, one has the following relation between (one-body) localization of h_n and (many-body) localization of H_n [27]:

Suppose that h_n is dynamically localized in the sense that there exist $C < \infty$ and $\eta > 0$ such that

$$\mathbb{E}\left(\sup_{t\in\mathbb{R}} \left\|\langle j|e^{-ith_n}|k\rangle\right\|\right) \le Ce^{-\eta|j-k|} \tag{7}$$

for all $n \in \mathbb{N}$ and $1 \le j,k \le n$. Then H_n satisfies

$$\mathbb{E}\left(\sup_{t\in\mathbb{R}} \|[\tau_t^n(A), B]\|\right) \le C'\|A\|\|B\|e^{-\eta|j-k|} \tag{8}$$

for all $1 \le j,k \le n$, $A \in \mathcal{A}_j$ and $B \in \mathcal{A}_k$.

Here $\langle j|e^{-ith_n}|k\rangle$ are 2×2-matrix-valued matrix-elements of e^{-ith_n}, $\tau_t^n(A) = e^{itH_n}Ae^{-itH_n}$ is the Heisenberg dynamics of A under H_n, $\mathcal{A}_j$ denotes the operators acting only on the j-th spin and $\mathbb{E}(\cdot)$ refers to averaging over the random parameters ν_j, $j = 1,\dots,n$.

(8) takes the form of a zero-velocity Lieb-Robinson bound, which is interpreted as expressing the absence of information transport through the spin chain under the Hamiltonian H_n. Thus it is a form of many-body dynamical localization as discussed in Section 2.1 above.

If $\gamma = 0$, i.e. for the isotropic XY chain, then h_n decouples into a direct sum of the Anderson model and the negative Anderson model. Thus (7) is well known for sufficiently regular distribution of the random variables, and thus many-body dynamical localization (8) for the isotropic XY chain in random exterior field follows. This rigorously proved and strengthened a result predicted in [13], where the importance of such bounds as a form of information localization was stressed.

For the anisotropic XY chain $\gamma \neq 0$, one-particle localization bounds such as (7) were previously unknown. Motivated by this as well as by applications in BCS theory, the localization theory of random block operators such as (5) has recently been a very active research field, see [35, 23, 22, 21, 16]. As a result, dynamical localization for the XY chain in random transversal field is now well understood.

Another result proven in [27], valid for very general classes of quantum spin systems, says that zero-velocity Lieb-Robinson bounds such as (8) imply exponential decay of ground state correlations (up to a logarithmic correction), a result previously only known for gapped spin systems. That this in turn leads to an area law for the entanglement of the ground state, also for quite general one-dimensional spin systems, was recently observed at the physical level of rigor in [11]. Both of these results can be understood as many-body versions of a well known fact from single particle theory (at least for the ground state), namely that dynamical localization implies eigenfunction localization (which, in the many-body context, can be measured via correlation and entanglement bounds).

2.3 Simple Models II: Disordered Oscillator Systems

Another frequently used many-body toy model are systems of interacting harmonic oscillators. While more general systems can be considered, a typical example is given by

$$H_L = \sum_{x\in\Lambda_L}\left(\frac{1}{2m_x}p_x^2 + \frac{k_x}{2}q_x^2\right) + \sum_{(x,y)} \lambda_{x,y}(q_x - q_y)^2, \tag{9}$$

acting on the Hilbert space $\mathcal{H}_L = \bigotimes_{x\in\Lambda_L} L^2(\mathbb{R}, dq_x)$. Here $\Lambda_L = [-L,L]^d \cap \mathbb{Z}^d$ is a sub-cube of the d-dimensional lattice and q_x and p_x the standard position and momentum operators. The second sum is over next neighbors in Λ_L. Disorder is introduced by choosing the parameters m_x and/or k_x and/or $\lambda_{x,y}$ as random variables.

Many-body localization properties for disordered quantum oscillator systems have been investigated in [50] and [51]. As for the XY spin chain, H_L can be studied in terms of an effective one-particle Hamiltonian (but unlike the XY chain, for oscillator systems this works in arbitrary dimension d). The effective Hamiltonian is given by $h_L = \mu^{1/2}\tilde{h}_L\mu^{1/2}$, where μ is the multiplication operator by $1/2m_x$, $x \in \Lambda_L$, and $\tilde{h}_L$ is characterized by its quadratic form

$$\langle f, \tilde{h}_L g\rangle = \sum_{(x,y)} \lambda_{x,y}\overline{(f(x)-f(y))}(g(x)-g(y)) + \sum_{x\in\Lambda_L} \frac{k_x}{2}\overline{f(x)}g(x). \tag{10}$$

Note that, in the case of constant m_x and $\lambda_{x,y}$ as well as i.i.d. k_x, h_L becomes the d-dimensional Anderson model.

The paper [50] proves two types of results for disordered oscillator systems. First, if the effective Hamiltonian (10) is dynamically localized at all energies (in sufficiently strong sense, which holds, e.g., for the Anderson model), then zero-velocity Lieb-Robinson bounds for H_L follow. Due to the fact that for oscillator systems the single site Hilbert space $L^2(\mathbb{R}, dq_x)$ is infinite-dimensional and the Hamiltonian H_L is unbounded, one can not deal with general local operators A, B as in (8). Instead, the Lieb-Robinson bounds hold for local unitary Weyl operators of the form $\exp(i(c_1 q_x + c_2 p_x))$, $c_1, c_2 \in \mathbb{R}$. Second, exponential decay of correlations for Weyl operators is proven, in the ground state as well as in thermal states. This only requires localization of h_L near the bottom of its spectrum.

In [51] an area law for the bi-partite entanglement entropy of the ground state is shown. More precisely, consider the orthogonal projection $P_L = |\Omega_L\rangle\langle\Omega_L|$ onto the non-degenerate ground state of H_L, fix a subsystem corresponding to sites $\Gamma \subset \Lambda_L$ and trace out the exterior degrees of freedom: $P_L^1 = \mathrm{tr}_{\mathcal{H}_2} P_L$, where $\mathcal{H}_2 = \bigotimes_{x \in \Lambda_L \setminus \Gamma} L^2(\mathbb{R})$. Then the von Neumann entropy $S(P_L^1) = -\mathrm{tr}(P_L^1 \ln P_L^1)$ satisfies the area law

$$\mathbb{E}(S(P_L^1)) \leq C|\partial\Gamma|, \tag{11}$$

as before under the assumption that h_L is localized at low energy. A similar result is shown for the entanglement of thermal states, using the logarithmic negativity (introduced in [59]) as a measure of entanglement for mixed states.

We remark that none of the results in [50] and [51] require the existence of a uniform (in the system size) ground state gap for H_L, an assumption under which similar results had been proven before, e.g. [19]. All that is required is what physicists refer to as a *mobility gap*.

2.4 Open Problems for Disordered Spin Systems and Oscillator Systems

As the rigorous examination of random many-body systems has only recently begun, there are still a wealth of interesting open question, even for simple models like the XY chain and lattice oscillators. For example, there are no results on correlation decay for thermal states corresponding to random XY chains. Moreover, an area law for the ground state of disordered XY chains is also unknown. In fact, for multi-dimensional XY systems where the Jordan-Wigner transformation no longer reduces the model to an effective one-particle Hamiltonian, there are no known localization results for these random models.

For random lattice oscillators, more localization results have been proven, but there are still open questions. As mentioned above, in [51] an upper bound demonstrating an area law for ground states (as well as a related result for thermal states) has been shown. It is unknown whether or not a matching lower bound can also be established. Also, the question of area laws for more general pure states in random oscillator systems has not been addressed.

One of the big open problems concerns results for disordered, genuinely many-body systems. An important goal is to establish techniques which verify localization properties directly in the random many-body setting, without requiring that the systems is governed by an effective single-particle Hamiltonian. Ideas for an attempt to understand this phenomenon for spin systems with large disorder was the topic of John Imbrie's talk at the workshop; see Section 3.1 below for some more detail.

2.5 Resonant delocalization for the Anderson model on tree graphs

The Anderson model on the Bethe lattice is historically among the first for which an energy regime of extended states and a separate regime of localized states could be established. It has also been proposed as an effective model for disordered many-particle models since their configuration space has an exponential volume-growth, e.g. [9]. The latter provided motivation for discussing the Anderson model on the Bethe lattice and more general tree graphs at the workshop, although understanding implications for many-body systems remains an open problem.

In recent years, some surprises have been discovered in the phase diagram on the Bethe lattice [3, 4]. Among them is that even at weak disorder, the regime of diffusive transport extends well beyond energies of the unperturbed model into the Lifshitz tails. The mechanism for the appearance of extended stated in this non-perturbative regime are disorder-induced resonances. It is conjectured that the corresponding eigenfunctions in this non-perturbative delocalization regime violate a heuristic version of the equidistribution principle. In particular, this would suggest Poisson statistics of the rescaled eigenvalues in this delocalized regime - a conjecture which is supported by numerics.

It is a challenge to determine whether the phenomenon of resonant delocalization may play a role in conduction within systems of many particles.

3 Survey of Workshop Activities

3.1 A Sampling of Talks

Below we discuss some, but certainly not all, of the interesting talks which were given at our workshop. The goal here is to give an indication of the scientific diversity and breadth of the researchers we gathered to work on this intriguing topic.

The talk of Dimitry Abanin, based on [56, 57], discussed a construction of a complete set of local integrals of motion that characterize the many-body localized phase. This approach relies on the assumption that local perturbations only act locally on the eigenstates in the interacting localized phase, and this notion was supported with numerical simulations of a random field XXZ spin chain. Additionally, this study provides a description of the structure of the many-body localized states. The implications of the local conservation laws for quantum dynamics in the interacting localized phase were shown to underlie an unusual, logarithmic in time, growth of entanglement entropy, as discussed in Conjecture 2 of Section 3.2 below. It was also argued that many-body localization can be used to protect coherence by preventing relaxation between eigenstates with different local integrals of motion.

Francois Huveneers discussed recent joint work with Wojciech De Roeck [20]. The topic was a theoretical approach to energy localization in both quantum and classical chains of oscillators. It is known that quenched disorder reduces, or sometimes completely suppresses, energy transport in systems like spin chains. When dealing with strongly anharmonic oscillators at high temperature, the notion that randomness coming from the Gibbs state could potentially play the same role was suggested. In this case, randomness evolves with time which would allow for resonances to travel into the system. Results on asymptotic localization of energy, valid for both quenched and thermal disorder, were presented.

The talk of John Imbrie, on joint work with Tom Spencer [33], presented a new method for proving exponential localization in the Anderson model. An explicit construction of eigenvalues and eigenfunctions was given using convergent expansions that exhibit the local dependence of these quantities on the random potential. This method is based on the idea of sequential diagonalization, a la Jacobi, with a mixture of perturbative and non-perturbative estimates. Prospects for this method as a tool for studying many-body localization, see for example Conjecture 1 in Section 3.2, in spin chains with large disorder were also discussed.

Abel Klein and Son Nguyen described recent work proving localization properties of multi-particle, continuous Anderson Hamiltonians [39, 40]. A bootstrap multi-scale analysis is developed and yields finite multiplicity of eigenvalues, dynamical localization, and decay of eigenfunction correlations for these random interacting models. Crucial in the argument is the well-known Wegner estimate which uses a unique continuation principle for spectral projections of Schrödinger operators [38]. An overview of this latter topic, as well as applications, was the focus of the lecture by Peter Hislop, see also [31].

The talk of Arijeet Pal examined the dynamical properties of an isolated, interacting quantum spin system in the presence of disorder, compare [53, 54]. A dynamical quantum phase transition at high energy densities was found that distinguishes between a thermal phase, which thermally equilibrates, and a localized phase, which fails to equilibrate or to serve as its own heat bath. The definition and motivation of different measures used to distinguish the two phases and the transition between them were discussed in detail. Based on a numerical study, evidence was given that suggests that the critical point may have infinite randomness.

Simone Warzel discussed recent results, joint work with Michael Aizenman, on the existence of extended states in the presence of disorder, and in particular, the occurrence of energy regimes in which extended states are formed from resonating local quasi-modes. This mechanism was found to play a key role in characterizing the phase diagram of random Schroedinger operators on tree graphs [3, 4]. Mira Shamis spoke on work in preparation concerning related results for the Bethe strip. In both contexts, the corresponding eigenstates are conjectured to be non-ergodic, in the sense that they violate a heuristic version of the equidistribution principle, yet they do not exhibit Anderson localization. A principle goal is to determine whether similar resonant delocalization may play a role in conduction within systems of many particles. In her talk, Simone Warzel also discussed joint work with Michael Aizenman and Mira Shamis on a toy model, namely the Anderson model on the complete graph, in which the existence of resonating quasi-modes can be rigorously established.

An introduction to the rigorous study of the interplay between interactions and disorder in the theory of Bose Einstein Condensation (BEC) was given by Jakob Yngvason. The talk provided a survey of the few known rigorous results (in both the deterministic and random setting) with particular emphasis on recent results concerning the effects of disorder on dilute Bose gases in the Gross-Pitaevskii limit; a joint work with Robert Seiringer and Valentin Zagrebnov [55]. Valentin Zagrebnov gave a related talk on BEC in the infinite-range hopping Bose-Hubbard model with repulsive on-site particle interaction and in the presence of an ergodic random one-site potential. It was shown that interaction produces a new phenomenon: in contrast to the usual enhancement of BEC for perfect bosons, for constant on-site repulsion and certain discrete single-site random potentials there is a suppression of BEC at some fractional densities which appears with increasing disorder.

3.2 Some Conjectures Formulated at the Workshop

Throughout the course of the workshop, a number of distinct signatures of many-body localization were discussed. Drawing from many different speakers a general consensus formed regarding what could possibly be proven for a variety of spin chains with random terms in the Hamiltonian. To state this more precisely, consider a simple example given by the spin-1/2 Heisenberg antiferromagnetic chain in a random magnetic field:

$$H_L = \sum_{x=-L}^{L-1} \mathbf{S}_x \cdot \mathbf{S}_{x+1} + \lambda \sum_{x=1}^{L} h_x S_x^3, \tag{12}$$

acting on the Hilbert space $\mathcal{H}_L = \bigotimes_{x=-L}^{L} \mathbb{C}^2$ of $2L+1$ spins. Here the interaction is

$$\mathbf{S}_x \cdot \mathbf{S}_{x+1} = S_x^1 S_{x+1}^1 + S_x^2 S_{x+1}^2 + S_x^3 S_{x+1}^3 \,, \tag{13}$$

$\lambda > 0$ is a disorder parameter, and $\{h_x\}$ is a real-valued sequence of random variables. Other examples, like the random XY model or the Ising model in a random transverse field, could also be considered and similar conjectures could be formulated.

Two concrete conjectures about such spin chains were discussed in detail:

Conjecture 1. Similarity by quasi-local unitary with an Ising spin chain having short-range interactions

Briefly, the above claim is that there exist quasi-local unitaries U_L acting on $\mathcal{H}_L$, and (real-valued) random interaction parameters $K_X^{(L)}$, with the following properties:

(i) The $K_X^{(L)}$ are well-defined and with probability 1 converge to $K_X(\omega)$ as $L \to \infty$. In addition, they are required to satisfy a short-range condition uniformly in L, so that an appropriately defined interaction norm is finite.

(ii) The unitaries satisfy a uniform Lieb-Robinson bound of the form: there exist constants $A, a > 0$, such that for all $A \in \mathcal{A}_X, B \in \mathcal{A}_Y$,

$$\|[U_L^* A U_L, B]\| \le C \|A\| \, \|B\| \max\{|\partial X|, |\partial Y|\} e^{-ad(X,Y)}. \tag{14}$$

(iii) Under these unitaries, the Hamiltonian transforms as

$$U_L^* H_L U_L = \sum_{X \subset [-L,L]} K_X^{(L)} \sigma_X, \tag{15}$$

where, for each finite subset X,

$$\sigma_X = \prod_{x \in X} \sigma_x^3. \tag{16}$$

with σ_x^3 being the third Pauli matrix for the spin at $x \in \mathbb{Z}$.

Conjecture 2. Logarithmic growth of the entanglement entropy of half-infinite chains

The second property, for which some numerical evidence was provided at the workshop, is concerned with the generation of bipartite entanglement by the Hamiltonian dynamics of the spin chain. Concretely, for typical pure initial states, ω_0, with zero entanglement between the left and right infinite half chains, say $(-\infty, 0]$ and $[1, +\infty)$, the entanglement grows logarithmically. Mathematically, this means that ω_0 is the product of a pure state of the chain on $(-\infty, 0]$ and another pure state on $[1, +\infty)$. One then considers the state $\omega^{(t)}_{(-\infty,0)}$ of the left half chain defined by

$$\omega^{(t)}_{(-\infty,0)}(A) = \lim_{L\to\infty} \omega_0(e^{itH_L} A e^{-itH_L}) \tag{17}$$

and the conjecture is that, for typical initial conditions, one has that the von Neumann entropy

$$S(\omega^{(t)}_{(-\infty,0)}) \sim c \log |t| \tag{18}$$

for a constant $c > 0$ and large $|t|$.

Proving these conjectures is one of the immediate goals for mathematical physicist working on many-body localization.

4 Scientific Progress and the Outcome of the Meeting

The experts participating in the meeting discussed recent progress and open challenges on a number of concrete topics related to the theory of disordered many-body systems. Among those topics were

- recent physical perspectives on the many-body localization transition,
- specific concepts such as Lieb-Robinson and entanglement bounds, which lend themselves to the mathematical investigation of the emerging physical picture,
- the phenomenon of resonance induced delocalization, which is conjectured to play a role in the transport theory of disordered many-body systems,
- simple toy models which can be used as test cases for relevant concepts (e.g. certain models of quantum spin systems, harmonic oscillator systems, as well as the Anderson model on tree graphs and complete graphs),
- recent progress on the multi-particle Anderson model, in part based on unique continuation properties of the Schrödinger equation,
- the interplay of interactions and disorder in the theory of Bose-Einstein condensation.

In addition to the specific topics mentioned above, one crucial point that this workshop helped to bring into focus is the fact that future progress on the theoretical understanding of disordered quantum many-body systems will require a joint effort from two groups of experts: those who generally work on deterministic many-body models and those who focus mainly on disordered systems. These two groups have had limited interaction in the past, but this workshop has helped to stimulate interest in problems situated firmly in the intersection of these two fields. In fact, the original motivation of physicists to investigate the Anderson model and related single-particle models came from wanting to understand transport in disordered many-body systems, in particular spin systems. Moreover, many-body localization has to be understood as information localization, i.e. the absence of information transport, which is quite different from the much studied localization properties of random Schrödinger-type operators. As a result, this workshop should lead to new collaborations between mathematical researchers with different areas of expertise. In addition, the re-orientation of some important research directions in mathematical physics, with an eye towards issues in disordered many-body systems, may also lead to a closer alignment with recent trends in physics as well as quantum information theory.

As is generally the case with mathematical research, specific outcomes (namely scientific articles or further research proposals/conferences) from the ideas inspired by the discussions at this workshop will, in all likelihood, take a year or two before they are completed. Two concrete conjectures, however, provide well-motivated and clear questions that are a direct result of the collaborative efforts generated by this workshop, see Section 3.2 for more details. Since so much interest has now been established, general progress can also be expected in the mathematical theory of disordered quantum spin systems. Here a number of works are currently under way to verify certain characteristics of many-body localization for larger, i.e. less trivial, classes of disordered spin systems. These expected results will be significant for applications, will greatly advance what is already known in this field, and will provide a mathematically rigorous justification for the physical observations and numerical evidence that were among the topics presented at this workshop.

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